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WORKS OF

PROF. HENRY T. BOVEY

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THEORY OF STRUCTURES

AND

STRENGTH OF MATERIALS

BY

aylor

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Etc., etc.

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DEDICATED

TO

William C. McDonald

WHOSE BENEFACTIONS TO McGILL UNIVERSITY

HAVE DONE SO MUCH TO ADVANCE THE CAUSE OF

SCIENTIFIC EDUCATION

PREFACE TO THE FOURTH EDITION.

THE present edition of this work has been so completely revised and enlarged that it is practically a new treatise. Numerous examples, drawn for the most part from actual practice, have been added to the several chapters, and a large number have been worked out in detail throughout the text so as to illustrate the principles of strength, construction, and design. In determining the stresses in framed structures Bow's method has been used, and the manner of its application has been shown in various examples of roofs and bridges. Chapter X on bridges presents the subject in a somewhat new manner, giving the principles now generally adopted in engineering design. The tables of strengths, elasticities, weights, etc., have been carefully revised, increased in number, and brought up to date. Many examples have been carefully selected from the works of well-known authors, but my special thanks are due to Professor Greenhill for permission to make use of his train problems in Chapter III, and also to Messrs. Waddell and Hedrick for important information on the preliminaries of bridge design, and for notes on problems which have arisen in the course of actual construction. To my colleague, Professor H. M. McKay, I desire to express my sincere thanks for assistance in revising proof-sheets and for many valuable suggestions.

HENRY T. BOVEY.

McGILL UNIVERSITY, October, 1905.

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THEORY OF STRUCTURES.

CHAPTER I.

FRAMES LOADED AT THE JOINTS.

1. Definitions.—*Frames* are rigid structures composed of straight struts and ties, jointed together by means of bolts, straps, mortises, and tenons, etc. *Struts* are members in compression, *ties* members in tension, and the term *brace* is applied to either.

The external forces upon a frame are the loads and the reactions at the points of support, from which may be found the resultant forces at the joints. The greatest care should be exercised in the design of the joints. The resultant forces should severally coincide in direction with the axes of the members upon which they act, and should intersect the joints in their centres of gravity. Owing to a want of homogeneity in the material, errors of workmanship, etc., this coincidence is not always practicable, but it should be remembered that the smallest deviation introduces a bending action. Such an action will also be caused by joint friction when the frame is insufficiently braced. The points in which the lines of action of the resultants intersect the joints are also called the *centres of resistance*, and the figure formed by joining the centres of resistance *in order* is usually a polygon, which is designated the *line of resistance* of the frame.

The *position* of the centres should on no account be allowed to vary. It is assumed, and is practically true, that the joints of a frame are flexible, and that the frame under a given load does not sensibly change in form. Thus an individual member is merely stretched or compressed in the direction of its length, i.e., along its line of resistance, while the frame as a whole may be subjected to a bending action.

The term *truss* is often applied to a frame supporting a weight.

2. Frame of Two Members.—OA, OB are two bars jointed at O and supported at the ends A, B. The frame in Fig. 1 consists of two ties, in Fig. 3 of two struts, and in Fig. 2 of a strut and a tie.

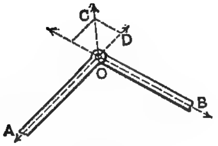


FIG. 1.

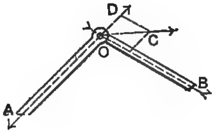


FIG. 2.

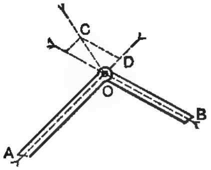


FIG. 3.

Let P be the resultant force at the joint, and let it act in the direction OC. Take OC equal to P in magnitude, and draw CD parallel to OB. OD is the stress along OA, and CD is that along OB.

Let the angle $AOB = \alpha$, and the angle $COD = \beta$.

Let S_1, S_2 be the stresses along OA, OB, respectively. Then

$$\frac{S_1}{P} = \frac{OD}{OC} = \frac{\sin(\alpha - \beta)}{\sin \alpha} \quad \text{and} \quad \frac{S_2}{P} = \frac{CD}{OC} = \frac{\sin \beta}{\sin \alpha}.$$

3. Frame of Three or More Members.—Let $A_1A_2A_3 \ldots$ be a polygonal frame jointed at A_1, A_2, A_3, . . . Let P_1, P_2, P_3, . . . be the resultant forces at the joints A_1, A_2, A_3, . . . , respectively. Let S_1, S_2, S_3, . . . be the forces along A_1A_2, A_2A_3, . . . , respectively.

Consider the joint A_1.

The lines of action of three forces, P_1, S_1, and S_6, intersect in this joint, and the forces, being in equilibrium, may be represented in direction and magnitude by the sides of the triangle Os_1s_6, in which s_1s_6 is parallel to P_1, Os_1 to S_1, and Os_6 to S_6.

Similarly, P_2, S_1, S_2 may be represented by the sides of the triangle Os_1s_2 which has one side, Os_1, common to the triangle Os_1s_6. and so on.

Thus every joint furnishes a triangle having a side common to each of the two adjacent triangles, and all the triangles together from a *closed* polygon $s_1s_2s_3 \ldots$ The sides of this polygon represent in magnitude and direction the resultant forces at the joints, and the radii from the pole O to the angles $s_1 s_2 s_3$, . . . represent in magnitude, direction, and character the forces along the several sides of the frame $A_1 A_2 A_3$ . . . The polygon $A_1 A_2 A_3$ . . . is the line of resistance of the frame, and is called the *funicular polygon* of the forces P_1, P_2, P_3, . . . with respect to the pole O.

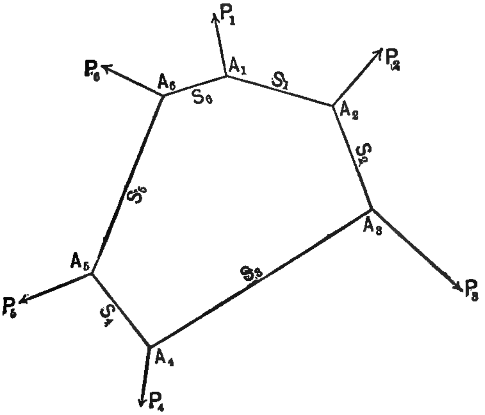


FIG. 4.

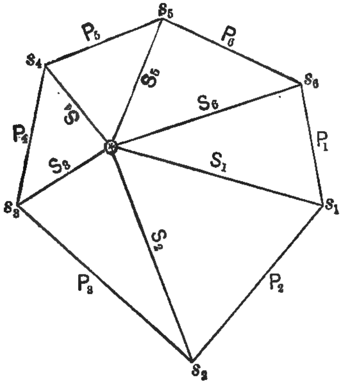


FIG. 5.

The two polygons are said to be *reciprocal*, and, in general, two figures in graphical statics are said to be *reciprocal* when the sides in the one figure are parallel or perpendicular to corresponding sides in the other.

A triangle or polygon is also said to be the reciprocal of a point when its sides are parallel or perpendicular to corresponding lines radiating from the point. Thus the triangle $Os_6 s_1$ is the reciprocal of the point A_1, and the polygon $A_1 A_2 A_3$ . . . is the reciprocal of the point O.

If more than *two* members meet at a joint, or if the joint is subjected to more than *one* load, the resulting force diagram will be a quadrilateral, pentagon, hexagon, . . . according as the number of members is 3, 4, 5, . . . or the number of loads 2, 3, 4, . . .

In practice it is usually required to determine the stresses in a number of members radiating from a joint in a framed structure. If the reciprocal of the joint can be drawn, its sides will represent in direction and magnitude the stresses in the corresponding members.

The *converse* of the preceding is evidently true. For if a system of forces is in equilibrium, the polygon of forces $s_1s_2s_3$ . . . must close, and therefore the polygon which has its sides respectively parallel to the radii from a pole O to the angles s_1, s_2, s_3, . . . and which has its angles upon the lines of action of the forces, must also close.

Ex. 1. Let O be a joint in a framed structure, and let Os_1, Os_2, Os_3, . . . be the axes of the members radiating from it. The polygon $A_1A_2A_3$ . . . is the reciprocal of O, the side A_1A_2 representing the stress along Os_1, the side A_2A_3 that along Os_2, etc., Fig. 5.

Ex. 2. Let the resultant forces at the joints be parallel. The polygon of forces becomes the straight line s_1s_5, which is often termed the *line of loads.* Thus, the forces P_2, P_3, . . . P_5 are represented by the sides s_1s_2, s_2s_3, . . . s_4s_5, which are in one straight line *closed* by s_1s_6 and s_5s_6, representing the remaining forces P_1 and P_6, while the triangles Os_1s_2, Os_2s_3, . . . are the reciprocals of the points A_2, A_3 . . . Draw OH perpendicular to s_1s_5. The projection of each of the lines Os_1, Os_2, Os_3, . . . perpendicular to s_1s_5 is the same and equal to OH, which therefore represents in magnitude and direction the stress which is the same for each member of the frame.

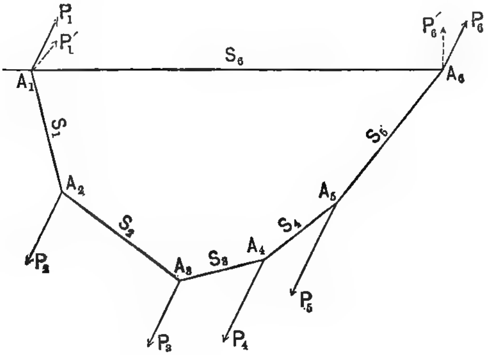


FIG. 6.

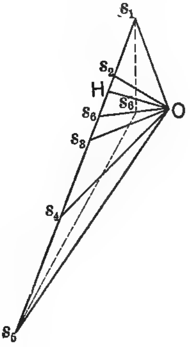


FIG. 7.

Let α_1, α_2, α_3, . . . be the inclinations of the members A_1A_2, A_2A_3, . . . respectively, to the line of loads. Then

$$OH = Hs_1 \tan \alpha_1 = Hs_5 \tan \alpha_5;$$

$$\therefore\ OH\ (\cot \alpha_1 + \cot \alpha_5) = Hs_1 + Hs_5 = s_1s_5 = P_2 + P_3 + P_4 + P_5 = P_1 + P_6,$$

and OH, in direction and magnitude, is equal to the stress common to each member. Also, the stress in any member, e.g., A_4A_5, $= Os_4 = OH$ cosec α_4.

Let the resultant forces at the joints A_1,A_6 be inclined to the common direction of the remaining forces, and act in the direction shown by the dotted lines. Let P_1', P_6' be the magnitudes of the new forces; draw s_1s_6' parallel to the direction of P_6' so as to meet Os_6 in s_6'; join s_5s_6'. Since there is equilibrium, $s_6's_5$ must be parallel to the line of action of P_6'. Thus $s_6's_5s_1$ is the force polygon.

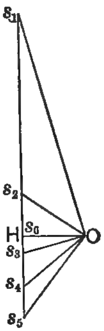


FIG. 8.

Ex. 3. The forces, or loads, P_2, P_3, ... P_5 are generally vertical, while P_1, P_6 are the vertical reactions of the two supports.

Suppose, e.g., that $A_1A_2 \dots A_6$ is a rope or chain suspended from the points A_1, A_6, in a horizontal plane and loaded at $A_2A_3 \dots$ with weights P_2, P_3, ... The chain will hang in a form dependent upon the magnitude of these weights. The points H and s_6 will coincide, and OH will represent the horizontal tension of the chain.

Let the polygon $A_1A_2 \dots A_6$ be inverted, and let the rope be replaced by rigid bars, A_1A_2, $A_2A_3 \dots$ The diagram of forces will remain the same, and the frame will be in equilibrium under the *given* loads. The equilibrium, however, is unstable, as the chain, and consequently the inverted frame, will change form if the weights vary. Braces must then be introduced to prevent distortion.

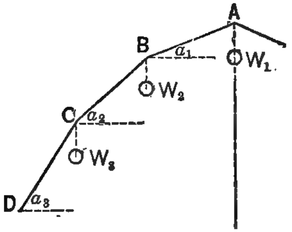


FIG. 9.

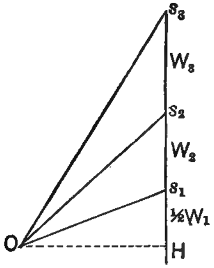


FIG. 10.

Take the case of a frame $DCBA \dots$ symmetrical with respect to a vertical through A, and let the weights at A, B, C, ... be W_1, W_2, W_3, ..., respectively.

Drawing the stress diagram in the usual manner, OH represents the horizontal thrust of the frame.

The portions s_1s_2, s_2s_3, ... of the line of loads give a definite relation between the weights for which the truss will be stable. The result may be expressed analytically, as follows:

Let α_1, α_2, α_3, ... be the inclinations of AB, BC, CD, ..., respectively, to the horizontal.

Let the horizontal thrust $OH = H$. Then

$$H = \frac{W_1}{2}\cot\alpha_1 = \left(\frac{W_1}{2} + W_2\right)\cot\alpha_2 = \left(\frac{W_1}{2} + W_2 + W_3\right)\cot\alpha_3 = \dots$$

If $W_1 = W_2 = W_3 = \dots,$

$$\cot\alpha_1 = 3\cot\alpha_2 = 5\cot\alpha_3 = \dots$$

If there are two bars only, viz., AB, BC, on each side of the vertical centre line, the frame will have a double slope, and in this form is employed to support a *Mansard* roof.

If there are a number of bars on each side of A, and if α_n, α_{n+1} are the inclinations to the horizon, of the nth and $(n+1)$th bars, respectively, counting from A,

$$\cot \alpha_{n+1} = \frac{2n-1}{2n+1} \cot \alpha_n.$$

If such a frame as the above is inverted, the stresses in the members are reversed in kind, but remain of the same magnitude.

4. Non-closing Polygons.—Let a number of forces P_1, P_2, P_3, . . . act upon a structure, and let these forces, *taken in order*, be represented in direction and magnitude by the sides of the unclosed figure $MNPQ$. . . This figure is the unclosed *polygon of forces*, and its closing line TM represents in direction and magnitude the resultant of the forces P_1, P_2, P_3, . . .

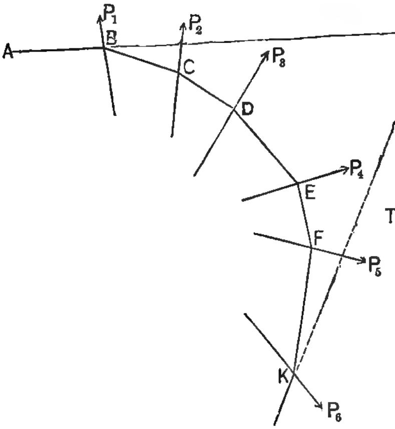


FIG. 11.

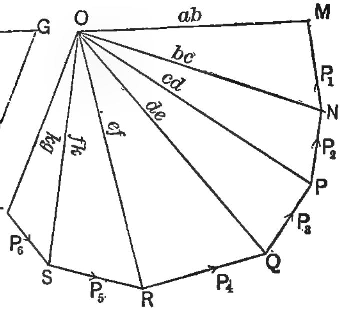


FIG. 12.

For PM is the resultant of P_1 and P_2, and may replace them; QM may replace PM and P_3, i.e., P_1, P_2, and P_3; and so on.

Take any point O and join OM, ON, OP, . . .

Draw a line AB parallel to OM and intersecting the line of action of P_1 in *any* point B. Through B draw BC parallel to ON and cutting the line of action of P_2 in C. Similarly, draw CD parallel to OP, DE to OQ, EF to OR, . . . The figure $ABCD$ . . . is called the *funicular polygon* of the given forces with respect to the *pole O.* The position of the pole O is *arbitrary,* and therefore an infinite number of funicular polygons may be drawn with different poles.

Also the position of the point B in the line of action of P_1 is arbitrary, and hence an infinite number of funicular polygons with their corresponding sides parallel, i.e., an infinite number of *similar* funicular polygons, may be drawn with the *same* pole.

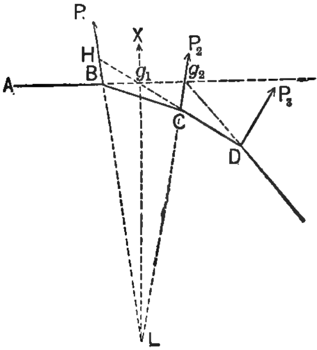


FIG. 13.

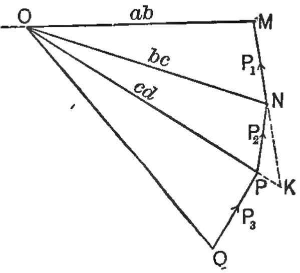


FIG. 14.

5. To show that the Intersection of the First and Last Sides of the Funicular Polygon (i.e., the Point G) is a Point on the Actual Resultant of the System of Forces P_1, P_2, P_3, . . .—First consider two forces P_1, P_2, MNP being the force and $ABCD$ the funicular polygon.

Let AB, DC, the first and last sides of the latter, be produced to meet in g_1; also let DC produced meet the line of action of P_1 in H.

Produce OP and MN to meet in K.

Let the lines of action of P_1 and P_2 meet in L.

By similar triangles,

$$\frac{KP}{KN}=\frac{HC}{HL}; \quad \frac{KN}{KO}=\frac{HB}{HC}; \quad \frac{KO}{KM}=\frac{Hg_1}{HB}.$$

Hence
$$\frac{KP}{KN}\,\frac{KN}{KO}\,\frac{KO}{KM}=\frac{HC}{HL}\,\frac{HB}{HC}\,\frac{Hg_1}{HB},$$

or
$$\frac{KP}{KM}=\frac{Hg_1}{HL},$$

and therefore, since the angle H is equal to the angle K, the line PM is parallel to the line Lg_1.

But PM represents in magnitude the resultant of the forces P_1, P_2, and is parallel to it in direction.

Therefore Lg_1 is also parallel to the direction of the resultant.

But L is evidently a point on the *actual* resultant of P_1, P_2. Hence g_1 must be a point on this resultant.

Next, let there be three forces, P_1, P_2, P_3.

Replace P_1, P_2 by their resultant X acting in the direction Lg_1. The force and funicular polygons for the forces X and P_2 are evidently MPQ and Ag_1DE, respectively; and g_2, the point of intersection of Ag_1 and ED produced, is, as already proved, a point on the actual resultant of X and P_3, i.e., of P_1, P_2, and P_3.

Hence the *first* and *last* sides, AB, ED, of the funicular polygon $ABCDE$ of the forces P_1, P_2, P_3, with respect to the pole O, intersect in a point which is on the *actual* resultant of the given forces.

The proof may be similarly extended to four, five, and any number of forces.

If the forces are all parallel, the force polygon of the two forces P_1, P_2 becomes a straight line, MNQ. Draw the funicular polygon $ABCD$ as before, and through g_1, the intersection of the *first* and *last* sides, draw g_1Y parallel to MQ, and cutting BC in Y.

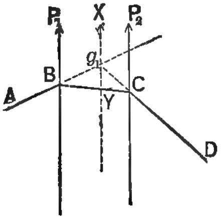


FIG. 15.

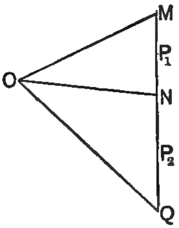


FIG. 16.

By similar triangles,

$$\frac{P_1}{ON}=\frac{MN}{ON}=\frac{g_1Y}{BY} \quad \text{and} \quad \frac{P_2}{ON}=\frac{QN}{ON}=\frac{g_1Y}{CY};$$

$$\therefore \frac{P_1}{P_2}=\frac{CY}{BY}.$$

Hence Yg_1, which is parallel to the direction of the forces P_1, P_2, divides the distance between their lines of action into segments which are inversely proportional to the forces, and must therefore be the line of action of their resultant. The proof may be extended to any number of forces, as in the preceding.

Funicular Curve.—Let the weights upon a beam AB become infinite in number, and let the distances between the weights diminish indefinitely.

The load then becomes continuous, and the funicular polygon is a curve, called the funicular curve.

The equation to this curve may be found as follows:

Let the tangents at two consecutive points P and Q meet in R. This point is on the vertical through the centre of gravity of the load upon the portion MN of the beam.

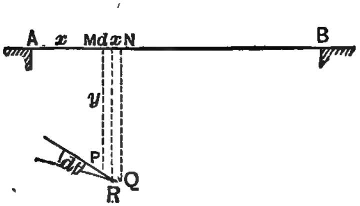


FIG. 17.

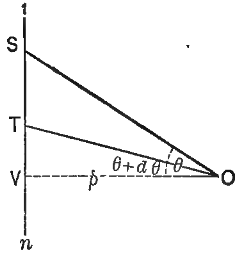


FIG. 18.

Let 1n be the line of loads, and let OS, OT be the radial lines from O, the pole, parallel to the tangents at P and Q. Take A as the origin, and let w be the intensity of the load.

Let θ be the inclination of the tangent at P to the beam, and let the polar distance $OV = p$.

The load upon the portion MN is wdx. Then

$$wdx = ST = SV - TV = p \tan \theta - p \tan (\theta + d\theta)$$
$$= -pd\theta, \text{ approximately.}$$

Therefore $$-w = p\frac{d\theta}{dx} = p\frac{d^2y}{dx^2}, \quad \text{since} \quad \theta = \frac{dy}{dx}.$$

Integrating twice,

$$py = -\int\int wdx^2 + c_1x + c_2,$$

c_1 and c_2 being constants of integration.

If w is constant,

$$py = -\frac{wx^2}{2} + c_1x + c_2,$$

and the curve is a parabola.

6. Centres of Gravity.—Let it be required to determine the centre of gravity of any plane area symmetrical with respect to an axis XX. Divide the area into suitable elementary areas a_1, a_2, a_3, . . . having known centres of gravity.

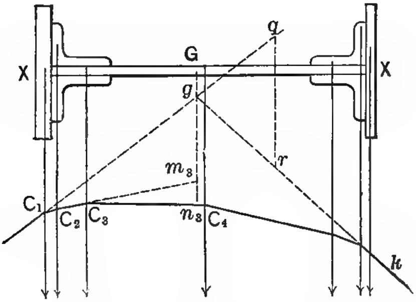


FIG. 19.

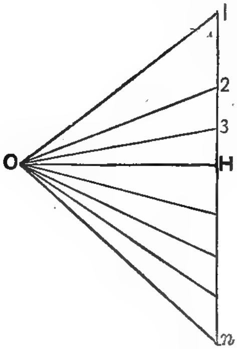


FIG. 20.

Draw the force (the line $1n$) and funicular polygons corresponding to these areas, and let g be the point in which the first and last sides of the funicular polygon meet. The line drawn through g parallel to $1n$ must pass through the centre of gravity of all the elementary areas and, therefore, of the whole area. Hence it is the point G in which this line intersects the axis XX.

Rail and similar sections may be divided into elementary areas by drawing a number of parallel lines at right angles to the axis of symmetry, and at such distances apart that each elementary figure may, without sensible error, be considered a rectangle of an area equal to the product of its breadth by its mean height.

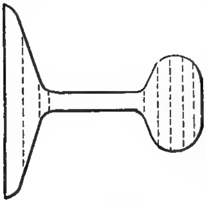


FIG 21.

In the case of a very irregular section, an accurate template of the section may be cut out of cardboard or thin metal. If the template is then suspended from a pin through a point near the edge, the centre of gravity of the section will lie in the vertical through the pin. By changing the point of suspension, a new line in which the centre of gravity lies may be found. The intersection of the two lines must, therefore, be the centre of gravity required. Another method of finding the centre of gravity is to carefully balance the template upon a needle-point.

The area of such a section may be determined either by means of a planimeter or by balancing the template against a rectangle cut out of the same material, the area of the rectangle being evidently the same as that of the section.

7. Moment of Inertia of a Plane Area.—Let any two consecutive sides, C_2C_3, C_3C_4, of the funicular polygon, Fig. 19, meet the line gG in the points m_3, n_3.

Let x_1, x_2, x_3, . . . be the lengths of the perpendiculars from the centres of gravity of a_1, a_2, a_3, . . . , respectively, upon gG.

Draw the line OH perpendicular to the line of loads, and let $OH=p$.

By the similar triangles $C_3m_3n_3$ and $O34$,

$$\frac{m_3n_3}{x_3}=\frac{34}{p}=\frac{a_3}{p}, \quad \text{or} \quad m_3n_3=\frac{a_3x_3}{p}$$

and
$$\frac{a_3}{p}\frac{x_3^2}{2}=m_3n_3\frac{x_3}{2}=\text{area of triangle } C_3m_3n_3.$$

But the total area A bounded by the funicular polygon $C_1C_2C_3$ . . . and the lines gC_1, gk is the sum of all the triangular areas C_1gm_1, $C_2m_2n_2$, $C_4m_4n_4$, . . . , described in the same manner as $C_3m_3n_3$. Therefore

$$A=\frac{a_1}{p}\frac{x_1^2}{2}+\frac{a_2}{p}\frac{x_2^2}{2}+\ldots=\frac{\Sigma(ax^2)}{2p}.$$

The sum $\Sigma(ax^2)$ is the moment of inertia, I, of the plane area with respect to gG. Hence

$$A=\frac{I}{2p}, \quad \text{or} \quad I=2Ap.$$

The moment of inertia I_y of the area, with respect to a parallel axis at distance y_1 from gG, is given by the equation

$$I_y=I+Sy_1^2,$$

where $S=A_1+A_2+\ldots$

Let the new axis intersect C_1g and kg in the points q and r. Since the triangles qgr and $O1n$ are similar,

$$\frac{qr}{y_1}=\frac{1n}{p}=\frac{S}{p},$$

and therefore the area A' of the triangle qgr

$$=\frac{qr}{2}y_1=\frac{Sy_1^2}{2p}.$$

Hence $$I_y=2pA+2pA'=2p(A+A').$$

Note.—If p be made $=\frac{1n}{2}=\frac{A}{2}$,

$$I=A^2 \quad \text{and} \quad Sy_1^2=AA',$$

and therefore $$I_y=A(A+A').$$

The angle $1On$ is also evidently a right angle.

8. Bow's Method.—An examination of the frame and stress diagrams, Figs. 4 and 5, shows:

(1) That if the lines representing external forces on parts of the frame meet in a point, the corresponding lines in the stress diagram form a closed polygon.

(2) That if the lines representing parts of the frame diagram form a closed polygon, the corresponding lines in the stress diagram meet in a point.

(3) That if lines meet in a point in the stress diagram, the corresponding lines in the frame diagram are contiguous and form either closing or non-closing polygons.

(4) That of all the stress diagrams which can be drawn for a given frame under given loads, there is only one which satisfies the three reciprocal relationships expressed in (1), (2), and (3), and it is called the reciprocal diagram of stress.

As a consequence of these relationships, Bow devised a system of notation which greatly facilitates the construction of the stress diagram.

Lines are drawn to represent the external forces acting at the joints. A letter is then assigned to each enclosed area of the frame and also to each space between the lines of action of the external forces. Thus each line in the frame diagram is defined by the two letters in the two spaces separated by the line in question. The corresponding line in the reciprocal diagram is parallel to this line and is similarly named.

Generally speaking, this method will be adopted in dealing with the stresses developed in framed structures.

9. **Roof-trusses.**—A roof consists of a covering and of the trusses (or frames) upon which it is supported. The covering is generally laid upon a number of *common rafters* which rest upon horizontal beams (or *purlins*), the latter being carried by trusses spaced at intervals varying with the type of construction, but averaging about 10 feet. The truss-rafters are called *principal* rafters, and the trusses themselves are often designated as *principals.*

In roofs of small span the trusses and purlins are sometimes dispensed with.

Types of Truss.—A roof-truss may be constructed of timber, of iron or steel, or of these materials combined. Timber is almost invariably employed for small spans, but in the longer spans it has been largely superseded by iron, in consequence of the combined lightness, strength, and durability of the latter.

Attempts have been made to classify roofs according to the mode of construction, but the variety of form is so great as to render it impracticable to make any further distinction than that which may be drawn between those in which the reactions of the supports are vertical and those in which they are inclined.

Fig. 22 is a simple form of truss for spans of less than 30 ft.

Fig. 23 is a superior framing for spans of from 30 to 40 ft.; it may be still further strengthened by the introduction of struts, Figs. 24 and 25, and with such modification has been employed to span openings of 90 ft. It is safer, however, to limit the use of the type shown by Fig. 24 to spans of less than 60 ft. Figs. 26, 27, 28, 29, and 30 are forms of truss suitable for spans of from 60 to 100 ft. and upwards.

Arched roofs, Figs. 29 and 30, admit of a great variety of treatment. They have a pleasing appearance, and cover wide spans without intermediate supports. The flatness of the arch is limited by the requirement of a minimum thrust at the abutments. The thrust may be resisted either by thickening the abutments or by introducing a tie. If the only load upon a roof-truss were its own weight, an arch in the form of an inverted catenary, with a shallow rib, might be used. But the action of the wind induces oblique and transverse stresses, so that a considerable depth of rib is generally needed. If the depth exceed 12 in., it is better to connect the two flanges by braces than by a solid web. Roofs of wide span are occasionally carried by ordinary lattice girders.

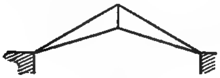


FIG. 22.

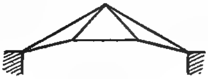


FIG. 23.

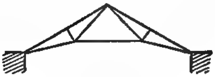


FIG. 24.

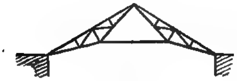


FIG. 25.

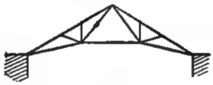


FIG. 26.

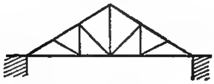


FIG. 27.

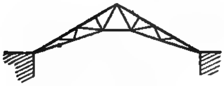


FIG. 28.

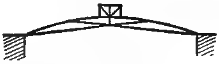


FIG. 29.

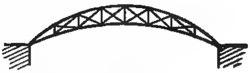


FIG. 30.

Principals, Purlins, etc.—The principal rafters in Figs. 22 to 28 are straight, abut against each other at the peak, and are prevented by tie-rods from spreading at the heels. When made of steel, tee (T), rail, and channel (both single — and double][) bars, bulb-tee (T) and rolled (I) beams, are all excellent forms.

Timber rafters are rectangular in section, and for the sake of economy and appearance are often made to taper uniformly from heel to peak.

The heel is fitted into a suitable cast-iron skew-back, or is fixed between wrought-iron angle-brackets and rests either directly upon the wall or upon a wall-plate.

When the span exceeds 60 ft., allowance should be made for alterations of length due to changes of temperature. This may be effected by interposing a set of rollers between the skew-back and wall-plate at one heel, or by fixing one heel to the wall and allowing the opposite skew-back to slide freely over a wall-plate.

The junction at the peak is made by means of a casting or by steel plates.

Light iron and timber beams as well as angle-irons are employed as purlins. They are fixed to the top or sides of the rafters by brackets, or lie between them in cast-iron shoes, and are usually held in place by rows of tie-rods, spaced at 6 or 8 ft. intervals between peak and heel, running the whole length of the roof.

The sheathing-boards and final metal or slate covering are fastened upon the purlins. The nature of the covering regulates the spacing of the purlins, and the size of the purlins is governed by the distance between the main rafters, which may vary from 4 ft. to upwards of 25 ft. When the interval between the rafters is so great as to cause an undue deflection of the purlins, the latter should be trussed. Each purlin may be trussed, or a light beam may be placed midway between the main rafters so as to form a supplementary rafter.

Struts are made of timber or iron. Timber struts are rectangular in section. Steel struts may consist of L bars, T bars, or light columns, while cast iron may be employed for work of a more ornamental character. The strut heads are attached to the rafters by means of cast caps, steel straps, brackets, etc., and the strut feet are easily designed both for pin and screw connections.

Ties may be of flat or round bars attached either by eyes and pins or by screw-ends, and occasionally by rivets. The greatest care is necessary in properly proportioning the dimensions of the eyes and pins to the stresses that come upon them.

To obtain greater security, each of the end panels of a roof may be provided with lateral braces, and wind-ties are often made to run the whole length of the structure through the feet of the main struts.

Due allowance must be made in all cases for changes of temperature.

10. Roof-Weights.—In calculating the stresses in the different members of a roof-truss two kinds of load have to be dealt with, the one *permanent* (or *dead*) and the other *accidental* (or *live*). The permanent load consists of the *covering,* the *framing,* and *accumulations of snow.*

Tables at the end of the chapter show the weights of various coverings and framings.

The weight of freshly fallen snow may vary from 5 to 20 lbs.

per cubic foot. English and European engineers consider an allowance of 6 lbs. per square foot sufficient for snow, but in cold climates, similar to that of North America, it is probably unsafe to estimate this weight at less than 12 lbs. per square foot.

The *accidental* or *live* load upon a roof is the wind pressure, the maximum force of which has been estimated to vary from 40 to 50 lbs. per square foot of surface *perpendicular to the direction of blow*. Ordinary gales blow with a force of from 20 to 25 lbs., which may sometimes rise to 34 or 35 lbs., and even to upwards of 50 lbs. during storms of great severity. Pressures much greater than 50 lbs. have been recorded, but are wholly untrustworthy. Up to the present time, indeed, all wind-pressure data are most unreliable, and to this fact may be attributed the frequent wide divergence of opinion as to the necessary wind allowance in any particular case. The great differences that exist in all recorded wind pressures are primarily due to the unphilosophic, unscientific, and unpractical character of the anemometers which give no correct information either as to pressure or velocity. The inertia of the moving parts, the transformation of velocities into pressures, and the injudicious placing of the anemometer, which renders it subject to local currents, all tend to vitiate the results.

It would be practically absurd to base calculations upon the violence of a wind-gust, a tornado, or other similar phenomena, as it is almost absolutely certain that a structure would not lie within its range. In fact, it may be assumed that a wind pressure of 40 lbs. per square foot upon a surface perpendicular to the direction of blow is an ample and perfectly safe allowance, especially when it is remembered that a greater pressure than this would cause the overthrow of nearly all existing towers, chimneys, etc.

11. Wind Pressure upon Inclined Surfaces.—The pressure upon an inclined surface may be obtained from the following formula, which was experimentally deduced by Hutton, viz.:

$$p_n = p \sin \alpha^{1.84 \cos\alpha - 1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

p being the intensity of the wind pressure in pounds per square foot upon a surface perpendicular to the direction of blow, and p_n being the normal intensity upon a surface inclined at an angle α to the direction of blow.

Let p_h, p_v be the components of p_n, parallel and perpendicular, respectively, to the direction of blow: Then

$$p_h = p_n \sin \alpha \quad \text{and} \quad p_v = p_n \cos \alpha.$$

Hence, if the inclined surface is a roof, and if the wind blows horizontally, α is the roof's pitch.

Again, let v be the velocity of a fluid current in feet per second, and be that due to a head of h feet.

Let w be the weight of the fluid in pounds per cubic foot.

Let p be the pressure of the current in pounds per square foot upon a surface perpendicular to its direction.

If the fluid, after striking the surface, is free to escape at right angles to its original direction,

$$p = 2hw = \frac{v^2}{g}w.$$

Hence for ordinary atmospheric air, since $w = .08$ lb., approximately,

$$p = \frac{.08}{32}v^2 = \left(\frac{v}{20}\right)^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

When the wind impinges upon a surface oblique to its direction, the intensity of the pressure is $\left(\frac{v \sin \beta}{20}\right)^2$, v being the absolute impinging velocity, and β being the angle between the direction of blow and the surface impinged upon.

Tables prepared from formulæ A and B are given at the end of the chapter.

12. Distribution of Loads.—Engineers have been accustomed to assume that the live load is uniformly distributed over the whole of the roof, and that it varies from 30 to 35 lbs. per square foot of covered surface for short spans, and from 35 to 40 lbs. for spans of more than 60 ft. But the wind may blow on one side only, and although its direction is usually horizontal, it may occasionally be inclined at a considerable angle, and be even normal to a roof of high pitch. It is therefore evident that the horizontal component (p_h) of the normal pressure (p_n) should not be neglected, and it may cause a complete *reversal* of stress in members of the truss, especially if it is of the arched or braced type.

If P_n is the total normal wind pressure on the side of a roof of pitch α, its horizontal component $P_n \sin \alpha$ will tend to push the roof horizontally over its supports. This tendency must be resisted by the reactions at the supports.

In roofs of small span the foot of each rafter is usually *fixed* to its support, and it may be assumed that each support exerts the same reaction, which should therefore be equal to $\frac{P_n \sin \alpha}{2}$. In roofs of large span the foot of one rafter is fixed, while that of the other rests upon rollers. The latter is not suited to withstand a horizontal force, and the whole of the horizontal component of the wind pressure must be borne at the fixed end, where the reaction should be assumed to be equal to $P_n \sin \alpha$.

In designing a roof-truss it is assumed that the wind blows on one side only, and that the total load is concentrated at the joints (or points of support) of the principal rafters.

For example, let the rafters AB, AC of a truss be each supported at two intermediate points (or joints), D, E and F, G, respectively, and let the wind blow on the side AB.

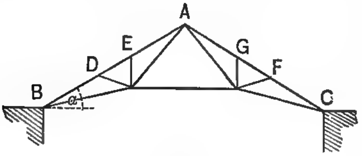


FIG. 31.

Take $BD = CF = l_1$, $DE = FG = l_2$, $EA = GA = l_3$; and let $l_1 + l_2 + l_3 = l$; $BC = 2l \cos \alpha$, α being the angle ABC.

Let W be the permanent (or dead) load per square foot of roof-surface. Le p_n be the normal wind pressure per square foot of roof surface. Let d be the horizontal distance in feet from centre to centre of trusses.

The total normal live load concentrated at $B = p_n d \frac{l_1}{2}$; at $D = p_n d \frac{l_1 + l_2}{2}$; at $E = p_n d \frac{l_2 + l_3}{2}$; at $A = p_n d \frac{l_3}{2}$.

The total vertical dead load concentrated at D and $F = wd \frac{l_1 + l_2}{2}$; at E and $G = wd \frac{l_2 + l_3}{2}$; at $A = wdl_3$.

Let R_1, R_2 be the *resultant* vertical reactions at B and C, respectively (i.e., the total vertical reactions less the dead weights, $wd \frac{l_1}{2}$, concentrated at these points).

Take moments about C. Then

$R_1 2l\cos\alpha$ = sum of moments of live loads about C + sum of moments of dead loads about C
= moment of resultant wind pressure about C + moment of resultant dead load about C

$$=p_n ld\left(\frac{l}{2}+l\cos 2\alpha\right)+wd(l_1+2l_2+2l_3)l\cos\alpha,$$

where $\frac{l}{2}+l\cos 2\alpha$ is the perpendicular from C upon the line of action of the resultant wind pressure which bisects AB normally.

The moment of the horizontal reaction at B or C about C is evidently *nil.*

R_2 may be found by taking moments about B.

The rafters AB, AC have one end, B, fixed and the other, C, on rollers, the reaction R_2 at C being vertical. A force P, inclined at α to the vertical, acts upon AB at D, and Q is the resultant vertical load on the rafters. Let P and Q meet in E. Taking $ED=P$, and the vertical $DF=Q$, then, on the same scale, FE is the resultant R_3 of P and Q. Let it make an angle θ with the vertical. Then

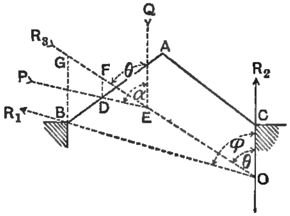


FIG. 32.

$$\sin\overline{\alpha-\theta}=\frac{Q}{P}\sin\theta=\frac{Q}{R_3}\sin\alpha.$$

Let R_3 and R_2 meet in O. Then OB is the direction of the resultant reaction R_1 at B. Taking the vertical $BG=R_2$, then, on same scale, $OG=R_3$ and $OB=R_1$, OBG being a triangle of forces. Let R_1 make an angle ϕ with the vertical. Then

$$\sin(\phi-\theta)=\frac{R_2}{R_1}\sin\theta=\frac{R_2}{R_3}\sin\phi.$$

If $BG=l$ and if p and q are the perpendiculars from B upon P and Q,

$$R_1 l=Pp+Qq.$$

13. Simple Frames of Various Types.—(*a*) *Jib-crane.*—Fig. 33 represents an ordinary jib-crane. OX is the post fixed in the ground at O, BC is the jib, and AC the tie. The jib-tie and gearing are so separated from the post as to admit of a free rotation round its axis. Fig. 34 is the stress diagram when the crane supports a weight W as indicated.

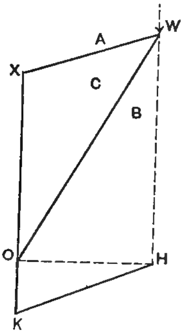


FIG. 33.

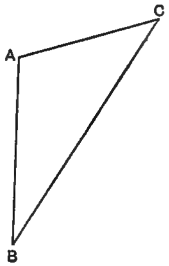


FIG. 34.

The weight W, however, is not suspended directly from B, but is raised or lowered by means of a chain passing over pulleys to a chain-barrel which is usually fixed to the post. Disregarding pulley friction, the tension in the chain is $\frac{W}{n}$, n being the number of *falls*. Let the dotted line, Fig. 35, indicate the direction in which

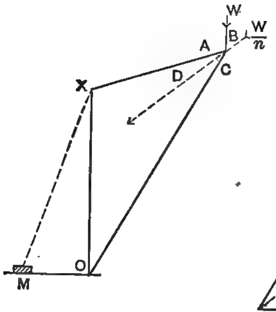


FIG. 35.

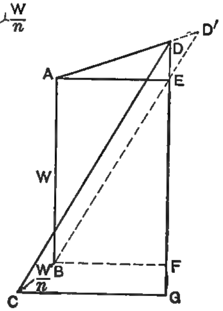


FIG. 36.

the chain passes to the chain-barrel. The loads on the apex are now W, acting vertically, and $\frac{W}{n}$, acting in the direction of the chain. Thus Fig. 36 is the stress diagram, and AC represents in direction and magnitude the resultant force at the apex. AD is the tension in the tie, and CD the compression in the jib. The dotted lines in Fig. 36 also show that generally the effect of chain tension is to increase the thrust on the jib and to diminish the tension in the tie.

Draw the horizontals AE and CG to meet the verticals at D in E and G respectively.

Then DG = vert. component of thrust in jib = pressure in post at O,
and DE = vert. component of tension in tie = tension along post.

Therefore $DG - DE = W + FG$ = resultant pressure in post at O.

Also, AE = horizontal component of tension AD in tie, and the moment of this force with respect to $O = AE \times XO$. This moment tends to upset the crane.

OH, the horizontal projection of the jib (or tie), is the *radius* or *throw* of the crane.

If the post revolves about its axis (as in *pit*-cranes), the jib and gearing are bolted to it, and the whole turns on a pivot at the toe K. In this case, the frame, as a whole, is kept in equilibrium by the weight W, the horizontal reaction H of the web plate at O, and the reaction R at G. The first two forces meet in H and therefore the reaction at K must also pass through H.

Hence, since OHK may be taken to represent the triangle of forces,

$$H = W\frac{OH}{OK} \quad \text{and} \quad R = W\frac{HK}{OK}.$$

In a portable crane the tendency to upset is counteracted by means of a weight placed upon a horizontal platform OM attached to the post and supported by the tie XM.

The horizontal pull at $X = AE$, Fig. 35, and if OM is taken to represent AE, then on the same scale OX is the counterweight at M.

(*b*) *Derrick-crane.*—The figure shows a combination of a derrick and crane, called a derrick-crane. It is distinguished from the jib-crane by having two backstays, AD, AE. One end of the jib is hinged at or near the foot of the post, and the other is held by a chain which passes over pulleys to a winch on the post, so that the jib may be raised or lowered as required.

The derrick-crane is generally made of wood, is simple in construction, is easily erected, has a vertical as well as a lateral motion, and a range equal to a circle of from 10 to 60 feet radius. It is therefore useful for temporary works, setting masonry, etc.

The stresses in the jib and tie are calculated as in the jib-crane, and those in the backstays and post may be obtained as follows:

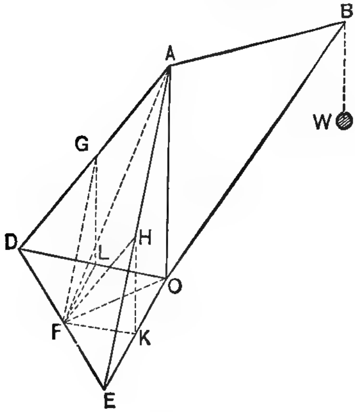


FIG. 37.

Let the plane of the tie and jib intersect the plane DAE of the two backstays in the line AF, and suppose the backstays replaced by a single tie AF. Take OF to represent the horizontal pull at A. The pull on the "imaginary" stay AF is then represented by AF and is evidently the *resultant* pull on the two backstays. Completing the parallelogram $FGAH$, AH will represent the pull on the backstay AE, and AG that upon AD, their horizontal components being OK, OL, respectively. The figure $OKFL$ is also a parallelogram.

If the backstays lie in planes at right angles to each other,

$OL = OF \cos\theta = T \sin\alpha \cos\theta$, and is a maximum when $\theta = 0°$,

and

$OK = OF \sin\theta = T \sin\alpha \sin\theta$, and is a maximum when $\theta = 90°$,

θ being the angle FOL, and α the inclination of the tie to the vertical.

Hence the stress in a backstay is a maximum when the plane of the backstay and post coincides with that of the jib and tie.

Again, let β be the inclination of the backstays to the vertical. The vertical components of the backstay stresses are

$$T \sin\alpha \cos\theta \cot\beta \quad \text{and} \quad T \sin\alpha \sin\theta \cot\beta;$$

and, therefore, the corresponding stress along the post is

$$T \sin\alpha \cot\beta\,(\cos\theta + \sin\theta),$$

which is a maximum when $\theta = 45°$.

c. Shear-legs (*or Shears*) *and Tripods* (*or Gins*) are often employed when heavy weights are to be lifted. The former consists of two struts, AD, AE, united at A and supported by a tie AC, which may be made adjustable so as to admit of being lengthened or shortened The weight is suspended from A, and the legs are capable of revolving

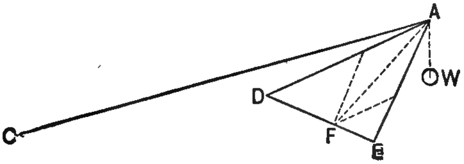


FIG. 38.

around DE as an axis. Let the plane of the tie and weight intersect the plane of the legs in AF, and suppose the two legs replaced by a single strut AF. The thrust along AF can now be easily obtained, and hence its components along the two legs.

In tripods one of the three legs is usually longer than the others. They are united at the top, to which point the tackle is also attached.

d. Bridge- and Roof-trusses of Small Span.—A single girder is the simplest kind of bridge, but is only suitable for very short spans. For longer spans the middle point of the girder may be supported

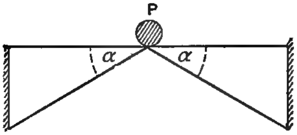


FIG. 39.

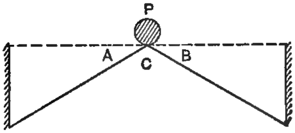


FIG. 40.

by struts, Fig. 39, through which a portion of the weight is transmitted to the abutments. Let P be the portion of the weight thus transmitted, as in Fig. 40. Fig. 41, in which $AB = P$ is the line of loads, is the stress diagram, and the thrust along each strut is $AC(=BC) = \frac{P}{2}$ cosec α. Drawing the horizontal CO, $AO(=BO)$ and CO are the components of the thrust at the foot of a strut. Thus the vertical and horizontal pressures on the masonry at the foot of a strut are $\frac{P}{2}$ and $\frac{P}{2}$ cot α respectively.

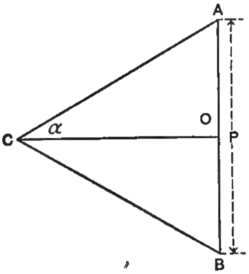


FIG. 41.

In Fig. 42 a *straining-cill* is introduced, and the girder is supported at two intermediate points. Let P be the portion of the weight at each of these points which is transmitted through the struts and the

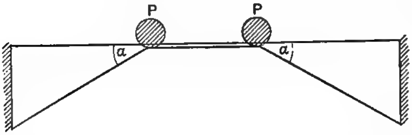


FIG. 42.

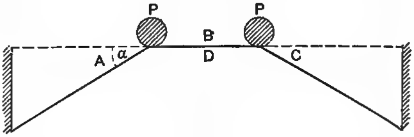


FIG. 43.

straining-cill, Fig. 43. Then Fig. 44 is the stress diagram, ABC being the line of loads. AD, CD are drawn parallel to the corresponding struts, and $AB = BC = P$. Draw DB horizontally.

The thrust along the straining-cill $= DB = P \cot \alpha$.

" " " a strut $= AD$ (or CD) $= P \operatorname{cosec} \alpha$.

By means of straining-cills the girders may be supported at

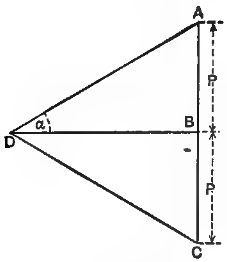


FIG. 44.

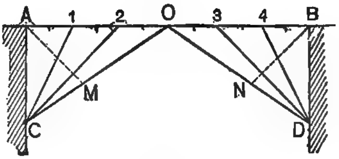


FIG. 45.

several points, 1, 2, . . . , and the weight concentrated at each may be assumed to be one half of the load between the two adjacent points of support. The calculations for the stresses in the struts, etc., are made precisely as above.

If the struts are very long, they are liable to bend, and counter-braces, AM, BN, are added to counteract this tendency.

e. The *triangle* is the only geometrical figure of which the form cannot be changed without varying the lengths of the sides. For this reason, all compound trusses for bridges, roofs, etc., are made up of triangular frames.

Fig. 46 represents the simplest form of roof truss. AC, BC are rafters of equal length inclined to the horizontal at an angle α, and each carries a uniformly distributed load W.

The rafters react horizontally upon each other at C, and their feet are kept in position by the tie-beam AB. Consider the rafter AC.

The resultant of the load upon AC, i.e., W, acts through the middle point D.

Let it meet the horizontal thrust H of BC upon AC in F. For equilibrium the resultant thrust at A must also pass through F.

Also, if γ is the angle FAE,

$$\cot\gamma = \frac{AE}{EF} = \frac{AE}{2DE} = \frac{1}{2}\cot\alpha.$$

AFE is evidently a triangle of forces for W, H, and R. Therefore

$$H = W\cot\gamma = \frac{W}{2}\cot\alpha,$$

$$R = W\operatorname{cosec}\gamma = W\sqrt{1+\frac{\cot^2\alpha}{4}},$$

$$R\cos\gamma = H = \text{tension in tie,}$$
$$R\sin\gamma = W = \text{pressure on support.}$$

It must of course be remembered that when the horizontal member acts as a tie the reactions at the two supports, due to the external loads on the truss, must necessarily be vertical.

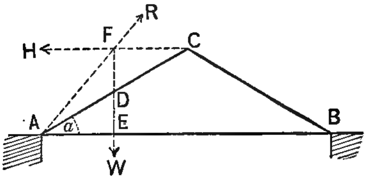


Fig. 46.

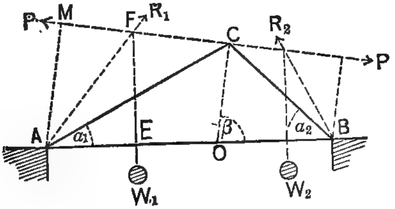


Fig. 47.

If the rafters AC, BC are unequal, let α_1, α_2 be their inclinations to A, B, respectively.

Let W_1 be the uniformly distributed load upon AC, W_2 that upon BC.

Let the direction of the mutual thrust P at C make an angle β with the vertical, so that if CO is drawn perpendicular to FC, the angle $COB = \beta$; the angle $ACF = 90° - ACO = 90° - (\beta - \alpha_1)$.

Draw AM perpendicular to the direction of P, and consider the rafter AC. As before, the thrust R_1 at A, the resultant weight W_1 at the middle point of AC, and the thrust P at C meet in the point F.

Take moments about A. Then

$$P \cdot AM = W_1 AE.$$

But $AM = AC \sin ACM = AC \cos(\beta - \alpha_1)$,

and $AE = \frac{AC}{2} \cos \alpha_1$. Therefore

$$P = \frac{W_1}{2} \frac{\cos \alpha_1}{\cos(\beta - \alpha_1)}.$$

Similarly, by considering the rafter BC,

$$P = \frac{W_2}{2} \frac{\cos \alpha_2}{\sin(\beta + \alpha_2 - 90^\circ)} = -\frac{W_2}{2} \frac{\cos \alpha_2}{\cos(\beta + \alpha_2)}.$$

Hence $$\frac{W_1}{2} \frac{\cos \alpha_1}{\cos(\beta - \alpha_1)} = P = -\frac{W_2}{2} \frac{\cos \alpha_2}{\cos(\beta + \alpha_2)},$$

and therefore $$\tan \beta = \frac{W_1 + W_2}{W_1 \tan \alpha_2 - W_2 \tan \alpha_1}.$$

The horizontal thrust of each rafter $= P \sin \beta$.

The vertical thrust upon the support $A = W_1 - P \cos \beta$.

The vertical thrust upon the support $B = W_2 + P \cos \beta$.

Sometimes it is expedient to support the centre of the tie-beam upon a column or wall, the king-post being a pillar against which the heads of the rafters rest in such a manner that the reaction upon AC at C is at right angles to AC.

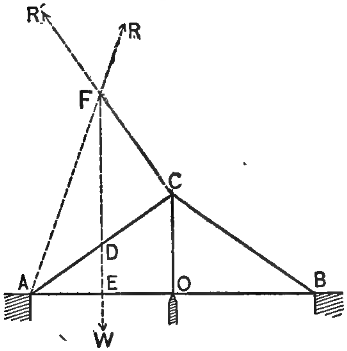


Fig. 48.

Consider the rafter AC.

The normal reaction R' of CO upon AC, the resultant weight W at the middle point D, and the thrust R at A meet in the point F.

Taking moments about A,

$$R' \cdot AC = W \cdot AE, \quad \text{or} \quad R' = \frac{W}{2} \cos \alpha.$$

Thus the total thrust transmitted through CO to the support at $O = W \cos^2 \alpha$.

The horizontal thrust on each rafter $=\frac{W}{2}\cos\alpha\sin\alpha=\frac{W}{4}\sin 2\alpha$.

14. King-post Truss.—The simple triangular truss may be modified by the introduction of a king-post, Fig. 49, which carries a portion P of the weight on the tie-beam, and transfers it through the rafters to the end of the tie, where it is again resolved into horizontal and vertical components, the former straining the tie in tension, and the latter causing a pressure on the supports.

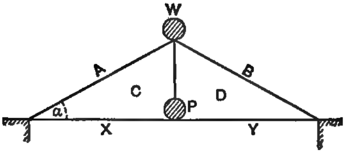


FIG. 49.

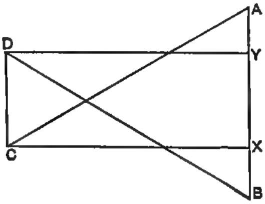


FIG. 50.

Of the weight W on each rafter, it is assumed that one-half is carried on a support, and the other half on the ridge. Thus the loads on the frame are W at the ridge, P at the foot of the post, and the two vertical reactions, each $=\frac{1}{2}(W+P)$, at the supports. Fig. 50 is evidently the stress diagram, AB being the line of loads. Then

$AB=W$; reaction at each support $=XA=YB=\frac{W+P}{2}$; $YX=P$.

Therefore

$$AC=\text{thrust along rafter}=\tfrac{1}{2}(W+P)\operatorname{cosec}\alpha,$$
$$CX=DY=\text{tension in tie}=\tfrac{1}{2}(W+P)\cot\alpha,$$
$$DC=YX=\text{tension in post}=P.$$

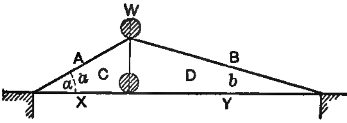


FIG. 51.

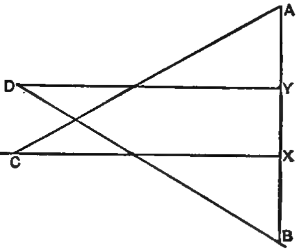


FIG. 52.

In Fig. 51 the post divides the tie-beam into two segments a and b. If W is the load on the ridge, assumed equal to *one* *half* of the load on the two rafters, and if P is the load concentrated at the foot of the post,

the reaction at the left support $= \dfrac{b}{a+b}(W+P)$ and

" " " " right " $= \dfrac{a}{a+b}(W+P)$.

Fig. 52 is evidently the stress diagram, AB being the line of loads.

Then $AB = W$; reaction at left support $= XA = W\dfrac{b}{a+b}$, at right support $= YB = W\dfrac{a}{a+b}$. Therefore

AC = thrust along one rafter $= W\dfrac{b}{a+b}\operatorname{cosec}\alpha$;

BD = " " the other rafter $= W\dfrac{a}{a+b}\operatorname{cosec}\beta$;

$CX = DY$ = tension in tie $= W\dfrac{a}{a+b}\cot\alpha = W\dfrac{b}{a+b}\cot\beta$;

$DC = YX$ = tension in post $= P$.

The tension in the tie diminishes as α increases, and is nothing when $\alpha = 90°$, i.e., when the rafter is vertical.

15. Incomplete Frames.—The frames discussed in the preceding articles will support, *without change of form*, any load consistent with strength, and the stresses in the several members can be found in terms of the load. It sometimes happens, however, that a frame is *incomplete*, so that it tends to change form under every distribution of load. An example of this class is the simple trapezoidal truss, consisting of the two horizontal members AB, DE, and the two equal inclined members AD, BE, Fig. 53.

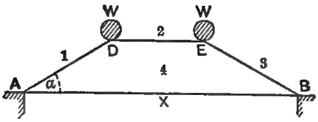


FIG. 53.

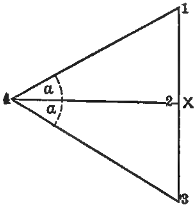


FIG. 54.

First, let there be a weight W at each of the points D, E.

Fig. 54 is the stress diagram in which $12 = 23 = W$.

Evidently also $X1 = X3 = W$ = vertical reaction at each support.

$14\ (=34) = W \operatorname{cosec} \alpha$ = thrust along each sloping member.

Tension in $AB = 24 = W \cot \alpha$ = thrust along DE.

Next let there be a weight W_1 at D and a weight $W_2 (< W_1)$ at E.

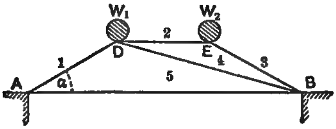


Fig. 55.

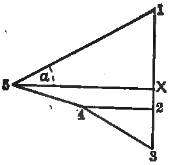


Fig. 56.

Let 123 be the line of loads, where $12 = W_1$ and $23 = W_2$.

The stress diagram for the joint E is the triangle 243, and if $1n$ is drawn parallel to AD, the point n evidently falls outside 4 and the stress diagram does not close. The frame will therefore be distorted unless a brace is introduced connecting B and D or A and E. In the former case the stress diagram becomes 12345, the line 45 being drawn parallel to BD.

Drawing the horizontal $5X$, the vertical reaction at A is $X1$, and at B is $X3$.

In general, if Fig. 57 represents any frame of four members, resting upon supports at A and B, and if W is a weight concentrated at D, then 124 is the stress diagram for the joint at D. Drawing 43 parallel to DE, it is evident that 23 is the only weight which can be supported at E without producing distortion.

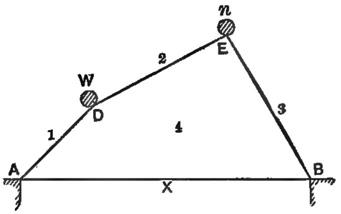


Fig. 57.

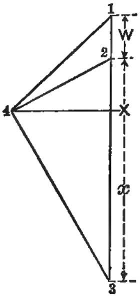


Fig. 58.

16. Queen-post Truss.—It is a common practice to modify the incomplete frame represented by Fig. 53 by introducing two vertical *queen-posts* (*queen-rods* or *queens*) DF and EG, Fig. 59, through which the loads are transmitted to D and E. The frame thus modified is still incomplete, and if there are no diagonal braces DG, EF, the distortion of the frame under an unevenly distributed load can only be prevented by the friction at the joints, the stiffness of the members, and by the queens being rigidly fixed to AB at F and G.

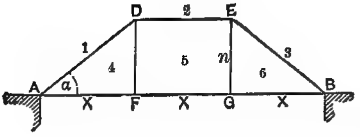


FIG. 59.

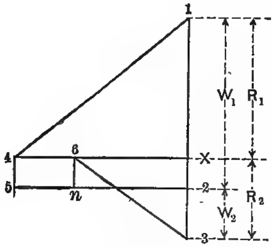


FIG. 60.

Let W_1 be the load at F transmitted through the queen FD to D.

Let W_2 $(<W_1)$ be the load at G transmitted through the queen GE to E.

If the frame is rigid, the reactions R_1 at A and R_2 at B, which will balance these weights, can easily be found by taking moments about B and A successively. Thus,

$$R_1 l = \frac{W_1}{2}(l+c) + \frac{W_2}{2}(l-c)$$

and

$$R_2 l = \frac{W_1}{2}(l-c) + \frac{W_2}{2}(l+c),$$

where $AB = l$ and $FG = c$.

In the line of loads, Fig. 60, $12 = W_1$ and $23 = W_2$.

Take $X1 = R_1$ and $X3 = R_2$. Then $X14$ and 12541 are the reciprocals of the joints at A and D respectively, so that there is

1st. A thrust $= 25 = X4 = R_1 \cot \alpha$ in DE, and

2d. A thrust $= 45 = X2 = \dfrac{W_1 - W_2}{2} \dfrac{l-c}{l}$ in DF

$=$ *downward* pressure at F on the tie AB.

Again, $X36$ and $236n2$ are the reciprocals of the joints at B and E respectively. For equilibrium this last should close with the reciprocal of the joint D, i.e., n should coincide with 5. But it appears that there is

1st. A thrust $=n2=6X=R_2 \cot \alpha$ in DE, and

2d. A tension $=6n=X2=\frac{W_1-W_2}{2}\frac{l-c}{l}$ in EG

$=$ an *upward* pull at G on the tie AB.

Thus in DE there is an unbalanced thrust

$$=(R_1-R_2) \cot \alpha=(W_1-W_2)\frac{c}{l} \cot \alpha=\frac{W_1-W_2}{2}\frac{(l-c)}{d}\frac{c}{l},$$

d being the depth of the truss.

The tie is also acted upon at F and G by two forces of equal magnitude but acting in opposite directions, thus forming a couple of moment $\frac{W_1-W_2}{2}\frac{l-c}{l}c$, which tends to cause a rotation of the tie-beam.

To neutralize the tendency to rotation and to take up the unbalanced force in DE, a brace may be introduced from D to G, Fig. 61, or from E to F. In the former case the brace will be in compression, and in the latter case in tension.

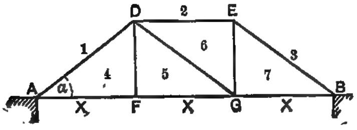


FIG. 61.

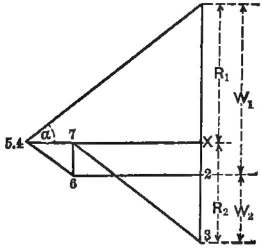


FIG. 62.

Fig. 62 is evidently the stress diagram for the modified truss, and the stress developed in DG by the unbalanced force is

$$56=\frac{W_1-W_2}{2}\frac{l-c}{l}\sec \beta=\frac{W_1-W_2}{2}\frac{l-c}{l}\frac{s}{d},$$

s being the length of the diagonal.

With a single collar-beam DE, Fig. 63, and a load $2W$ uniformly distributed over the rafters, it may be assumed that $\frac{W}{2}$ is concentrated at each of the joints D, C, and E. Then Fig. 64 is the stress diagram, 1234 being the line of loads.

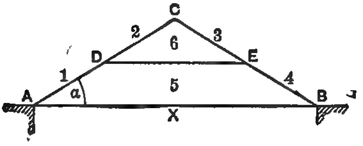


FIG. 63.

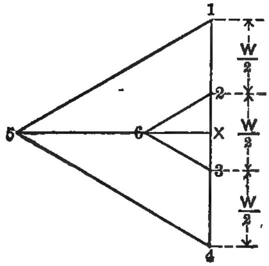


FIG. 64.

The stress in the tie-beam $= X5 = \frac{3}{4}W \cot \alpha$,
" " " " collar-beam $= 56 = \frac{1}{2}W \cot \alpha$,
" " " AD (or BE) $= 15 = \frac{3}{4}W \operatorname{cosec} \alpha$,
" " " CD (or CE) $= 26 = \frac{1}{4}W \operatorname{cosec} \alpha$.

With a collar-beam DE, two queen-posts DF, EG, Fig. 65, and a uniformly distributed load of $2W$ over the rafters, the stresses in the members at the joints D and E become indeterminate. A further condition is therefore required, and it is sometimes assumed that the component of each of the weights $\frac{W}{2}$ at D and E, normal to the rafter on which it is concentrated, is taken up by the collar-beam and the queen.

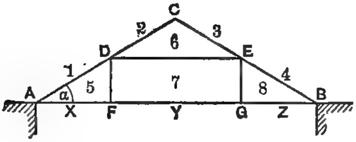


FIG. 65.

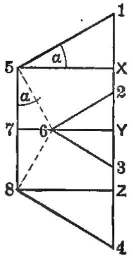


FIG. 66.

Fig. 66 is evidently the stress diagram, 1234 being the line of loads.

$$56\ (=68) = \frac{W}{2}\cos\alpha = \text{component of } \frac{W}{2} \text{ normal to rafter,}$$

$$57\ (=78) = \frac{W}{2}\cos^2\alpha = \text{thrust along each queen.}$$

The effect upon the tie-beam is the same as if a load $\frac{W}{2}\cos^2\alpha$ is concentrated at each of the joints F and G, the loads being directly borne by the supports at A and B.

$$67 = \frac{W}{2}\cos^2\alpha\tan\alpha = \frac{W}{4}\sin 2\alpha = \text{thrust along collar-beam,}$$

$$X5 = Y7 = Z8 = \text{tension in tie-beam,}$$

$$X1 = Z4 = \tfrac{3}{2}W - XZ = \tfrac{3}{2}W - 58 = \tfrac{3}{2}W - W\cos^2\alpha = \text{vertical reaction at each support,}$$

$$26\ (=36) = \frac{W}{4}\operatorname{cosec}\alpha;\ 15\ (=48) = \frac{3}{4}W\operatorname{cosec}\alpha.$$

This frame belongs to the *incomplete* class, and if it has to support an *unequally* distributed load, braces must be introduced from D to G and from E to F.

The truss ABC, Fig. 67, having the rafters supported at two intermediate points, may be employed for spans of from 30 to 50 feet.

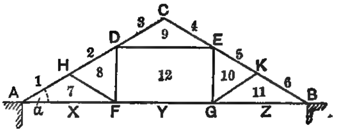


Fig. 67.

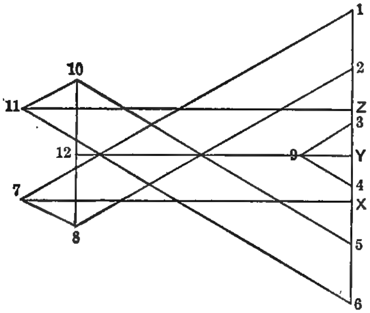


Fig. 68.

Suppose that these intermediate joints of support trisect the rafters, and let each rafter carry a uniformly distributed load W.

It may be assumed that $\frac{W}{3}$ is concentrated at each of the joints H, D, C, E, K. Let P be the load borne directly at each of the joints F and G.

Fig. 68 is evidently the stress diagram, 16 being the line of loads, and $12 = 23 = 34 = 45 = 56 = \frac{W}{3}$.

Also, $X1 = \frac{5}{6}W + P = Z6 =$ vertical reaction at each support;

$$ZY = P = YX.$$

Then

$$17 = (\tfrac{5}{6}W + P) \operatorname{cosec} \alpha = 6 \cdot 11,$$
$$78 = \frac{W}{6} \operatorname{cosec} \alpha = 10 \cdot 11,$$
$$28 = (\tfrac{2}{3}W + P) \operatorname{cosec} \alpha = 5 \cdot 10,$$
$$X7 = (\tfrac{5}{6}W + P) \cot \alpha = Y \cdot 6,$$
$$Y \cdot 12 = (\tfrac{2}{3}W + P) \cot \alpha,$$
$$9 \cdot 12 = \left(\frac{W}{2} + P\right) \cot \alpha.$$

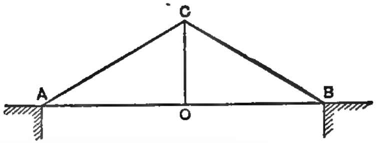


FIG. 69.

17. King-post Roof-truss. (Fig. 69.)—This truss is a simple and economical frame for spans of not more than 30 feet. To eliminate excessive bending and to diminish the danger of transverse failure, the middle points of the comparatively long rafters are supported by struts OD and OE. A portion of the weight on the rafters is then transmitted through these struts to the vertical tie (king-post or rod) CO, which again transmits it through the rafters to act partly as a vertical pressure upon the supports and partly as a tension on the tie-beam. The main purpose, indeed, of struts and ties is to transform transverse into longitudinal stresses.

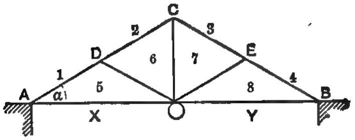


FIG. 70.

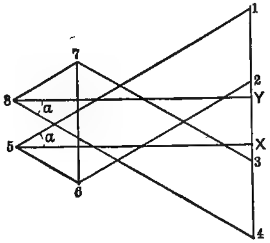


FIG. 71.

If W is the uniformly distributed load on each rafter, it may be assumed that *one-half* of the load upon AD and upon BE is borne directly by the supports at A and B, and that, of the remainder of the load, $\frac{W}{2}$ is concentrated at each of the joints D, C, and E.

Let P be the load concentrated at O.

Then Fig. 71 is evidently the stress diagram, 1234 being the line of loads in which $12 = 23 = 34 = \frac{W}{2}$.

Also, $X1 = \frac{3}{4}W + \frac{P}{2} = Y4 =$ vertical reaction at each support.

The tension in the tie $= X5 = X1 \cot\alpha = \left(\frac{3}{4}W + \frac{P}{2}\right)\cot\alpha = Y8$,

" " " king-post $= 67 = \frac{W}{2} + P$,

" thrust " DA (or BE) $= 15 = \left(\frac{3}{4}W + \frac{P}{2}\right)\operatorname{cosec}\alpha$,

" " " CD (or CE) $= 26 = \left(\frac{1}{4}W + \frac{P}{2}\right)\operatorname{cosec}\alpha$,

" " " DO (or EO) $= 56 = \frac{W}{4}\operatorname{cosec}\alpha$.

18. Bent Crane.—Fig. 72 shows a convenient form of crane when much headroom is required near the post. The crane is merely a girder with one end fixed, and the sections of the members and other details of construction are governed by the lifting power required.

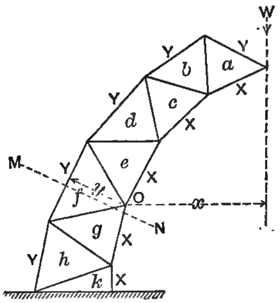


FIG. 72.

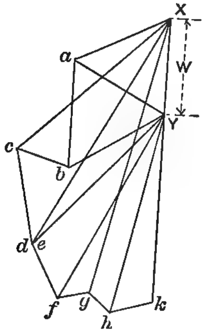


FIG. 73.

For light loads, say not exceeding 10 tons, the flanges may be braced together as shown in Fig. 72. The flanges may be kept at the same distance apart throughout, or the distance between them may be gradually diminished from the base to the peak.

Let W be the weight concentrated at the peak.

Fig. 73 is the stress diagram, and the magnitudes of the stresses obtained from this diagram may be checked by the method of moments.

Take $W=10$ tons, and consider the equilibrium of the portion of the crane above the section MN.

Let x, y be the length of the perpendiculars from O upon Yf and the line of action of W respectively. Then

$$Yf \times y = 10 \times x, \quad \text{and} \quad \text{therefore} \quad Yf = 10\frac{x}{y} = \frac{150}{6\frac{2}{3}} = 22\tfrac{1}{2} \ \text{tons},$$

which is the value obtained for Yf from the stress diagram.

19. Roof-trusses of Considerable Span.—(*a*) Each of the joints in the principal rafters of the roof-truss represented by Fig. 74 is loaded with a weight W. The line of loads is NU, Fig. 75, in which $NO = OP = W = \ldots = TU$, and Fig. 75 is the diagram giving the stresses in all the members of the truss.

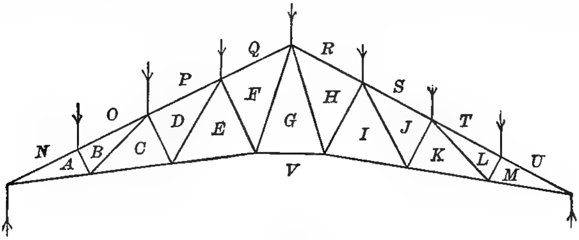


FIG. 74.

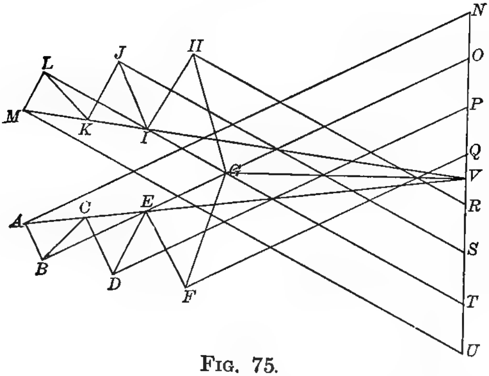


FIG. 75.

(*b*) Next, let the truss be subjected to loads which act on the joints of one of the principal rafters and which are inclined to the vertical.

(*c*) Again, let the frame Fig. 76 be subjected to inclined loads at the joints. The reactions due to these loads may be found as follows:

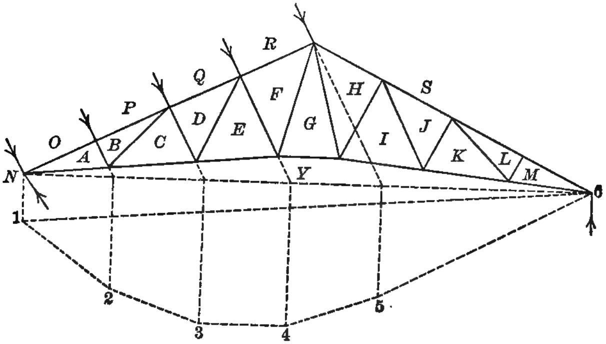


FIG. 76

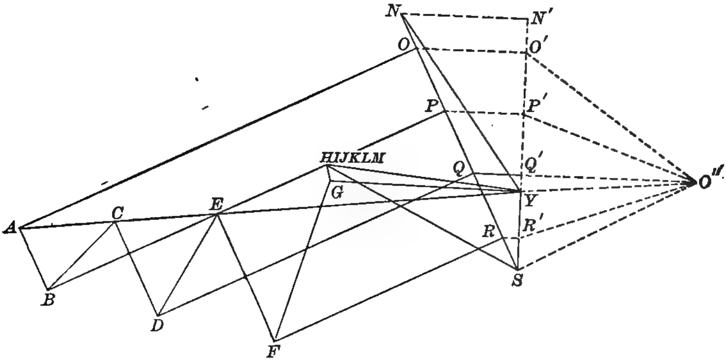


FIG. 77.

Produce the lines of action of the loads to meet the springing-line.

Resolve the loads NO, OP, PQ, QR, and RS into their vertical and horizontal components, the former being represented by $N'O'$, $O'P'$, $P'Q'$, $Q'R'$, and $R'S$, Fig. 77.

Take any pole O'' and join it with O', P', Q' R', S.

Construct the funicular polygon 123456 and draw $O''Y$ parallel to the closing line 16. Then SY and YN are the actual reactions at the supports, and the reciprocal diagram Fig. 77 can be easily constructed, as already described.

In this figure the points *H*, *I*, *J*, *K*, *L*, and *M* coincide, and there can be no stress in *HI*, *IJ*, *JK*, *KL*, or *LM*.

(*d*) *Lock-joint Trusses.*—In certain trusses the stresses at two or more joints, called lock-joints, are indeterminate, and the reciprocal figure cannot be directly drawn. A further condition is required, and it is sometimes assumed that the stresses in two of the members meeting at a lock-joint are equal. The stresses are also determinate if the relative *yield* of the bars is known. A simple and independent method may be illustrated by means of the truss shown in Fig. 78.

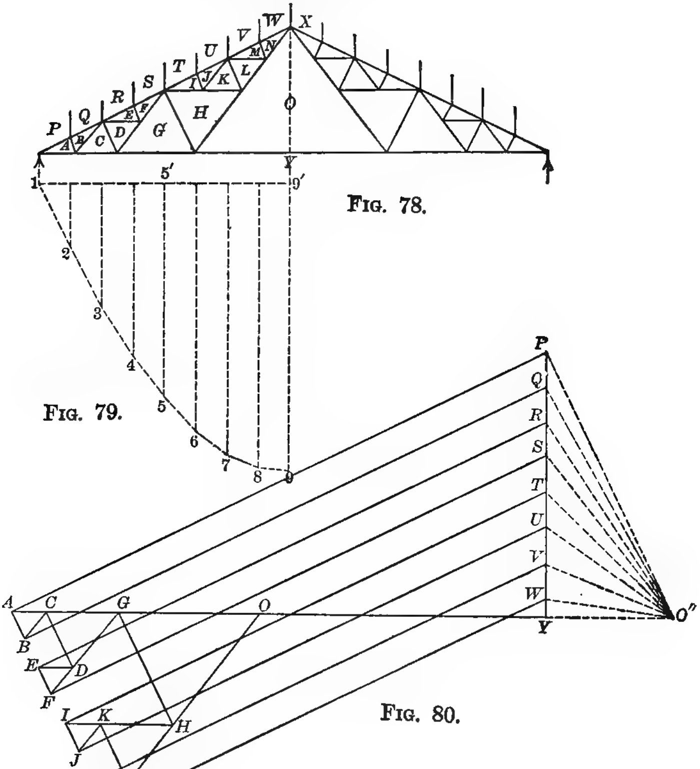


FIG. 78.

FIG. 79.

FIG. 80.

Taking any pole O'', Fig. 79 is the semi-funicular polygon of the loads on the roof, and the bending moment at any point is measured by the intercept between the closing-line 19′ and the line 123 . . . 9 (Chapter II).

Let F_1 be the force in OY;

x be the vertical distance between OY and the truss apex.

$$\text{Then} \quad F_1 x = \text{the bending moment at the apex}$$
$$= 99' \times O''Y.$$

Take $O''Y = x$;

$$\text{then} \quad F_1 = 99',$$

and therefore 99′ is the stress in the member OY.

Let F_2 be the force in GY.

$$\text{Then} \quad F_2 \frac{x}{2} = \text{bending moment at apex of space } G$$
$$= 55' \times O''Y$$
$$= 55' \times x.$$
$$\text{Therefore} \quad F_2 = 2 \times 55',$$

and thus the stress in GY is twice the intercept 55′.

In Fig. 80 take $YO = 99$ and $YG = 2 \times 55'$. The reciprocal diagram can now be easily constructed.

Consider the truss Fig. 81 with two lock-joints. In this truss the rise is about one fifth of the span; the struts FG and ST are normal to the rafters, and about one tenth of the span in length. The members AB and BC, CD and DE, . . . , JK and KL meet

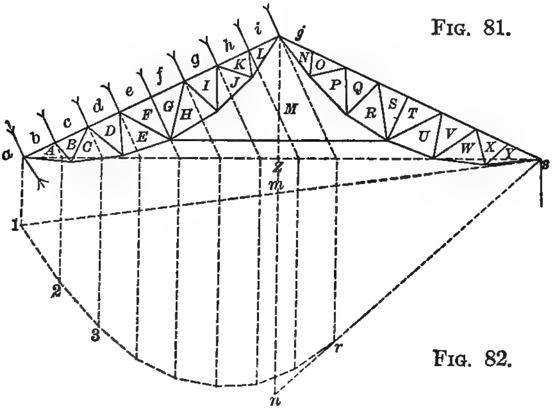


FIG. 81.

FIG. 82.

on a circular arc drawn through the inner end of the strut *FG* and through the ends of the rafter.

Let x be the distance between the member *MZ* and the apex of the truss.

Let the truss be fixed at the left support and rest on rollers at the right support.

Consider the effect of a normal wind pressure.

At the points of intersection of the springing-line with the lines of action of the loads resolve the loads into vertical and horizontal components, and represent the former by $a'b'$, $b'c'$, $c'd'$, $\ldots i'j$. Take the pole O'' at a distance from the load-line $a'j$ equal to the vertical distance between the ridge and *MZ*. Draw the funicular polygon $123\ldots rs$ for the vertical components of the loads, and draw $O''Z$ parallel to the closing-line $s1$. Then jZ is the reaction at the right support, and is of course vertical. Hence if the vertical through the truss apex intersects the closing-line $1s$ in m and sr produced in n, mn is the total stress in *MZ*, and thus the position of the point *M* in Fig. 83 is defined. The diagram can now be completed in the usual manner.

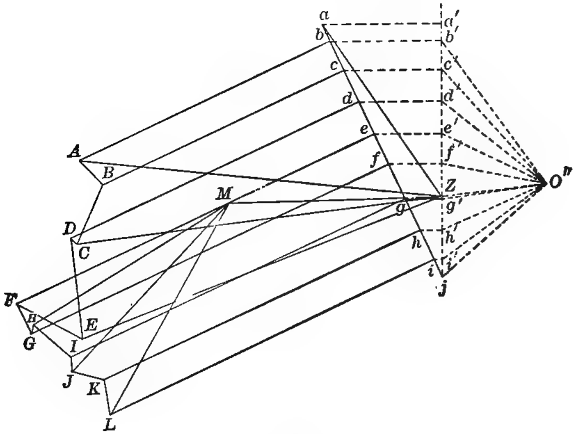


Fig. 83.

(*e*) The principal rafters of the truss represented by Fig. 84 are each l ft. long and are spaced d ft. centre to centre. They carry a dead weight of w lbs. per square foot of roof-covering, and upon the rafter *AB* there is a normal wind pressure of p_n lbs. per square foot. The end B is fixed, and rollers are placed under the foot of AC.

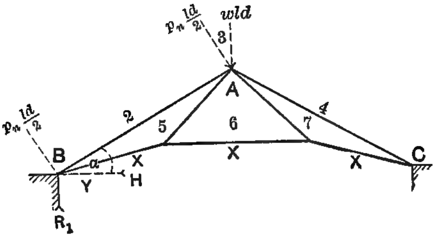


FIG. 84.

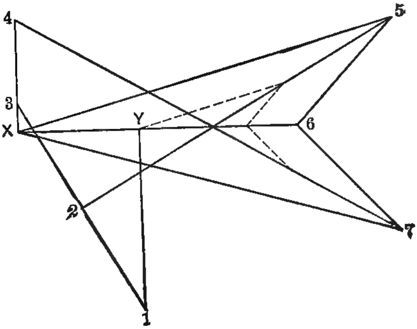


FIG. 85.

Resolve the reaction at B into the horizontal component H and the vertical component R.

Then H = total horizontal component of the wind pressure $= p_n ld \sin\alpha$.

Also, taking moments about C,

$$R_1 \cdot 2l\cos\alpha = wld \cdot l\cos\alpha + p_n ld\left(\frac{l}{2} + l\cos\alpha\right),$$

and therefore $$R_1 = \frac{wld}{2} + \frac{p_n ld}{\cos\alpha}\left(\frac{1}{4} + \frac{\cos 2\alpha}{2}\right).$$

Again, since there are rollers at C, the reaction at C is vertical, and taking moments about B,

$$R_1 2l\cos\alpha = wld \cdot l\cos\alpha + p_n ld \cdot \frac{l}{2},$$

or $$R_1 = \frac{wld}{2} + \frac{p_n ld}{4\cos\alpha}.$$

Then Fig. 85 is the stress diagram, 1234 being the "line of loads," in which

$$LI3 = wld; \quad 32 = p_n\frac{ld}{2} = 21.$$

Also, $$R_1 = 1Y, \quad YX = H, \quad \text{and} \quad X4 = R_2.$$

The dotted lines show how the stress diagram is modified when rollers are placed under B. The stresses in the members are diminished, and hence, in designing such a roof-truss, the wind pressure should be assumed to act upon the side on which the foot of the principal is fixed to its support.

(*f*) A roof-truss for a larger span may have the middle points of its principal rafters supported by struts, as in Fig. 86. The vertical dead load at each of the points F, A, $G=\frac{wld}{2}$.

The normal wind pressure over AB

$$=\frac{p_nld}{4} \text{ at } A \text{ and at } B,$$

and

$$=\frac{p_nld}{2} \text{ at } F.$$

The horizontal component of the wind pressure

$$=p_nld \sin\alpha = \text{horizontal reaction at } B.$$

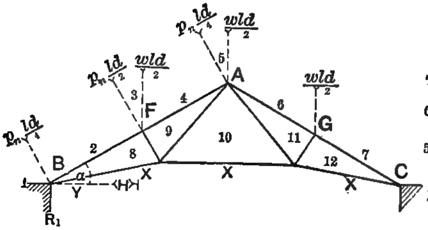


FIG. 86.

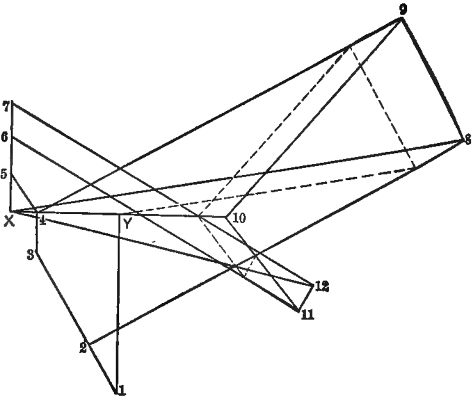


FIG. 87.

The vertical reactions R_1 and R_2 at B and C can be found at once by taking moments about C and B successively, and are

$$R_1=\tfrac{3}{4}\,wld+\frac{p_nld}{2\cos\alpha}\left(\cos 2\alpha+\tfrac{1}{2}\right),$$

$$R_2=\tfrac{3}{4}\,wld+\frac{p_nld}{4\cos\alpha}.$$

Fig. 87 is the stress diagram, 1234567 being the "line of loads" in which $12 = 45 = \frac{p_n ld}{4}$, $23 = \frac{p_n ld}{2}$, $34 = \frac{wld}{2} = 56 = 67$.

Also, $1Y = R_1$, $YX = H$, $X7 = R_2$.

The dotted lines show how the stresses are modified when the rollers are under B, and also show that the straining is greatest when the end of the rafter on which the wind blows is fixed.

(*g*) The principal rafters of the truss Fig. 88 are supported at two intermediate points.

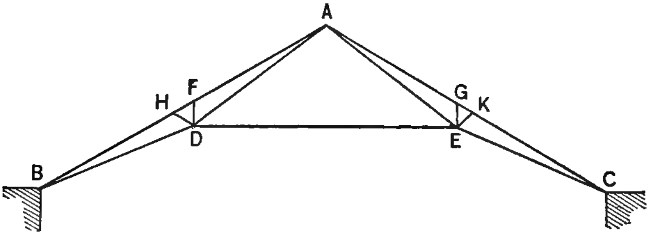


FIG. 88.

Data.—Pitch $= 30°$; $AD = BD = AE = CE = 23$ ft.; trusses $= 13$ ft. centre to centre; dead weight $= 8$ lbs. per square foot of roof-surface; wind pressure on the side AB normal to roof-surface $= 28$ lbs. per square foot; $DF = DH = EG = EK$; DF and EG are vertical; rollers under the end C; span $= 79$ ft.; $AF = BH = 21$ ft.; $FH = 3\frac{1}{2}$ ft.

The wind pressure $= 4459$ lbs. $\left(= \frac{24\frac{1}{2}}{2} \cdot 13 \cdot 28\right)$ at each of the points F, H,

and $= 3822$ lbs. $\left(= \frac{21}{2} \cdot 13 \cdot 28\right)$ at each of the points, A, B.

The *dead* load $= 1274$ lbs. $\left(= \frac{24\frac{1}{2}}{2} \cdot 13 \cdot 8\right)$ at each of the points F, H, K, G,

and $= 2184$ lbs. $(= 21 \cdot 13 \cdot 8)$ at the point A.

The resultant reaction at B

$$= \frac{1}{2}(4 \times 1274 + 2184) + \frac{16562}{\sqrt{3}} = 13201.8 \text{ lbs.}$$

The horizontal reaction at B

$$= 16562 \sin 30° = 8281 \text{ lbs.}$$

Fig. 89 is the stress diagram 123 . . . 10, being the "line of loads" in which $12=67=3822$ lbs.; $23=45=4459$ lbs.; $34=1274=56=89=9 \cdot 10$; $78=2184$ lbs. Also, $1Y=13{,}202$ lbs.; $YX=8281$ lbs.

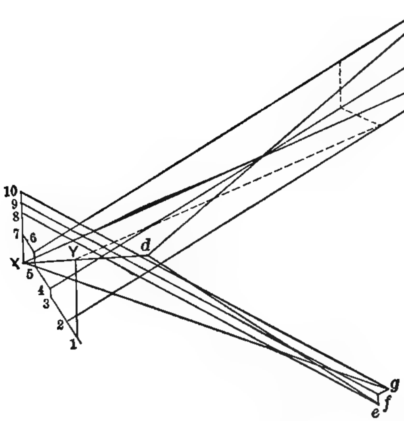


Fig. 89.

The dotted lines show the modified stresses when rollers are under B, and again show that the straining is greatest when B is fixed.

(*h*) *Bowstring Roof-truss.*—In the truss represented by Fig. 90 the joints of the upper and lower chords lie on curves which are either circular or parabolic arcs, and it is assumed that the portion of a chord between any two consecutive joints is straight. Under a uniformly distributed load the reciprocals of the joints at the feet of the verticals do not close, and the truss evidently belongs to the class of incomplete frames. Equilibrium is impossible and deformation is only prevented by the stiffness of the truss and friction of the joints. To render the stresses determinate, the truss is modified by the introduction of the sloping members shown in Fig. 91, and the stress diagram for the uniformly distributed load is Fig. 92, 123456 being the line of loads, in which $12=W=23=34=45=56=$ load concentrated at each of the upper-chord joints.

Next let there be a given wind pressure on the left half of the truss, and let it be assumed that the truss is fixed at the left support and rests upon rollers at the right support. At the latter the reaction is vertical and can be found by taking moments about the left support. The reaction at the left support may be resolved into a vertical and a horizontal component, and the latter must necessarily be equal in magnitude to the horizontal component of the total wind pressure. The vertical component is now easily obtained by taking moments about the left support.

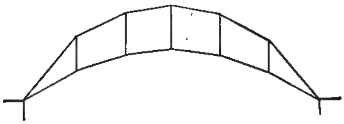


FIG. 90.

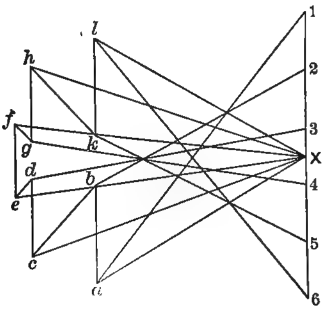


FIG. 92.

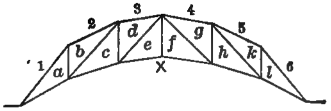


FIG. 91.

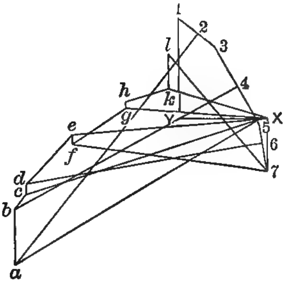


FIG. 94.

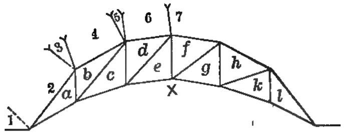


FIG. 93.

A reversal of stress in any of the sloping members or verticals on the right half of the truss can be prevented by introducing the counter-braces shown in Fig. 93. The stress diagram for the wind effect is Fig. 94, 1234567 being the "line of load," in which

$1Y$ = vertical reaction at left support,
YX = horizontal reaction at left support,
$X7$ = vertical reaction at right support,

and also $12=\frac{1}{2}$ normal pressure on $2a=23$
$34=\frac{1}{2}$ " " " $4b=45$
$56=\frac{1}{2}$ " " " $6d=67$.

21. Bridge-trusses.—A bridge-truss proper consists of an upper chord (or flange), a lower chord (or flange), and an intermediate portion, called the web, connecting the two chords. Its depth is made as small as possible consistent with economy, strength, and stiffness. Its purpose is to carry a distributed load which, as in roof-trusses, is assumed to be concentrated at the joints, or *panel-points*, of the upper and lower chords.

(*a*) Fig. 97 is evidently the stress diagram for the bridge-truss represented by Fig. 95, 1234 being the line of loads, in which

$$12 = W_1, \quad 23 = W_2, \quad 34 = W_3,$$

the loads concentrated at the panel-points *FKG* respectively.

Fig. 95.

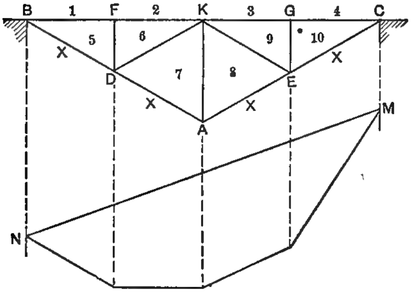


Fig. 96.

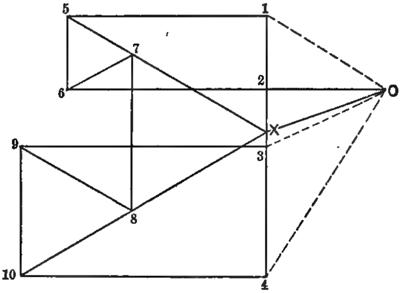


Fig. 97.

Drawing *OX* parallel to the closing line *MN* of the funicular polygon, then *X*1 and *X*4 are the vertical reactions at *B* and *C* respectively.

Also by the method of moments it follows at once that

$$X1 = \tfrac{3}{4}W_1 + \tfrac{1}{2}W_2 + \tfrac{1}{4}W_3$$

and

$$X4 = \tfrac{1}{4}W_1 + \tfrac{1}{2}W_2 + \tfrac{3}{4}W_3.$$

This truss *inverted* is sometimes used for bridge purposes, and may be constructed entirely of timber. The stresses remain the same in magnitude, but are, of course, reversed in character.

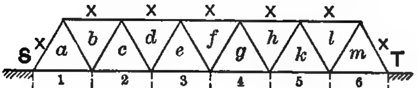
(*b*) Fig. 98 represents a through-bridge truss of the Warren type, and is composed of a number of equilateral triangles.

Let 123456 be the line of loads, in which

$$12 = W_1, \quad 23 = W_2, \quad 34 = W_3, \quad 45 = W_4, \quad \text{and } 56 = W_5.$$

With any pole O describe the funicular polygon and draw OX parallel to the closing line QR. Then $X1$ and $X6$ are the vertical reactions at S and T respectively.

FIG. 98.



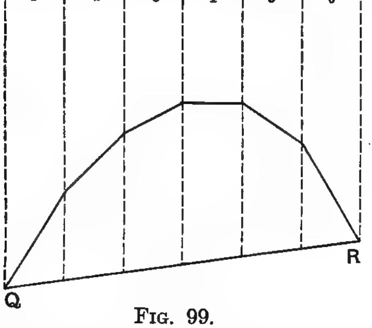


FIG. 99.

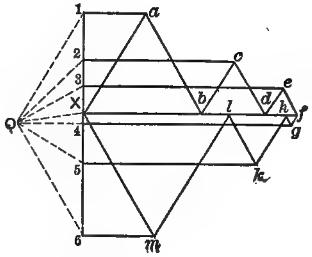


FIG. 100.

By the method of moments

$$X1 = \tfrac{5}{6}W_1 + \tfrac{4}{6}W_2 + \tfrac{3}{6}W_3 + \tfrac{2}{6}W_4 + \tfrac{1}{6}W_5,$$
$$X6 = \tfrac{1}{6}W_1 + \tfrac{2}{6}W_2 + \tfrac{3}{6}W_3 + \tfrac{4}{6}W_4 + \tfrac{5}{6}W_5.$$

If the truss is inverted, the loads are carried on the upper chord and the bridge is one of the *deck* type. The stresses in the several members remain the same in magnitude but are opposite in kind.

(*c*) The Howe truss, Fig. 101, is suitable for bridges of the *through* type, has been widely used, and may be constructed of timber, of iron, or of timber and iron combined.

Under a uniformly distributed load composed of W concentrated at each panel-point, Fig. 103 is the stress diagram, 123 . . . 8 being the line of loads. Also $X1 = 3\frac{1}{2}W = X8$, OX being drawn parallel to the closing line of the funicular polygon, Fig. 102.

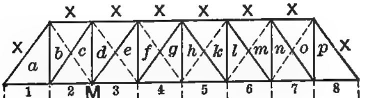
If the load is *unevenly* distributed the stresses in certain of the members may be reversed. For example, let the truss carry a single load concentrated at the panel-point M, Fig. 101.

Take any pole O and draw the funicular polygon, Fig. 104. If OX is now drawn parallel to the closing line,

$X1$ = reaction at left support $= \frac{3}{4}P$,
and $X8$ = " " right " $= \frac{1}{4}P$.

These results may also be verified by the method of moments. The stress diagram is Fig. 105, and it is evident that the stresses in the verticals and ties on the right of *M* are reversed in kind. Such a reversal may be provided for by giving sufficient sectional area to the members thus strained or by the introduction of counter-braces which are indicated by the dotted lines in Fig. 101.

Fig. 101.



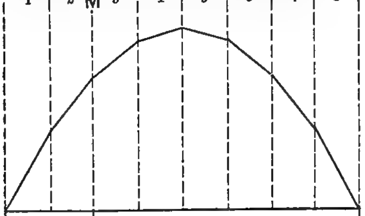


Fig 102.

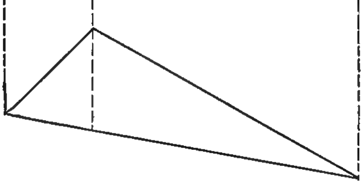
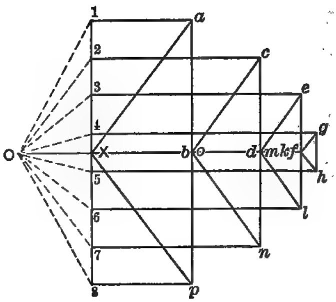


Fig. 104.

Fig. 103.



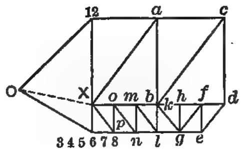


Fig. 105.

Under a live load, as when a train passes over a bridge, counter-braces may be introduced on each side of the centre, and although they may not be required in every panel, they give increased stiffness to the truss.

(*d*) The Pratt deck truss, Fig. 106, is merely an inverted Howe truss, and the stresses under the same uniformly distributed load are the same in magnitude but are reversed in kind. Thus in the Howe truss the upper chord is in compression and the lower in tension, the verticals are ties and the sloping members struts. In

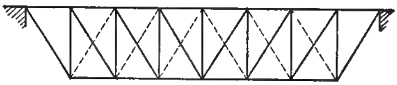


Fig. 106.

the Pratt deck truss the upper chord is in compression and the lower in tension, the verticals are struts and the sloping members ties. The dotted lines are counter-braces introduced to provide for the effects of a varying or *live* load.

Note.—The directions of the lines of action of the stresses in the several members at a joint in any framed structure, and hence also their character, i.e., whether they are tensions or compressions, are easily determined by following *in order* the sides of the reciprocals of the joints.

(*e*) *Petit Truss.*—This type of truss is suitable for a bridge of long span. Fig. 107 represents the truss for a through bridge and its stress diagram for a load W concentrated at each panel-point is Fig. 108.

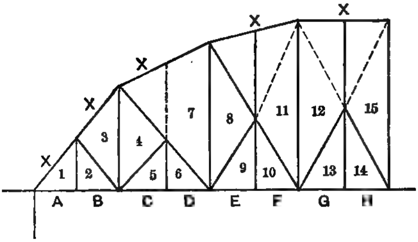


FIG. 107.

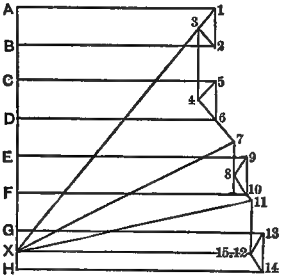


FIG. 108.

In the stress diagram $ABCD\dots$ is the "line of loads" and

$$AB = W = BC = CD = \dots$$

Also, XA = vertical reaction at the left support

$$= 6\tfrac{1}{2}W.$$

In drawing the reciprocals of the joints p, q, $r\dots$, it must be remembered that *one half* of the loads concentrated at the feet of the verticals 56, 910, and 1314 are transmitted through the members 45, 89, and 1213, respectively.

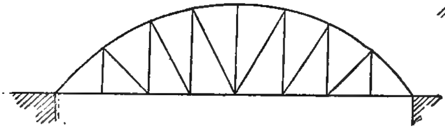


FIG. 109.

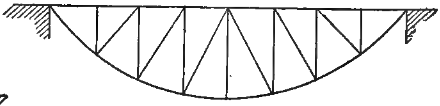


FIG. 110.

22. Bowstring Truss.—In its simplest form the bowstring truss is represented by Fig. 109 for a through and by Fig. 110 for a deck bridge, the axis of the curved chord being either the arc of a circle or of a parabola.

In practice the portion of the curved chord between consecutive joints is usually straight (Figs. 111, 112), and, under a uniformly distributed load of W concentrated at each panel-point of the horizontal chord, the stress diagram for one half the truss is Fig. 113, if the joints lie in a circular arc, and Fig. 114 if they lie in the arc of a parabola. In the latter case the diagonals are unstrained and might be dispensed with, so that the uniformly distributed load is transmitted to the bow through the verticals only.

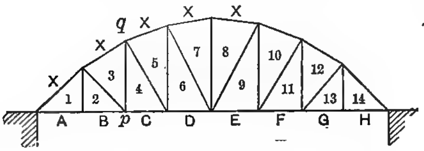


FIG. 111.

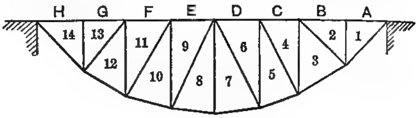


FIG. 112.

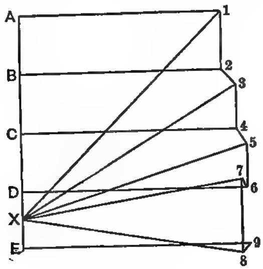


FIG. 113.

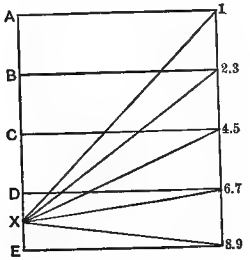


FIG. 114.

In the stress diagrams $AB = W = BC = CD = DE =$ load concentrated at each panel-point and $XA =$ vertical reaction at a support $= 3\frac{1}{2}W$.

Under a moving (or *live*) load the stresses in web members may be reversed in kind. Suppose, for example, that a load P is concentrated at the panel-point p. Then Fig. 116 is the stress diagram, $BC432B$ and $X345X$ being the reciprocals of the joints p and q. Following the sides of these reciprocals *in order*, the stresses in 34 and 45 due to P are found to be a tension and compression respectively, while under the uniformly distributed load the corresponding stresses are a compression and a tension. If the stresses due to

P exceed the latter in magnitude, the members 34 and 45 must be designed to bear safely stresses which may be alternately tensile and compressive. This can be done either by giving each member in question a sufficient sectional area or by introducing the *counter-braces* shown by the dotted lines in Figs. 117, 118.

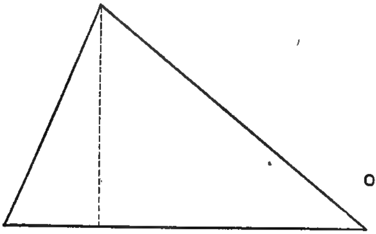


FIG. 115.

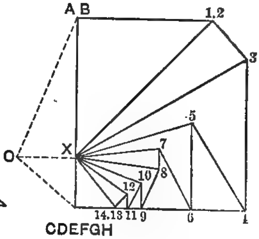
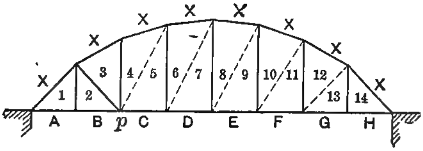


FIG. 116.

FIG. 117.



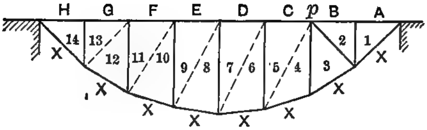


FIG. 118.

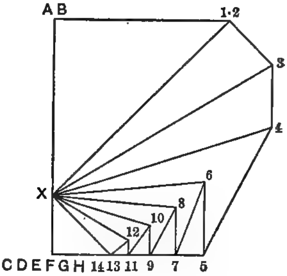


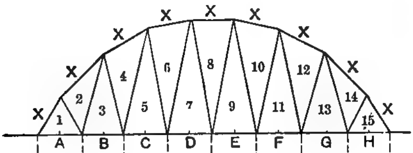
FIG. 119.

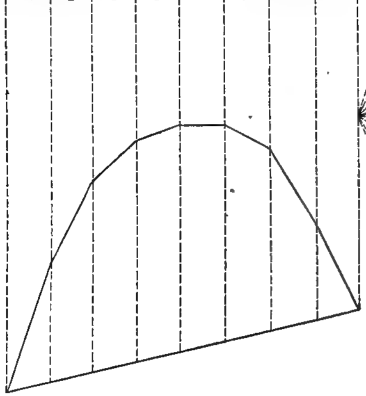
The stress diagram for the latter case, with the load P concentrated at p, is Fig. 119.

It will of course be noted that the truss represented by Fig. 118 is merely the truss Fig. 117 inverted. Under the same load the stresses in corresponding members are of the same magnitude, but are reversed in kind. Thus the curved chord is in compression in Fig. 117 and in tension in Fig. 118.

The truss Fig. 120 is a bowstring truss with isosceles bracing, the upper chord being straight between consecutive joints.

Fig. 120.





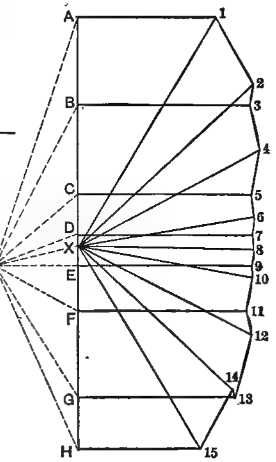


Fig. 121. Fig. 122.

Under any given loads *AB*, *BC*, *CD*, *DE*, *EF*, *FG*, and *GH*, concentrated at the panel-points, Fig. 122, is the stress diagram, *XA* and *XH* being the reactions at the supports. The members 12 and 14.15 are always in tension but, under a live load, every other web member may be placed in tension or in compression and must therefore be designed for a reversal of stress.

23. **Piers.**—Fig. 123 represents one of the braced piers for supporting a deck bridge.

Height of pier = 50 ft.; of truss = 30 ft.

Width of pier at top = 17 ft.; at bottom = $33\frac{3}{4}$ ft.

The bridge when most heavily loaded throws a weight of 100 tons on each of the points *A* and *B*.

Weight of half pier = 30 tons.

The increased weight at each of the points *C*, *D* and *E*, *F* from the portions *AD* and *CF* of the pier = 5 tons.

Resultant horizontal wind pressure on train = 40 tons at $87\frac{1}{2}$ feet above base.

Resultant horizontal wind pressure on truss $=20$ tons at 65 feet above base.

Resultant horizontal wind pressure on pier $=2\frac{1}{2}$ tons at each of the points C and E.

With the wind pressure acting as in the figure, the diagonals CB, ED, and GF are required. When the wind blows on the other side, the diagonals D to A, F to C, and H to E are brought into play. The moment of the couple tending to overturn the pier

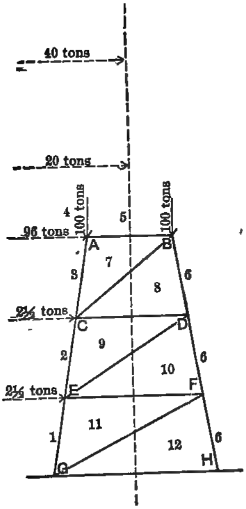


FIG. 123.

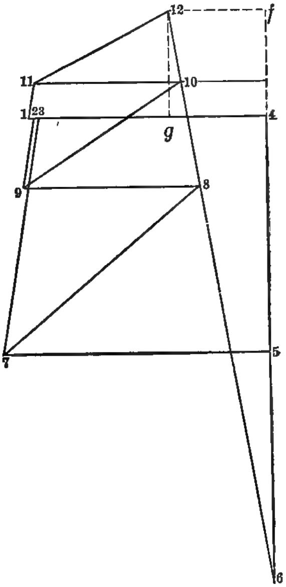


FIG. 124.

$$=40\times 87\tfrac{1}{2}+20\times 65+4\times 25=4900 \text{ ton-feet.}$$

The moment of stability $=(200+30)\times\dfrac{33\frac{2}{3}}{2}=3871\frac{2}{3}$ ft.-tons.

Thus the difference, $=4900-3871\frac{2}{3}=1028\frac{1}{3}$ ft.-tons, must be provided for in the anchorage. The pull on a vertical anchorage-tie at $G=\dfrac{1028\frac{1}{3}}{33\frac{2}{3}}=30\frac{55}{101}$ tons.

Again, if H be the horizontal force upon the pier at A due to wind pressure,

$$H\times 50=40\times 87\tfrac{1}{2}+20\times 65=4800;$$

$$H=96 \text{ tons.}$$

Fig. 124 is the stress diagram, 123456 being the load lines, in which $12 = 2$ tons $= 23$; $34 = 96$ tons; $45 = 100$ tons $= 56$.

Project 12.6 upon the vertical $f6$. Then

$$f6 = \text{vertical pressure at } H,$$

and $$12,f = \text{outward horizontal thrust at } H.$$

Project 1,11,12 upon the vertical $64f$, and upon the horizontal $1g4$. Then

$$f4 = \text{uplifting force at } G,$$

and $$1g = 14 - f,12 = \text{horizontal pull at } G.$$

In computing the stresses in the leeward posts of a braced pier it is common practice to assume that the maximum load is upon the bridge and that the wind exerts a pressure of 30 lbs. per square foot upon the surfaces of the train and structure, or a pressure of 50 lbs. per square foot upon the surface of the structure when unloaded. The negative (or reversed) stresses in the windward posts of the pier are determined when the minimum load is on the pier, the wind pressure remaining the same.

24. **Fink Truss.**—In the truss represented in the accompanying figure the length of the beam *AB* is so great that the single triangular truss *ACB* with a single central strut *CO* is an insufficient support. The two halves are therefore strengthened by the simple triangular trusses *AGO* with a central strut *GF* and *BPO* with a central strut *PN*.

Again, each quarter length, viz., *AF*, *FO*, *ON*, *NB*, is similarly trussed. The subdivisions may, if necessary, be carried still farther.

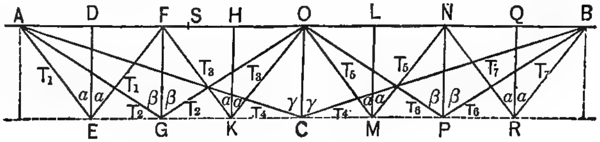


Fig. 125.

This truss in *four*, *eight*, *sixteen*, . . . divisions or *panels* is known as the Fink truss, and has been widely employed in America, the number of panels usually being eight or sixteen.

The members shown by the dotted lines may be introduced for stiffness, and the platform may be either at the top or bottom. The weight directly borne by a strut is usually determined from the loads upon the two adjacent panels by assuming the corresponding portions of the beam to be independent beams supported at the ends. Thus if there be a weight W at the point S in the panel FH, the portion of W borne by the strut GF at F is $W\frac{SH}{FH}$, and the portion borne by the strut KH at H is $W\frac{FS}{FH}$.

Let W_1, W_2, W_3, W_4, W_5, W_6, W_7 be the weights upon the struts (or posts) DE, FG, HK, OC, LM, NP, QR, respectively;

Let P_1, P_2, P_3, P_4, P_5, P_6, P_7 be the compressions to which these posts are severally subjected;

Let α, β, γ be the inclinations to the vertical of AE, AG, AC, respectively;

Let T_1, T_2, T_3, . . . be the tensions in the ties, as in Fig. 125.

The tensions in the ties meeting at the foot of a post are evidently equal.

Each triangular truss may be considered separately.

From the truss AEF, $2T_1 \cos\alpha = P_1 = W_1$;

from the truss AGO, $2T_2 \cos\beta = P_2 = W_2 + (T_1 + T_3)\cos\alpha$;

from the truss FKO, $2T_3 \cos\alpha = P_3 = W_3$;

from the truss ACB,

$$2T_4 \cos\gamma = P_4 = W_4 + (T_2 + T_6)\cos\beta + (T_3 + T_5)\cos\alpha;$$

from the truss OMN, $2T_5 \cos\alpha = P_5 = W_5$;

from the truss OPB, $2T_6 \cos\beta = P_6 = W_6 + (T_5 + T_7)\cos\alpha$;

from the truss NRB, $2T_7 \cos\alpha = P_7 = W_7$.

Hence

$$T_1 = \frac{W_1}{2}\sec\alpha, \quad T_2 = \frac{1}{2}\left(W_2 + \frac{W_1 + W_3}{2}\right)\sec\beta,$$

$$T_3 = \frac{W_3}{2}\sec\alpha,$$

$$T_4 = \frac{1}{2}\left(W_4 + \frac{W_2 + W_6}{2} + \frac{W_1 + 3W_3 + 3W_5 + W_7}{4}\right)\sec\gamma,$$

$$T_5 = \frac{W_5}{2}\sec\alpha,$$

$$T_6 = \frac{1}{2}\left(W_6 + \frac{W_5 + W_7}{2}\right)\sec\beta, \quad T_7 = \frac{W_7}{2}\sec\alpha.$$

Again, the thrust along $AF = T_1 \sin \alpha + T_2 \sin \beta + T_4 \sin \gamma$;
" " at $F = T_2 \sin \beta + T_4 \sin \gamma$;
" " along $FO = T_2 \sin \beta + T_4 \sin \gamma + T_3 \sin \alpha$;
" " at $O = T_4 \sin \gamma$;
etc., etc.

If the truss carries a uniformly distributed load W,

$$W_1 = W_2 = W_3 = W_4 = W_5 = W_6 = W_7 = \frac{W}{8},$$

$$T_1 = T_3 = T_5 = T_7 = \frac{W}{16} \sec \alpha,$$

$$T_2 = T_6 = \frac{W}{8} \sec \beta, \quad T_4 = \frac{W}{4} \sec \gamma.$$

If the above diagram is inverted, it will represent another type of truss in which the obliques are struts and the verticals ties.

25. Bollman Truss.—Fig. 126 represents a beam trussed by a number of *independent* triangular trusses, the vertical posts being equidistant. The weight concentrated at the head of each post may be found by the method described in Ex. 2, which in fact is generally applicable to all bridge- and roof-trusses.

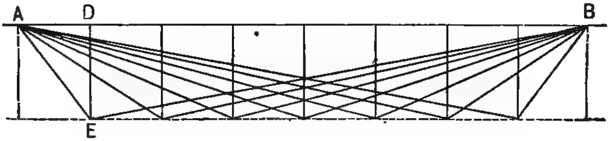


FIG. 126.

Let T_1, T_2 be the tensions in AE, BE, respectively;
" W_1 be the weight at D;
" α_1, α_2 be the inclinations of AE, BE, respectively, to the vertical.

Then $$T_1 = W_1 \frac{\sin \alpha_2}{\sin (\alpha_1 + \alpha_2)}, \quad T_2 = W_1 \frac{\sin \alpha_1}{\sin (\alpha_1 + \alpha_2)}.$$

Similarly, the stress in any other tie may be obtained.

The compression in the top chord is the algebraic sum of the horizontal components of all the stresses in the ties which meet at one end.

The verticals are always struts and the obliques ties.

This truss has been used for bridges of considerable span, but the ties may prove inconveniently long.

26. Method of Sections.—It often happens that the stresses in the members of a frame may be easily obtained by the method of sections. This method depends upon the following principle:

If a frame is divided by a plane section into two parts, and if each part is considered separately, the stresses in the bars (or members) intersected by the secant plane must balance the external forces upon the part in question.

Hence the *algebraic* sums of the horizontal components, $\Sigma(X)$, of the vertical components, $\Sigma(Y)$, and of the moments of the forces with respect to any point, $\Sigma(M)$, are severally zero; i.e., analytically,

$$\Sigma(X)=0, \quad \Sigma(Y)=0, \quad \text{and} \quad \Sigma(M)=0.$$

These equations are solvable, and the stresses therefore determinate, if the secant plane does not cut more than three members.

Example 1. ABC is a roof-truss of 60 ft. span and 30° pitch. The strut $DF=GH=5$ ft.; the angle $FDA=90°$. Also $AF=FB=AG=GC$.

The vertical reaction at $B=5$ tons. The weight concentrated at $D=4\frac{1}{2}$ tons.

Let the angle $ABF=\alpha$.

Then $$AB=30\sec 30°=20\sqrt{3}, \quad \cot\alpha=\frac{10\sqrt{3}}{5}=2\sqrt{3}.$$

$$\therefore \sin\alpha=\frac{1}{\sqrt{13}}, \quad \cos\alpha=\frac{2\sqrt{3}}{\sqrt{13}}.$$

If the portion of the truss on the right of a secant plane MN be removed, the forces C, T_2, T_1, in the members AD, AF, FG must balance the external forces 5 tons and $4\frac{1}{2}$ tons in order that the equilibrium of the remainder of the truss may be preserved.

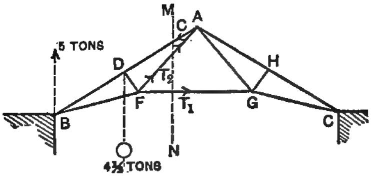


Fig. 127.

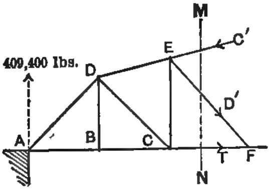


Fig. 128.

Hence, resolving horizontally and vertically,

$$T_1 + T_2 \cos(\alpha + 30°) - C \sin 60° = 0,$$

$$T_2 \sin(\alpha + 30°) - C \cos 60° + 5 - 4\tfrac{1}{2} = 0.$$

Taking moments about F,

$$C\,5 - 5BF \cos(30° - \alpha) + 4\tfrac{1}{2}DF \sin 30° = 0.$$

But

$$\cos(\alpha + 30°) = \frac{5}{2\sqrt{13}}, \quad \sin(\alpha + 30°) = \frac{3\sqrt{3}}{2\sqrt{13}}, \quad \cos(30° - \alpha) = \frac{7}{2\sqrt{13}}.$$

$$BF = BD \sec \alpha = 5\sqrt{13}, \quad \text{and} \quad DF = 5 \text{ ft.}$$

Therefore

$$T_1 + T_2 \frac{5}{2\sqrt{13}} - C \cdot \frac{\sqrt{3}}{2} = 0,$$

$$T_2 \frac{3\sqrt{3}}{2\sqrt{13}} - C \cdot \tfrac{1}{2} + \tfrac{1}{2} = 0.$$

$$C\,5 - 5 \times 5\sqrt{13} \frac{7}{2\sqrt{13}} + 4\tfrac{1}{2} \times 5 \times \tfrac{1}{2} = 0.$$

Hence $C = 15\tfrac{1}{4}$ tons, $T_1 = 9.89$ tons, and $T_2 = 6.35$ tons.

Ex. 2. Fig. 128 represents a portion of a bridge-truss cut off by a plane MN and supported at the abutment at A.

The vertical reaction at $A = 409{,}400$ lbs.
The weight at $B = 49{,}500$ lbs.
" " " $C = 38{,}700$ lbs.

$$AB = BC = 24 \text{ ft.}; \quad BD = 24 \text{ ft.}; \quad CE = 29\tfrac{1}{3} \text{ ft.}$$

The forces C', D', T in the members met by MN must balance the external forces at A, B, C.

Resolving horizontally and vertically,

$$T + D' \cos \alpha - C' \cos \beta = 0,$$

$$D' \sin \alpha + C' \sin \beta - 409400 + 49500 + 38700 = 0,$$

α and β being the inclinations to the horizon of EF, DE, respectively.

Taking moments about E,

$$-T \times 29\tfrac{1}{3} + 409400 \times 48 - 49500 \times 24 = 0.$$

But

$$\tan \alpha = \frac{29\tfrac{1}{3}}{24} = \frac{11}{9} \quad \text{and} \quad \tan \beta = \frac{5\tfrac{1}{3}}{24} = \frac{2}{9}.$$

$$\therefore \sin \alpha = \frac{11}{\sqrt{202}}, \quad \cos \alpha = \frac{9}{\sqrt{202}}, \quad \sin \beta = \frac{2}{\sqrt{85}}, \quad \cos \beta = \frac{9}{\sqrt{85}}.$$

Hence $T = 629{,}427\tfrac{3}{11}$ lbs.,

$$C' = \frac{9814500}{117}\sqrt{85} = \frac{1090500}{13}\sqrt{85}\text{ lbs.},$$

$$D' = \frac{1994600}{13}\sqrt{202}\text{ lbs.}$$

The results in the two preceding examples can also be easily verified by drawing the stress diagrams of the portions of the trusses under consideration.

27. Three-hinged Braced Arch.—For station roofs and for sheds of wider spans than those for which simple trusses are found to be economical, it is a common practice to employ two braced trusses, as, e.g., Figs. 129, 130, connected by a central hinge and hinged also at the two abutments. The use of these hinges makes the stresses determinate and the stress diagram for any given loading may be drawn as soon as the resultant reactions at the hinges are known. They can be easily calculated by the "method of sections." Fig. 130 represents one of the three-hinged braced arches which support the roof of a large hall.

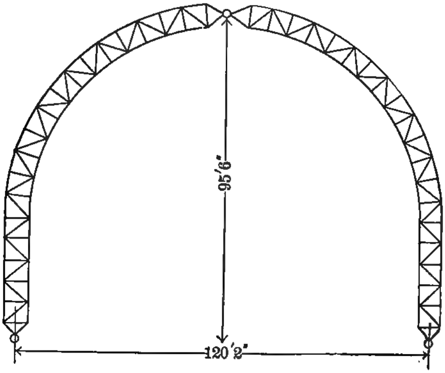


Fig. 129.

A load P at any point p of the left truss developes a reaction at B, and the direction of this reaction must necessarily pass through the centre hinge C. If the direction falls above or below C, the hinge is subjected to a bending moment and rotation or deformation can only be prevented by the inherent stiffness of the trusses. Let the lines of action of P and the reaction along BC meet in the point r.

Then, for the equilibrium of the left truss, the resultant reaction at A must also pass through the point r.

So, again, a load Q at any point q on the right truss developes a reaction along AC, and if the direction of the reaction meets the line of action of Q in the point s, then Bs is the direction of the resultant reaction at B due to Q.

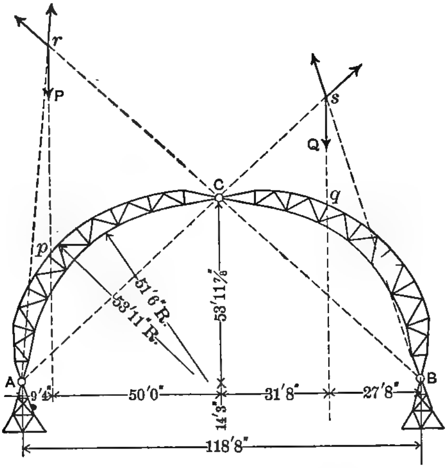


Fig. 130.

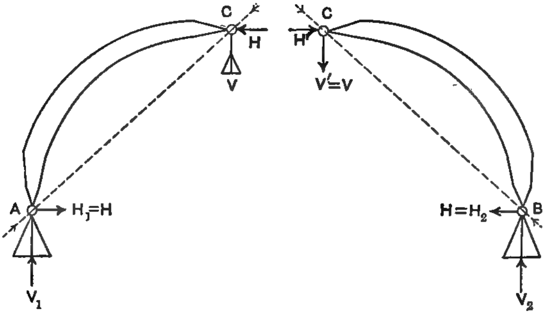


Fig. 131.

At C resolve the reaction along BC due to P into its horizontal and vertical components H and V.

At C resolve the reaction along AC due to Q into its horizontal and vertical components H' and V'.

Then, evidently $H = H'$ and $V + V' = 0$.

Consider the equilibrium of each truss separately.

In Fig. 131, considering the left truss, the hinge C is acted upon by the horizontal reaction H and the upward vertical force (or shear) V. The resultant reaction at A may be also resolved into its vertical and horizontal components, viz., V_1 and H_1.

As H and H_1 are the only two horizontal forces acting upon the truss in this case, they are equal in magnitude, but act in opposite directions, and the $H_1(=H)$ at A may be either taken up by the abutments or by a tie-rod connecting A and B, which may often be conveniently placed below the floor.

Take moments about A. Then,

$$H \times 53.99 + V \times 59.33 = P \times 9.33$$

or

$$H + V \times 1.099 = P \times .173.$$

Similarly, by taking moments about B for the right truss,

$$H \times 53.99 - V \times 59.33 = Q \times 27.67$$

or

$$H - V \times 1.099 = Q \times .513.$$

Hence

$$H = P \times .087 + Q \times .256$$

and

$$V = P \times .08 - Q \times .24.$$

Also,

$$V_1 + V = P \quad \text{and} \quad V_2 - V = Q,$$

V_2 being the vertical component of the reaction at B.

Therefore

$$V_1 = P \times .9236 + Q \times .229$$

and

$$V_2 = P \times .0764 + Q \times .771.$$

The principle of the three hinges is also sometimes, but less frequently, adopted in *bridge work* (Fig. 132). The two halves are connected by a central hinge and are also hinged at the abutments. A serious objection to this type of bridge is the large deflection due to changes of temperature. The elimination of the hinge at C greatly diminishes the deflection and stiffens the structure, but at the same time developes temperature stresses in the several members.

The reactions at the three hinges are determined in precisely the same manner as in the preceding case.

A load P on the left truss necessarily produces a reaction along the sloping chord CB and if the line of action of P meets BC produced in s, As is the direction of the resultant reaction at A due to P. So Bt is the direction of the resultant reaction at B due to a load Q on the right truss.

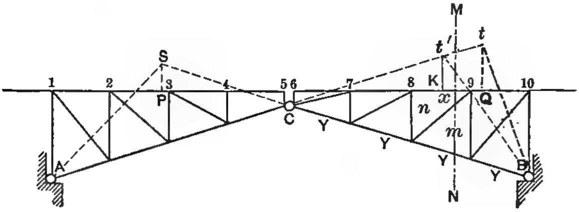


FIG. 132.

If the resultant reaction passes through a panel-point, e.g. g, and intersects ACt in t', a weight placed on the horizontal chord at x produces no stress in the member Ym. This can be easily proved by considering the equilibrium, under the weight, of the portion of the truss on the right of a vertical section MN. The moments of the stresses in Kn and nm, and of the resultant reaction at B, are all *nil*, as their lines of action pass through the point 9. Hence the moment of the stress in Ym and therefore the stress itself is also *nil*.

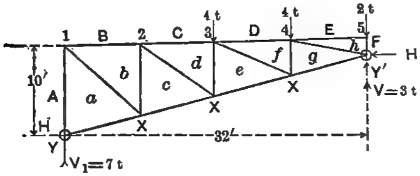


FIG. 133.

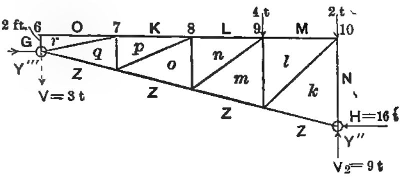


FIG. 134.

Let Figs. 133, 134 represent the two halves of a 3-hinged bridge of 64 ft. span, 2 ft. deep at the centre and 10 ft. deep at the ends;

Let loads of 4, 4, 2, 2, 4 tons be concentrated at the panel-points 3, 4, 5, 9, and 10, respectively;

Let H be the horizontal reaction at each hinge;

Let V_1, V_2 be the vertical reactions at A and B respectively;

Let $\pm V$ be the vertical shear at the central hinge. Consider the equilibrium of each half separately.

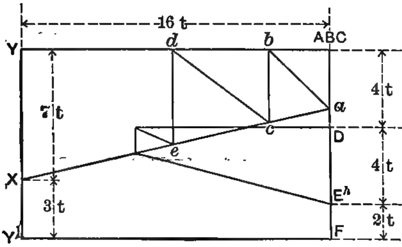


FIG. 135.

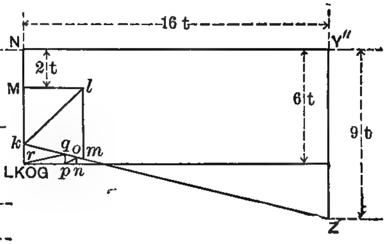


FIG. 136.

Taking moments about A,

$$-V\cdot 32 - H\cdot 8 + 4(16+24) + 2\times 32 = 0,$$

or

$$H + 4V = 28.$$

Taking moments about B,

$$-V\cdot 32 + H.8 - 4\times 8 = 0,$$

or

$$H - 4V = 4.$$

Therefore $\quad H = 16$ and $V = 3$.

Also, $\quad V_1 = 10 - V = 7$ tons and $V_2 = 6 + V = 9$ tons.

Then Figs. 135, 136 are the stress diagram of Figs. 133, 134, respectively.

If the three-hinged braced arch, Fig. 132, is inverted, the bridge becomes one of the suspension type and the load is carried on the lower chord. The stresses under a given load remain the same in magnitude but are reversed in kind.

If the sloping chord, instead of being straight, has its joints lying on the arc of a parabola with its vertex at the central hinge, it can easily be shown, either graphically or analytically, that, under a uniformly distributed load, *no stress* is developed in the horizontal chords.

Although it is the usual practice to connect together the sloping chords by means of the central hinge, it is really more rational to connect the horizontal chords together, and the truss then requires no depth at the centre.

Table of Weights of Roof-coverings.

Description of Covering.	Weight of Covering in lbs. per sq. ft. of Covered Area.	Dead Weight of Roof in lbs. per sq. ft. of Covered Area.
Boarding ($\frac{3}{4}$-inch)	2.5 to 3	
Boarding and sheet iron	6.5	
Cast-iron plates ($\frac{3}{8}$-inch)	15	
Copper	8 to 1.25	
Corrugated iron	1 to 3.75	
Felt, asphalted	1	
Felt and gravel	8 to 10	
Galvanized iron	1 to 3.5	
Lath and plaster	9 to 10	
Pantiles	10	
Sheet lead	5 to 8	
Sheet zinc	1.25 to 2	
Sheet iron (corrugated)	3.4	8 without boards and 11 with boards for spans up to 75 ft.
“ “ “	3.4	12 without boards and 15 with boards for spans from 75 to 150 ft.
“ “ ($\frac{1}{16}$ inch thick)	3	
“ “ (16 W.G.) and laths	5	
Shingles (16-inch)	2	10 on laths for spans up to 75 ft.
“ (long)	3	14 on laths for spans from 75 to 150 ft.
Sheathing (pine, 1-inch yellow, Northern)	3	
Sheathing (pine, 1-inch yellow, Southern)	4	
Sheathing (1-inch chestnut and maple)	4	
Sheathing (1-inch ash, hickory, oak)	5	
Sheathing (spruce, 1 inch thick)	2	
Slates (ordinary)	5 to 9	13 without boards or on laths and 16 on $1\frac{1}{4}$-in. boards for spans up to 75 ft.
Slates (large)	9 to 11	17 without boards or on laths and 20 on $1\frac{1}{4}$-in. boards for spans from 75 to 150 ft.
Slates and iron laths	10	
Shingles, pine	2	
Sheet lead	5 to 8	
Thatch	6.5	
Tiles, flat	15 to 20	
“ grooves, and fillets	7 to 10	
Tiles and mortar	20 to 30	
Timbering of tiled and slate roofs (additional)	5.5 to 6.5	
Tin	.7 to 1.25	
Zinc	1 to 2	

Snow: 20 to 25 lbs. per horizontal square foot.
Wind: 30 to 50 “ “ vertical square foot.

The Carnegie Steel Company gives the following table of approximate loads (including weight of truss) per square foot for roofs and spans under 75 ft.:

Roof covered with corrugated sheets, unboarded....	8 lbs.
“ “ “ “ “ on boards.....	11 “
“ “ “ slate, on laths................	13 “
Same, on boards 1¼ inch thick....................	16 “
Roof covered with shingles, on laths..............	10 “
Add to above, if plastered below rafters...........	10 “
Snow, light, weighs per cubic foot.................	5 to 12 “

For spans over 75 ft. add 4 lbs. to the above loads per square foot.

It is customary to add 30 lbs. per square foot to the above for snow and wind when separate calculations are not made.

TABLE OF THE VALUES OF P_n, P_v, P_h, IN POUNDS PER SQUARE FOOT OF SURFACE WHEN $P=40$, AS DETERMINED BY THE FORMULA $P_n=P \sin a^{1.84 \cos a-1}$.

Pitch of Roof.	P_n	P_v	P_h
5°	5.0	4.9	0.4
10°	9.7	9.6	1.7
20°	18.1	17.0	6.2
30°	26.4	22.8	13.2
40°	33.3	25.5	21.4
50°	38.1	24.5	29.2
60°	40.0	20.0	34.0
70°	41.0	14.0	38.5
80°	40.4	7.0	39.8
90°	40.0	0.0	40.0

TABLE PREPARED FROM THE FORMULA $p=\left(\frac{v}{20}\right)^2$

Velocities in feet per second.	Velocities in miles per hour.	Pressure in lbs. per sq. ft.
10	6.8	0.25
20	13.6	1.00
40	27.2	4.00
60	40.8	9.00
70	47.6	14.25
80	54.4	16.00
90	61.2	20.25
100	68.0	25.00
110	74.8	30.25
120	81.6	36.00
130	88.4	42.25
150	102.0	56.25

EXAMPLES.

1. A number of coplanar forces have to be in equilibrium. The lines of action and magnitudes of all the forces with the exception of two are known; and of these two, the line of action of one of them and a point on the line of action of the other are also known. Show how you would graphically determine the two unknown forces.

2. Show that the locus of the poles of the funicular polygons, of which the first and last sides pass through two fixed points on the closing line, is a straight line parallel to the closing line.

3. The first and last sides of a funicular polygon of a system of forces intersect the closing line in two fixed points. Show that for any position of the pole each side of the polygon will pass through a fixed point on the closing line.

4. If the pole of a funicular polygon describe a straight line, show that the corresponding sides of successive funicular polygons with respect to successive positions of the pole will intersect in a straight line which is parallel to the locus of the pole.

5. Forces act upon a rectangle in the manner shown by Fig. 137. Determine the magnitude, position, and direction of the resultant.

6. Fig. 138 represents a jointed frame, the lengths of the links being as indicated. The frame is acted upon by six forces, of which three are given, and the lines of action of all the forces bisect respectively the angles marked by a dotted arc. Find the forces X, Y, and Z and determine the directions in which they act. Also find the force along each link, distinguishing between struts and ties.

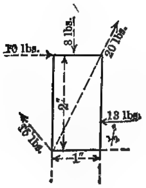


Fig. 137.

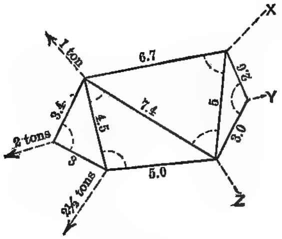


Fig. 138.

7. Draw the funicular polygon for the six forces of 1, 2, 3, 4, 5, and 6 tons, acting at angular distances of 60°, as in Fig. 139. Also determine graphically

the moment of these loads with respect to any given point on the line of action of the 6-ton force.

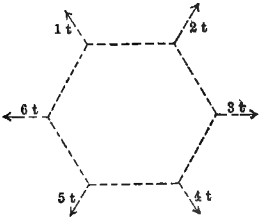


FIG. 139.

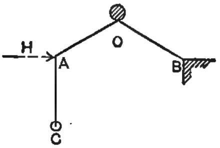


FIG. 140.

8. A rigid frame *CAOB* is hinged at *C* and rests upon rollers at *B*. Find the magnitude and direction of the resultant reaction at *C*, and also the vertical reaction at *B*, due to the horizontal force *H* at *A*, Fig. 140.

9. Sketch the reciprocal diagram of a frame consisting of four equal rods placed so as to form a parallelogram with angles of 30°, and braced by two diagonals crossing each other without joint, the rods being pinned at the corners of the figure only. The frame is free from external forces, but in such constraint that the longer diagonal is under a tension of 2 tons.

10. Four bars of equal weight and length, freely articulated at the extremities, form a square *ABCD*. The system rests in a vertical plane, the joint *A* being fixed, and the form of the square is preserved by means of a horizontal string connecting the joints *B* and *D*. If *W* be the weight of each bar, show (*a*) that the stress at *C* is horizontal and $=\frac{W}{2}$, (*b*) that the stress on *BC* at *B* is $W\frac{\sqrt{5}}{2}$ and makes with the vertical an angle $\tan^{-1}\frac{1}{2}$, (*c*) that the stress on *AB* at *B* is $W\frac{\sqrt{13}}{2}$ and makes with the vertical an angle $\tan^{-1}\frac{3}{2}$, (*d*) that the stress upon *AB* at *A* is $\frac{5}{2}W$, (*e*) that the tension of the string is $2W$.

11. Five bars of equal length and weight, freely articulated at the extremities, form a regular pentagon *ABCDE*. The system rests in a vertical plane, the bar *CD* being fixed in a horizontal position, and the form of the pentagon being preserved by means of a string connecting the joints *B* and *E*. If the weight of each bar be *W* show that the tension of the string is $\frac{W}{2}(\tan 54° + 3 \tan 18°)$, and find the magnitudes and directions of the stresses at the joints.

12. Six bars of equal length and weight ($=W$), freely articulated at the extremities, form a regular hexagon *ABCDEF*.

First, if the system hang in a vertical plane, the bar AB being fixed in a horizontal position, and the form of the hexagon being preserved by means of a string connecting the middle points of AB and DE, show that (a) the tension of the string is $3W$, (b) the stress at C is $\frac{W}{2\sqrt{3}}$ and horizontal, (c) the stress at D is $W\sqrt{\frac{13}{12}}$ and makes with the vertical an angle $\cot^{-1} 2\sqrt{3}$.

Show that the stresses at C and F remain horizontal when the bars AF, FE, BC, CD are replaced by any others which are all equally inclined to the horizon.

Second, if the system rest in a vertical plane, the bar DE being fixed in a horizontal position, and the form of the hexagon being preserved by means of a string connecting the joints C and F, show that (a) the tension of the string is $W\sqrt{3}$, (b) the stress at C is $W\sqrt{\frac{31}{12}}$ and makes with CB an angle $\sin^{-1}\sqrt{\frac{8}{124}}$, ($c$) the stress at B is $W\sqrt{\frac{7}{12}}$ and makes with CB an angle $\sin^{-1}\sqrt{\frac{8}{28}}$.

Third, if the system hang in a vertical plane, the joint A being fixed and the form of the hexagon being preserved by means of strings connecting A with the joints E, D, and C, show that (a) the tension of each of the strings AE and AC is $W\sqrt{3}$, (b) the tension of the string AD is $2W$, and determine the magnitudes and directions of the stresses at the joints, assuming that the strings are connected with pins distinct from the bars.

13. A system of heavy bars, freely articulated, is suspended from two fixed points; determine the magnitudes and directions of the stresses at the joints. If the bars are all of equal weight and length, show that the tangents of the angles which successive bars make with the horizontal are in arithmetic progression.

Ans. If α_r, α_{r+1} are the slopes of the rth and $(r+1)$th bars and W_r, W_{r+1} their weights, and if H is the horizontal stress in each bar, the stress in the rth bar $= H\sec\alpha_r = \frac{W_r + W_{r+1}}{2} \frac{\cos\alpha_{r+1}}{\sin(\alpha_r - \alpha_{r+1})}$.

14. In a mansard roof of 12 ft. rise, the upper triangular portion (of 4 ft. rise) has its rafters inclined at 60° to the vertical. The rafters of the lower portion are inclined at 30° to the vertical. If there is a load of 1000 lbs. at the ridge, find the load at each intermediate joint necessary for equilibrium, and the thrust of the roof.

A load of 2000 lbs. is concentrated at each of the intermediate joints and a brace is inserted between these joints. Find the stress in the brace.

Ans. 1000 lbs.; thrust $= 500\sqrt{3}$ lbs.; $333\frac{1}{3}\sqrt{3}$ lbs.

15. A chain of equal links is suspended and loaded as shown in Fig. 141. The joints lie in a circular arc of 20 ft. radius. The loads at E and F are inclined at 30° to the vertical. Determine the loads at the other joints and the stresses in the links.

16. A chain of $2m-1$ equal rods (each of weight W), linked at the ends, is suspended from two points in the same horizontal plane. Prove (a) that the horizontal component of the stress at each joint is constant, (b) that the

vertical component at the junction of the nth and $(n+1)$th (reckoned from one end) is $W(m-\frac{1}{2}-n)$, and (c) that the tangent of the inclination of the nth rod to the horizon is proportional to $m-n$.

17. A bicycle, Fig. 142, with 30-in. wheels carries a rider weighing 150 lbs. The member BC is horizontal, and A, the intersection of BC and ED, is vertically above the centre of the front wheel. Determine the stresses in the several members (a) when all the weight is carried on the saddle, (b) when the weight is carried on the treadle.

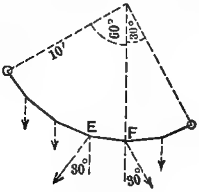


FIG. 141.

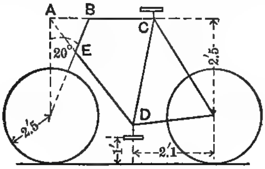


FIG. 142.

18. The lengths of the post, tie-rod, and jib of a crane are 15, 20, and 30 ft., respectively. If the crane lifts a weight of 5 tons, determine the stresses in the several members when the chain passes (a) along the jib, (b) along the tie. *Ans.* (a) 5; 1.65; 8.34 tons. (b) 8; 6.65; 13.48 tons.

19. The post OA of a portable hand crane makes an angle of 45° with the jib OB, and an angle of 120° with the tie AB. The backstay AC makes an angle of 45° with the horizontal strut OD. A weight of 10 tons is suspended from the end B of the jib. Find the amount and kind of stress in each of the members OB, AB, and AC. Also determine the counter balance-weight required at C. *Ans.* 33.5 27.3; 33.5; 23.7 tons.

20. The post of a jib-crane is 10 ft.; the weight lifted $=W$; the jib is inclined at 30° and the tie at 60° to the vertical. Find (a) the stresses in the jib and tie, and also the B. M. at the foot of the post.

How (b) will these stresses be modified if the chain has *four* falls, and if it passes to the chain-barrel in a direction bisecting the angle between the jib and tie?

Ans. (a) Stress in tie $=W$; in jib $=W\sqrt{3}$; B.M. $=W5\sqrt{3}$ ft.-tons.
(b) " " " $=.87\ W$; in jib $=1.87\ W$.

21. An ordinary jib-crane is required to lift a weight of 10 tons at a horizontal distance of 9 ft. from the axis of the post. The hanging part of the chain is in *four* falls; the jib is 15 ft. long and the top of the post is $16\frac{1}{2}$ ft. above ground. Find the stresses in the jib and tie when the chain passes (1) along the jib, (2) along the tie.

The post turns round a vertical axis. Find the direction and magnitude of the pressure at the toe, which is 3 ft. below ground.

Ans. (1) Stress in tie $=6.1$ tons; in jib $=11\frac{13}{22}$ tons.
(2) " " " $=3.6$ tons; in jib $=9\frac{1}{11}$ tons.

Pressure on toe $=10\sqrt{10}$ tons, inclined to vertical at an angle $\tan^{-1}3$.

22. Fig. 143 is a crane for lifting 4 tons, the chain being in *four* falls and passing from A to E. Draw the stress diagram, (a) when a member BD is introduced, (b) when a member CE is introduced. Also find the vertical uplift at E.

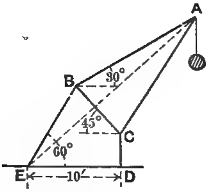


FIG. 143.

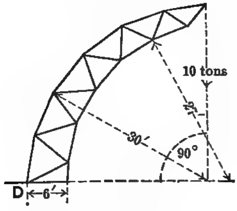


FIG. 144.

23. The bracing of the crane, Fig. 144, consists of isosceles triangles having equal bases upon an outer arc of 30′ radius subtending an angle of 90° at the centre. The radius of the inner arc is 27 ft. and the flanges are 6 ft. apart at the ground surface. Determine the stresses in all the members when a weight of 10 tons is being lifted. Also find the stresses in the anchorage-bars.

24. The post of a derrick-crane is 30 ft. high; the horizontal traces of the two backstays are at right angles to each other, and are 15 ft. and 25 ft. in length. Show that the angle between the shorter trace and the plane of the jib and tie, when the stress in the post is a maximum, is 30° 58′.

Also find the greatest stresses in the different members of the crane when the jib, which is 50 ft. long and is hinged at the foot of the post, is inclined at 45° to the vertical, the weight lifted being 4000 lbs.

Ans. Stress in jib = $6666\frac{2}{3}$ lbs.; in tie = 4768.4 lbs.; max. thrust along post = 10991.5 lbs.; max. stress on long backstay = 7362.7 lbs.; on short backstay = 10539 lbs.

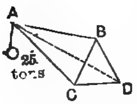


FIG. 145.

25. In the crane represented by the figure $AB = AC = 35$ ft., $BC = 20$ ft., $BD = 20$ ft., the weight lifted = 25 tons, AC slopes at 45°, the chain hangs in four falls and passes from A to D. Find the stresses in all the members and the upward pull at D.

Ans. Stress in $BC = 25.8$; $AC = 48.25$; $AB = 28.27$; $BD = 32.4$ tons.

Vertical pull at $D = 31.1$ tons.

26. The figure represents the framing of an hydraulic crane. $AB = BD = DF = FG = HK = 5$ ft.; $KG = BC = 2\frac{1}{2}$ ft. Find the stresses in the members of the crane when the weight (1 ton) lifted is (a) at A; (b) at B; (c) at D. Also (d) find the stresses when there is an additional weight of $\frac{1}{2}$ ton at each of the points B, D, F, and G.

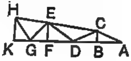


FIG. 146.

	Stress.	Case *a*.	Case *b*.	Case *c*.	Case *d*.
Ans.	$Aa=Bb$	2	0	0	0
	$Cd=De$	$\frac{42}{11}$	$\frac{28}{11}$	$\frac{14}{11}$	$\frac{21}{11}$
	Eg	$\frac{117}{26}$	$3\frac{1}{2}$	$2\frac{1}{2}$	$\frac{67}{18}$
	Xa	$\sqrt{5}$	0	0	0
	Xc	$\frac{20}{9}\sqrt{2}$	$\frac{10}{9}\sqrt{2}$	0	$\frac{5}{9}\sqrt{2}$
	Xf	$\frac{40}{18}\sqrt{2}$	$\frac{80}{18}\sqrt{2}$	$\frac{20}{18}\sqrt{2}$	$\frac{80}{18}\sqrt{2}$
	ab	0	1	0	$\frac{1}{2}$
	de	0	0	0	$\frac{1}{2}$
	bc	$\frac{5}{9}\sqrt{5}$	$\frac{7}{9}\sqrt{5}$	0	$\frac{10}{18}\sqrt{5}$
	dc	$\frac{5}{99}\sqrt{317}$	$\frac{7}{99}\sqrt{317}$	$\frac{1}{11}\sqrt{317}$	$\frac{8}{99}\sqrt{317}$
	fe	$\frac{5}{148}\sqrt{317}$	$\frac{7}{148}\sqrt{317}$	$\frac{9}{148}\sqrt{317}$	$\frac{27}{286}\sqrt{317}$
	fg	$\frac{5}{26}\sqrt{5}$	$\frac{7}{26}\sqrt{5}$	$\frac{9}{26}\sqrt{5}$	$\frac{10}{18}\sqrt{5}$

27. The post AB of a jib-crane, Fig. 147, is 20 ft.; the jib AC is inclined at 30° and the tie BC at 45° to the vertical; the weight lifted is 5 tons. Find the stresses in the jib and tie when the chain passes (a) along the jib, (b) along the tie, (c) horizontally from C to the post.

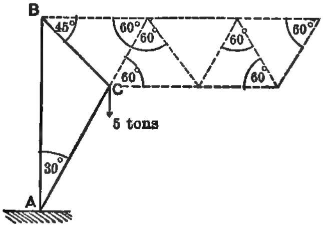


FIG. 147.

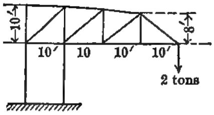


FIG. 148.

The chain has *two* falls.

Also draw the stress diagram for the three cases when the dotted members are added.

		Case *a*.	Case *b*.	Case *c*.
Ans.	Stress in jib,	6.16	3.66	5.49
	" " tie,	2.59	.09	.35

28. Fig. 148 represents a 2-ton travelling crane running upon a pair of rails. Draw the stress diagram and find the reactions on the rails.

29. In the crane ABC the vertical post $AB=15'$, the jib $AC=23'$, and the angle $BAC=30°$. Find (a) the stresses in the jib and tie, and also the bending moment at the foot of the post when the crane lifts a weight of 4 tons.

The throw is increased by adding two horizontal members, CE, BD and an inclined member DE, the figure BE being a parallelogram and the diagonal CD coincident in direction with CA. Find (b) the stresses in the several members of the crane as thus modified, the weight lifted being the same.

In the latter case show (c) how the stresses in the members are affected when the chain, which is in *four* falls, passes from E to B and then down the post.

Ans. (*a*) Tension in tie $=3\frac{1}{3}$ tons; thrust in jib $=6\frac{2}{15}$ tons.
(*b*) Stress in $CE=9.34$; in $ED=10.16$; in $CB=13.49$; in $CD=6.15$; in $DA=10.7$; in $BD=7$ tons.
(*c*) Stress in $CE=8.9$; in $ED=10.7$; in $CB=12.9$; in $CD=5.8$; in $DA=10.7$; in $BD=7.4$ tons.

30. The horizontal traces of the two backstays of a derrick-crane are x and y feet in length, and the angle between them is β. Show that the stress in the post is a maximum when $\frac{\cos(\beta-\theta)}{\cos\theta}=\frac{x}{y}$, θ being the angle between the trace y and the plane of the jib and tie.

31. The two backstays of a derrick-crane are each 38′ long, and the angle between their horizontal traces $2\tan^{-1}\frac{5}{42}$; height of crane-post $=32'$; the length of the jib $=40'$; the throw of the crane $=20'$; the weight lifted $=4$ tons. Determine the stresses in the several members and the upward pull at the foot of each backstay when the plane of the jib and post (*a*) bisects the angle between the horizontal traces of the backstays, (*b*) passes through a backstay.

Ans. In jib $=5$; in tie $=2.52$ tons; in backstay in (*a*) $=1.77$, in (*b*) $=3.267$ tons.

32. Find the stresses in the members of the crane represented by Fig. 149; also find balance-weight at C.

Ans. Stress in $BE=25$; $DE=26.9$; $DB=18.85$; $DA=26.08$; $BA=1.67$; $BC=16.49$ tons; counterweight at $C=11.67$ tons.

33. In the crane represented by Fig. 150, draw the stress diagram when a load of 4000 lbs. is suspended from A. If the chain passes along BA over a pulley at A and is in *four* falls, determine the stresses in the several members.

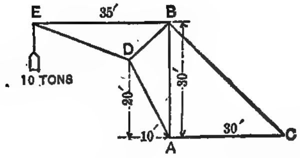


FIG. 149.

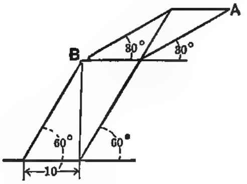


FIG. 150.

34. A pair of shear-legs, each 25 ft. long, with the point of suspension 20 ft. vertically above the ground surface, is supported by a tie 100 ft. long; distance between feet of legs $=10\sqrt{5}$ ft. Find the thrusts along the legs and the tension in the tie when a weight of 2 tons is being lifted.

Ans. Tension in tie $=1.137$ tons; compression in each leg $=1.39$ tons.

35. $ABCD$ is a quadrilateral truss, AB and CD being horizontal and 15 and 30 ft. in length, respectively. The length of AC is 10 ft., and its in-

clination to the vertical is 60°. A weight W_1 is placed at C, and W_2 at D. What must be the relation between W_1 and W_2 so that the truss may not be deformed? For any other relation between W_1 and W_2, explain how you would modify the truss to prevent deformation, and find the stresses in all the members.

Ans. $W_2 = 1.366\ W_1$.

36. Draw the stress diagram for the frame, Fig. 151, under a load of 1 ton at the apex. Also find the vertical forces in the anchorages.

37. In the frame, Fig. 152, $AC = AB = 10$ ft., and a load of 1000 lbs. is concentrated at A. What load at B will prevent distortion? If the member BC is introduced and a load of 3000 lbs. is concentrated at B, draw the stress diagram.

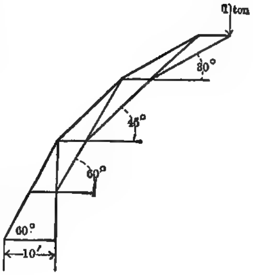


FIG. 151.

38. In a triangular truss of the dimensions shown in Fig. 153, a load of 1 ton is concentrated at A and at B. What must be the magnitude of the load P at C so that there may be no distortion of the frame?

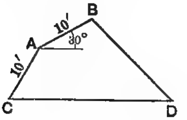


FIG. 152.

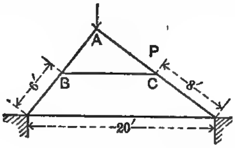


FIG. 153.

39. In a quadrilateral truss $ABCD$, AD is horizontal, AB and BC are inclined at angles of 60° and 30° respectively to the horizontal, and CD is inclined at 45° to the horizontal. What weight must be concentrated at C to maintain the equilibrium of the frame under a weight W at B?

If a weight W is placed at C as well as at D, what member must be introduced to prevent distortion? What will be the stress in that member?

Ans. 1st, $1.366\ W$; 2d, stress in $BD = .134W \operatorname{cosec}(60° + \alpha)$, where $\alpha = BDA$.

40. ACE is a triangular truss supported at A and E; $AE = 30$ ft.; $AC = 24$ ft.; $CE = 18$ ft. FB and FD are two struts from a point F in the tie AE vertically below C, B and D being the middle points of AC and CE respectively. Find the load which must be placed at D to prevent distortion (a) when there is a vertical load of $\frac{1}{4}$ ton at B and at C, (b) when there is a normal wind pressure of $\frac{1}{4}$ ton at B and a load of $\frac{1}{8}$ ton at B and at C, assuming the end A to be fixed and the end E to rest upon rollers.

Ans. (a) .23 ton; (b) .425 ton.

41. In the frame, Fig. 154, the bars are each of the same length and slope as shown, BC being horizontal. A load of 1 ton is placed at A. What load may be concentrated at B and at C without producing distortion? If the member OB is introduced and a load of 1 ton is placed at A and at B, what must be the load at C so that there might be no distortion? If the member

OC is also introduced, draw the stress diagram for a load of 1 ton at each of the joints *A*, *B*, and *C*.

42. In the triangular truss, Fig. 155, $AB=20'$, $AC=15'$, and $BC=25'$. The tie *BC* is horizontal and is supported at *B* and *C*. The middle points of *AB* and *AC* are supported by struts *DE* and *DF* from a point *D* vertically below *A*. Loads of 400 and 800 lbs. are concentrated at *E* and *A* respectively. What load must be concentrated at *F* to prevent distortion?

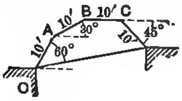


FIG. 154.

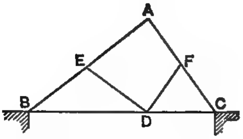


FIG. 155.

43. Three bars, freely articulated, form an equilateral triangle *ABC*. The system rests in a vertical plane upon supports at *B* and *C* in the same horizontal line, and a weight *W* is suspended from *A*. Determine the stress in *BC*, neglecting the weight of the bars. *Ans.* $\frac{W}{2\sqrt{3}}$.

44. Three bars, freely articulated, form a triangle *ABC*, and the system is kept in equilibrium by three forces acting on the joints. Determine the stress in each bar.

What relation holds between the stresses when the lines of action of the forces meet (*a*) in the centroid, (*b*) in the orthocentre *O* of the triangle?

Ans. (*a*) $BC:CA:AB$; (*b*) $OA:OB:OC$.

45. A triangular truss of white pine consists of two equal rafters *AB*, *AC*, and a tie-beam *BC*; the span of the truss is 30 ft. and its rise is 7½ ft.; the uniformly distributed load upon each rafter is 8400 lbs. Determine the stresses in the several members.

Ans. Stress in $BC=8400$ lbs., in $AB=4200\sqrt{5}$ lbs.

46. A roof-truss of 20 ft. span and 8 ft. rise is composed of two rafters and a horizontal tie-rod between the feet. The load upon the truss $=500$ lbs. per foot of span. Find the pull on the tie. What would the pull be if the rod were raised 4 ft.? *Ans.* 3125 lbs.; 6250 lbs.

47. The rafters *AB*, *AC* of a roof are unequal in length and are inclined at angles α, β to the vertical; the uniformly distributed load upon $AB=W_1$, upon $AC=W_2$. Find the tension on the tie-beam.

Ans. $\frac{W_1+W_2}{2}\frac{\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$.

48. In the preceding example, if the span $=10$ ft., $\alpha=60°$ and $\beta=45°$, find the tension on the tie, the rafters being spaced 2½ ft. centre to centre and the roof load being 20 lbs. per square foot. *Ans.* 198 lbs.

49. The equal rafters *AB*, *AC* for a roof of 10 ft. span and 2½ ft. rise are spaced 2½ ft. centre to centre; the weight of the roof-covering, etc. $=20$ lbs.

per square foot. Find the vertical pressure and outward thrust at the foot of a rafter.

Ans. Total vertical pressure $= 125\sqrt{5}$ lbs. = horizontal thrust.

50. The lengths of the tie-beam and two rafters of a roof-truss are in the ratios of 5:4:3. Find the stresses in the several members when the load upon each rafter is uniformly distributed and equal to 100 lbs.

Ans. Stress in tie = 48 lbs.; in one rafter = 60 lbs.; in other = 80 lbs.

51. In a triangular truss the rafters each slope at 30°; the load upon the apex = 100 lbs. Find the thrust of the roof and the stress in each rafter.

Ans. 100 lbs.; 86.6 lbs.

52. A roof-truss is composed of two equal rafters and a tie-beam, and the span = 4 times the rise; the load at the apex = 4000 lbs. Find the stresses in the several members.

Secondly, if a man of 150 lbs. stands at the middle of a rafter, by how much will the stress in the tie-beam be inceased?

Ans. 1. Stress in tie = 4000 lbs.; in each rafter $= 2000\sqrt{5}$ lbs.
2. 75 lbs.

53. A king-post truss for a roof of 30 ft. span and 7½ ft. rise is composed of two equal rafters *AB*, *AC*, the horizontal tie-beam *BC*, the vertical tie *AD*, and the struts *DE*, *DF* from the middle point *D* of the tie-beam to the middle points of the rafters; the roof load = 20 lbs. per square foot, of roof surface and the rafters are spaced 10 ft. centre to centre. Find the stresses in the several members. Second, find the altered stresses due to a man of 150 lbs. weight standing on the ridge. Third, find the stresses due to a weight on the tie-beam of 12 lbs. per square foot.

Ans.	1st.	2d.	3d.
BE	5625	$75\sqrt{5}$	$900\sqrt{5}$
EA	3750	$75\sqrt{5}$	$900\sqrt{5}$
BD	$2250\sqrt{5}$	150	1800
DE	1875	0	0
AD	$750\sqrt{5}$	0	1800

54. The triangular truss *ABC*, Fig. 156, is of the dimensions and loaded as indicated. Draw the stress diagram, and show how the stresses are modified when the strut *DE* has been removed. The horizontal component of the normal load on *AB* is to be divided equally between the two supports.

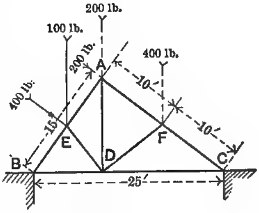


FIG. 156.

55. A queen-truss for a roof consists of two horizontal members, the lower 48 ft. long, the upper 16 ft. long; two inclined members *AB*, *DC*, and two queens *BE*, *CF*, each 8 ft. long; the points *E*, *F* divide *AD* into three equal segments; the load upon the members *AB*, *BC*, *CD* is 120 lbs. per lineal foot. Find (*a*) the stresses in the several members. How (*b*) will these stresses be modified if struts are introduced from the feet of the queens to the middle points *G*, *H* of the inclined members? In this latter case also, determine (*c*) the stresses due to a wind pressure of 120 lbs.

per lineal foot normal to AB, assuming that the horizontal reaction is equally divided between the two supports at A and D.

Ans. (*a*) Stress in lbs. in $AE = 4066.56 = EF = DF = BC$;
$AB = 4546.56 = CD$; $BE = 0 = CF$.
(*b*) Stress in lbs. in $AE = 5139.84 = DF$;
$BC = 4066.56 = AF$; $AG = 5746.56 = DH$;
$BG = 4546.56 = CH$; $EG = 1200 = FH$;
$CD = 536.64 = CF$.
(*c*) Additional stress in $AG = 1040\sqrt{5}$; $BG = 680\sqrt{5}$;
$GE = 600\sqrt{5}$; $AE = 2320$; $BE = 600$; $BF = 400\sqrt{5}$;
$BE = 400\sqrt{5}$; $CF = 400$; $CB = 800$; $EF = 1120$;
$FD = 320$.

(In case (*c*) the brace BF is introduced to prevent distortion.)

56. The rafters AB, AC of a factory roof are 18 and 24 ft. in length respectively. The tie BC is horizontal and 30 ft. long. The middle points of the rafters are supported by struts DE, DF from the middle point D of the tie BC; the point D is supported by the tie-rod AD. The truss carries a load of 500 lbs. at each of the points E, A, and F. Find the stresses in all the members. Secondly, find the stresses in the members when the rafter AB is subjected to a normal pressure of 300 lbs. per lineal ft., rollers being at C.

Ans. (1) Stress in $BE = 1112\frac{1}{2}$; $EA = 712\frac{1}{2}$; $AF = 716\frac{2}{3}$; $FC = 1016\frac{2}{3}$;
$BD = 667\frac{1}{2}$;
$CD = 813\frac{1}{3}$; $DE = 300$; $DF = 300$; $AD = 520\frac{5}{6}$ lbs.
(2) Stress in $BE = 1012\frac{1}{2}$; $EA = 1012\frac{1}{2}$; $AF = 2700 = FC$; $BD = 3847\frac{1}{2}$;
$CD = 2160$; $DE = 2700$; $DF = 0$; $AD = 1687\frac{1}{2}$ lbs.

If it be assumed in the first part that the whole of the weight is concentrated at the points E and F, draw the stress diagram.

57. The rafters AB, AC are supported at the centres by the struts DE, DF; the centre of the tie-beam is supported by the tie AD; $BC = 30$ ft., $AD = 7\frac{1}{2}$ ft.; the load upon AB is 4000 lbs., that upon AC 1600 lbs. Find (*a*) the stresses in all the members. By an accident the strut DE was torn away; how (*b*) were the stresses in the other members affected?

		(*a*)	(*b*)		(*a*)	(*b*)
Ans.	BE	$2400\sqrt{5}$	$1400\sqrt{5}$;	BD	4800	2800
	EA	$1400\sqrt{5}$	$1400\sqrt{5}$;	DC	3600	3600
	AF	$1400\sqrt{5}$	$1400\sqrt{5}$;	DE	$1000\sqrt{5}$	
	FC	$1800\sqrt{5}$	$1800\sqrt{5}$;	DF	$400\sqrt{5}$	$400\sqrt{5}$
	AD	1400	400;			

58. A triangular frame ABC, in which $AB = AC$ and BC is horizontal, is supported at B and C and carries a weight at A. If T and C are the tensile and compressive strength of the material of the frame, show that the economy of material will be greatest when $\tan ABC = \sqrt{\frac{C+T}{T}}$.

59. A roof-truss consists of two equal rafters AB, AC inclined at 60° to

the vertical, of a horizontal tie-beam BC of length l, of a collar-beam DE of length $\frac{l}{3}$, and of queen-posts DF, EG at each end of the collar-beam; the truss is loaded with a weight of 2600 lbs. at the vertex, a weight of 4000 lbs. at one collar-beam joint, a weight of 1200 lbs. at the other, and a weight of 1500 lbs. at the foot of each queen; the diagonal DG is inserted to provide for the unequal distribution of load. Find the stresses in all members.

Ans. Stress in $BD = 11733\frac{1}{3}$; $BF = 5866\frac{2}{3}\sqrt{3}$; $DF = 1500$;
$DA = 2600$; $DE = 3633\frac{1}{3}\sqrt{3}$; $DG = 1866\frac{2}{3}$;
$GC = 4933\frac{1}{3}\sqrt{3}$; $GE = 2433\frac{1}{3}$; $CE = 9866\frac{2}{3}$;
$AE = 2600$ lbs.

60. A triangular truss consists of two equal rafters AB, AC and a tie-beam BC, all of white pine; the centre D of the tie-beam is supported from A by a wrought-iron rod AD; the uniformly distributed load upon each rafter is 8400 lbs., and upon the tie-beam is 36,000 lbs.; determine (*a*) the stresses in the different members, BC being 40 ft. and AD 20 ft. What (*b*) will be the effect upon the several members if the centre of the tie-beam be supported upon a wall, and if for the rod a post be substituted against which the heads of the rafters can rest? Assume that the pressure between the rafter and post acts at right angles to the rafter.

Ans. (*a*) Stresses in $BD = 13200$; $AD = 18000$; $AB = 13200\sqrt{2}$ lbs.
(*b*) " " $= 4200$; " $= 4200$; " $= 6300\sqrt{2}$ lbs.

61. A triangular truss of white pine consists of a rafter AC, a vertical post AB, and a horizontal tie-beam BC; the load upon the rafter is 300 lbs. per lineal foot; $AC = 30$ ft., $AB = 6$ ft. Find the resultant pressure at C, upon AC assuming that the pressure upon AC at A is normal to AC.

Find the stresses in the several members when the centre D of the rafter is also supported by a strut from B.

Ans. 4762 lbs.; stress in $BC = 4500\sqrt{6}$; $CD = 11250$; $DB = 11250$;
$DA = 0$; $AB = 2250$ lbs.

62. A white-pine triangular truss consists of two rafters AB, AC of unequal length and a tie-beam BC. A vertical wrought-iron rod from A, 10 ft. long, supports the tie-beam at a point D, dividing its length into the segments $BD = 10$ ft. and $CD = 20$ ft. The load upon each rafter is 300 lbs. per lineal ft.; the load upon the tie-beam is 18,000 lbs., uniformly distributed. Determine the stresses in the several members.

Ans. In $AB = 9650\sqrt{2}$ lbs.; $AC = 4825\sqrt{5}$ lbs.; $BD = CD = 9650$ lbs.;
$AD = 9000$ lbs.

63. A frame is composed of a horizontal top-beam 40 ft. long, two vertical struts 3 ft. long, and three tie-rods of which the middle one is horizontal and 15 ft. long. Find the stresses produced in the several members when a single load of 12,000 lbs. is concentrated at the head of each strut.

Ans. Stress in horizontal members = 50000 lbs.
" " sloping " = 51418 "
" " struts = 12000 "

64. The rafters AB, AC of a roof-truss are 20 ft. long, and are supported at the centres by the struts DE, DF; the centre D of the tie-beam BC is supported by a tie-rod AD 10 ft. long; the uniformly distributed load upon AB is 8000 lbs., and upon AC is 2400 lbs. Determine (*a*) the stresses in all the members.

What will be the effect (*b*) upon the several members if AB be subjected to a horizontal pressure of 156 lbs. per lineal foot?

Ans. (*a*) Stress in $BD = 4600\sqrt{3}$; $BE = 9200$; $EA = 5200$; $ED = 4000$; $AD = 2600$; $DF = 1200$; $AF = 5200$; $CF = 6400$; $CD = 3200\sqrt{3}$.
(*b*) Tens. in $BE = 520\sqrt{3}$; $AD = 260\sqrt{3}$; compres. in $ED = 520\sqrt{3}$; $AC = 520\sqrt{3}$; $DC = 780$.
No stresses in BD, AE, and DF.

Determine the stresses in all the members of the truss, assuming the tie-beam to be also loaded with a weight of 600 lbs. per lineal foot.

Ans. Stress in AB increased by $3000\sqrt{3}$ lbs.; in BC by 9000 lbs.; in AD by $6000\sqrt{3}$ lbs.

65. A horizontal beam of length l is trussed and supported by a vertical strut at its middle point. If a weight W roll across the beam, show that the stress in each member increases proportionately with the distance x of the wheel from the end.

Ans. Stress in horizontal tie $= \frac{W}{l} x \cot \theta$; in sloping tie $= \frac{W}{l} x \operatorname{cosec} \theta$; in strut $= 2\frac{W}{l}x$, θ being angle between the horizontal and the sloping members.

66. If a wheel loaded with 12,000 lbs. travel over the top-beam in the last question, what members must be introduced to prevent distortion? What are the maximum stresses to which these members will be subjected?

Ans. 19122 lbs.

67. A beam of 30 ft. span is supported by an inverted queen-truss, the queens being each 3 ft. long and the bottom horizontal member 10 ft. long. Find the stresses in the several members due to a weight W at the head of a queen, introducing the diagonal required to prevent distortion. Also find the stresses due to a weight W at centre of beam.

Ans. (1) Stress in $AB = \frac{20}{9}W = EF$; $AE = 2.32W$; $BE = \frac{2}{3}W$; $BF = 1.16W = DF$; $BC = \frac{10}{9}W$; $CF = 0$.
(2) Stress in $AB = \frac{5}{8}W = EF$; $AE = 1.74W = BE = \frac{W}{2}$; $BF = 0$.

68. The platform of a bridge for a clear span of 60 ft. is carried by two queen-trusses 15 ft. deep; the upper horizontal member of the truss is 20 ft. long; the load upon the bridge $= 50$ lbs. per square foot of platform, which is 12 ft. wide. Find the stresses in the several members.

Ans. Stress in vertical $= 6000$ lbs.; in each sloping member $= 10000$ lbs.; in each horizontal member $= 8000$ lbs.

69. If a single load of 3000 lbs. pass over the bridge in the last question, and if its effect is equally divided between the trusses, find (*a*) the greatest stress in the members of the truss, and also (*b*) in the members which must be introduced to prevent distortion. Also find (*c*) the stresses when one half the bridge carries an additional load of 50 lbs. per square foot of platform.

Ans. (*a*) In sloping end strut $= 3333\frac{1}{3}$ lbs.; horizontal tie $=$ $2666\frac{2}{3}$ and $1333\frac{1}{3}$ lbs.; horizontal strut $= 2666\frac{2}{3}$ lbs.
(*c*) In sloping end strut $= 6250$ lbs.; horizontal tie $=$ 5000 lbs.; horizontal strut $= 5000$ and 3000 lbs.
(*b*) In case (*a*) $= 1666\frac{2}{3}$ lbs.; in case (*c*) $= 2500$ lbs.

70. The platform of a bridge for a clear span of 60 ft. is carried by two trusses 15 ft. deep, of the type shown by the accompanying diagram; the load upon the bridge is 50 lbs. per square foot of platform, which is 12 ft. wide. Find the stresses in the several members.

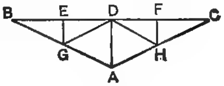


FIG. 157.

Ans. Stress in $BE = 13500$; $BG = 6750\sqrt{5}$; $EG = 4500$; $ED = 13500$; $GD = 2250\sqrt{5}$; $GA = 4500\sqrt{5}$; $AD = 9000$ lbs.

71. If a single weight of 2000 lbs. pass over a truss similar to that shown in the preceding question, find the stresses in the several members when the load is (1) at E, (2) at D.

Ans. (1) Stress in $BG = 1500\sqrt{5}$; $BE = 3000 = ED$; $EG = 2000$; $GD = 1000\sqrt{5}$; $AG = 500\sqrt{5} = AH = CH$; $DH = 0 = FH$; $DF = 1000 = FC$ lbs.
(2) Stress in $BA = 1000\sqrt{5} = CA$; $BD = 2000 = DC = AD$ lbs.. Stresses in other members $= 0$.

72. The feet of the equal roof-rafters AB, AC are tied by rods BD, CD which meet under the vertex and are joined to it by a rod AD. If W_1, W_2 are the uniformly distributed loads in pounds upon AB, AC, respectively, and if S is the span of the roof in feet, find the weight of metal (wrought iron) in the ties.

Ans. $\frac{5}{6}\frac{W_1 + W_2}{f} S \cot \beta$, f being inch-stress in pounds and β the angle ABD.

(*a*) If $AB = AC = 20$ ft., $AD = 5$ ft., the angle $BAD = 60°$, find the stresses in the several members when a weight of 3500 lbs. is concentrated at the vertex.

Ans. 7000 lbs.; 6309.8 lbs.; 3500 lbs.

(*b*) The roof in (*a*) is loaded with 10 lbs. per square foot on one side and 33 lbs. per square foot on the other, the trusses being 13 ft. centre to centre. Determine (*a*) the stresses in the several members. Examine (*b*) the effect of a horizontal pressure of 14 lbs. per square foot on the most heavily loaded side, assuming that the reaction is equally divided between the two supports.

Ans. (*a*) 11180 lbs.; 10077.65 lbs.; 5590 lbs.

73. In the truss represented in the accompanying figure, the load on $AB = W_1$, on $AC = W_2$; the angle $ABD = \beta$; $AD = BD = AE = CE$. Find the total weight of metal (wrought iron) in the tie-rods.

Ans. $\frac{5}{6}\frac{W_1+W_2}{f} S \cot \beta$, S being the span and f the inch-stress.

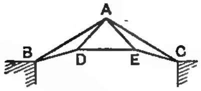


FIG. 158.

(*a*) If the stress in BD or EC is equal to the stress in DE, show that $\beta = 60° - \frac{\alpha}{3}$, α being the angle ABC.

(*b*) The trusses are 12 ft. centre to centre; the span is 40 ft.; the horizontal tie is 16 ft. long. the rafters are inclined at 60° to the vertical; the dead weight of the roof, including snow, is estimated at 10 lbs. per sq. ft. of roof surface. Determine the stress in each member when a wind blows on one side with a force of 30 lbs. per sq. ft. normal to the roof surface, assuming that the horizontal reaction (1) is wholly borne at B, (2) is equally divided between the supports.

Ans. (1) Stress in $AB =$ 8956.8 lbs.; $BD = 10015.2$ lbs. $= EC$;
$AD =$ 2503.8 lbs. $= AE$; $DE = 8196$ lbs.;
$AC = 11356.8$ lbs.
(2) Stress in $AB =$ 7756.8 lbs.; $BD = 6840.9$ lbs. $= EC$;
$AD =$ 1710 lbs. $= AE$; $AC = 10156.8$ lbs.

74. In the truss represented by the accompanying figure, the load upon $AB = W_1$, upon $AC = W_2$; the angle $ABD = \beta$; the span $BC = S$; the ties AD, BD, AE, CE are equal; F and G are the middle points of the rafters. Find the amount of metal in the tie-rods (wrought iron).

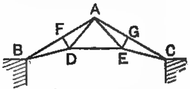


FIG. 159.

Ans. $\frac{5}{6}\frac{S}{f}\frac{W_1 + (W_1 + W_2)\cos^2\beta}{\sin\beta\cos\beta}$.

(*a*) The struts DF and EG are each 5 ft.; the angle $ABC = 30°$; the dead weight of the roof, including snow, is 9 lbs. per square foot of roof surface, and the trusses are 12 ft. centre to centre. Determine the stresses in the several members when a wind blows with a force of 30 lbs. per square foot of roof surface normal to the side AB. The span $= 60$ ft., and the end C rests upon rollers.

Secondly, determine the stresses produced in the members of the truss in the preceding question when a single weight of 3000 lbs. is suspended from G.

Ans. (1) Stresses in BD; DA; DE; EA; EC;
31238.55; 19852.35; 12633.6; 8613.5; 19999.09;
BF; FA; FD; CG; GA; GE.
29620.44; 28685.16; 7855.2; 22420.44; 21485.16; 1620 lbs.
(2) Stresses in BD; DA; DE; EA; EC;
$375\sqrt{39}$; $125\sqrt{39}$; $1000\sqrt{3}$; $875\sqrt{39}$; $1125\sqrt{39}$;
BF; FA; FD; CG; GA; GE.
2625; 2625; 0; 7875; 6375; $1500\sqrt{3}$ lbs.

(*b*) The rafters AB, AC are of unequal length and make angles of 60° and

45° respectively with the vertical; the strut $DF = 7\frac{1}{2}$ ft.; the tie DE is horizontal; the dead load upon each rafter = 100 lbs. per lineal foot; the wind pressure normal to $AB = 300$ lbs. per lineal foot, rollers are placed at C. Find the stresses in all the members. The rafter $AB = 45$ ft.

Show by dotted lines how the stress diagram will be modified:

(1) If the rollers are placed at B.

(2) If the strut DF is omitted.

(3) If a single weight of 500 lbs. is concentrated at D.

(*c*) If it is assumed that the horizontal reaction is equally divided between B and C, show that the stress in DE due to a horizontal wind pressure upon AB is nil, the angle ABC being 30°.

(*d*) In a given roof the rafters are of pitch-pine, the tie-rods of wrought iron; the span is 60 ft.; the trusses are 12 ft. centre to centre; $DF = 5$ ft. $= EG$; the angle $ABC = 30°$; the dead weight of the roof, including snow, is 9 lbs. per square foot of roof surface; rollers are placed at C; a single weight of 3000 lbs. is suspended from F, and the roof is also designed to resist a normal wind pressure of 26.4 lbs. per square foot of roof surface on one side AB. Determine the stresses in the several members.

75. In the truss represented in the accompanying figure, the struts DF DH, EG, EK are equal, and the ties BD, AD, EA, EC are also equal; the load upon AB is W_1, and upon AC is W_2. Find the weight of metal (wrought iron) in the ties.

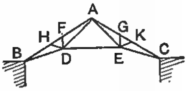


FIG. 160.

Ans. $\dfrac{5}{18}\dfrac{S}{f}\dfrac{4W_1+3(W_1+W_2)\cos^2\beta}{\cos\beta\sin\beta}$.

(*b*) The rafters AB, AC are inclined at 60° to the vertical and are each 40 ft. in length. The foot C rests on rollers, and the foot B is fixed. The strut DF is vertical, is 10 ft. long, and is equal to the strut DE in length. Also $AF = HF = 10$ ft. The dead load carried by the rafters is 120 lbs. per lineal foot. Provision has also to be made for a normal wind pressure upon AB of 300 lbs. per lineal foot. Draw the stress diagram, and show how it will be modified if the strut DF is removed.

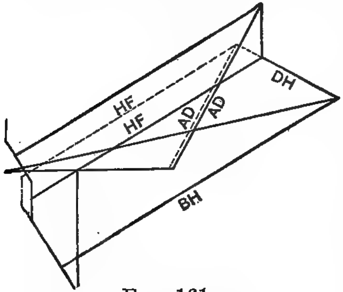


FIG. 161.

Ans. Vertical reaction at $B = 10528$ lbs. both before and after DF is removed. Horizontal reaction at $B = 6000$ lbs. The dotted lines show the modified stresses for one half of the truss.

76. The boom AB of the accompanying truss is supported at five intermediate points dividing the length into six segments each 10 ft. long. The depth of the truss = 10 ft. Draw stress diagrams for the following cases:

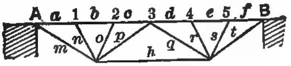


FIG. 162.

(*a*) A weight of 100 lbs. at each intermediate point of support.

(*b*) Weights of 100, 200, 300, 400, 500 lbs. in order at these points.

Ans. (*a*) Stress in $a=375$; $b=325$; $c=375$; $h=450$;
$m=125\sqrt{13}$; $n=50\sqrt{5}$; $o=50\sqrt{5}$;
$p=25\sqrt{13}$ lbs.
(*b*) Stress in $a=875$; $b=825$; $c=925$; $h=1350$; $d=13$
$e=1125$; $f=1375$; $n=50\sqrt{5}$; $o=100\sqrt{5}$
$p=141\frac{2}{3}\sqrt{13}$; $q=8\frac{1}{3}\sqrt{13}$; $r=200\sqrt{5}$;
$s=250\sqrt{5}$; $t=458\frac{1}{3}\sqrt{13}$; $m=291\frac{2}{3}\sqrt{13}$ l

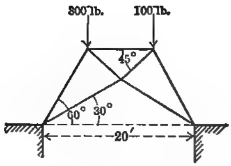


FIG. 163.

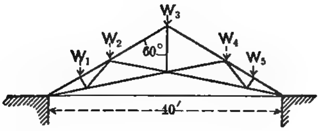


FIG. 164.

77. Determine the stresses in every member of the frame shown in Fig. 1

78. Draw the stress diagram for the frame Fig. 164, loaded, and of dimensions as shown.

79. Each of the joints *A*, *B*, *C*, *D*, *E*, *F*, *G*, Fig. 165, is loaded with bs. Find the stresses in the several members and distinguish between str and ties.

80. The dead load upon a roof-truss of type Fig. 166 consists of 1 lbs. at *F*, 1000 lbs. at *K*, and 500 lbs. at *G*; the wind pressure is a nor

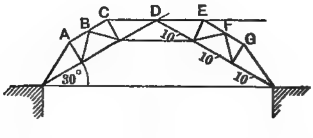


FIG. 165.

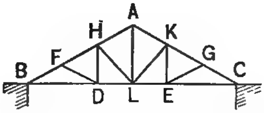


FIG. 166.

force of 30 lbs. per square foot of roof surface upon *AB*; the span $=90$ the rise $=25$ ft.; the trusses are 25 ft. centre to centre. Find the stres in the several members when rollers are (*a*) at *C*, (*b*) at *B*.

Ans. (*a*) Reaction (vertical) at $C=12291\frac{2}{3}$ lbs.; vertical reaction
$B=23958\frac{1}{3}$ lbs.; horizontal reaction at $B=18750$ lbs.
Tension in $BD=48625$; $DL=34475$; $LE=21675$; $EC=2212$
$DH=7861\frac{1}{9}$; $AL=15888\frac{8}{9}$; $KE=250$ lbs.
Compression in $BF=3666\frac{2}{3}\sqrt{106}$; $FH=2788\frac{8}{9}\sqrt{106}$;
$HA=1977\frac{7}{9}\sqrt{106}$; $AK=2325\sqrt{106}$;
$KG=2408\frac{1}{3}\sqrt{106}$; $GC=2458\frac{1}{3}\sqrt{106}$;
$DF=1572\frac{2}{9}\sqrt{106}$; $LH=1505\frac{5}{9}\sqrt{181}$;
$LK=83\frac{1}{3}\sqrt{181}$; $EG=50\sqrt{106}$ lbs.

(*b*) Only alteration in stresses is that each stress in the different sections of the horizontal tie is diminished by 18750 lbs.; all the remaining stresses are unchanged.

81. In the accompanying roof-truss, angle $ABC = 30°$; the span $= 90\frac{1}{3}$ ft.; $DF = EG = 10\frac{1}{2}$ ft.; each rafter is divided into four equal segments by the points of support; the trusses are 20 ft. centre to centre; the weight of a bay of the roof $= 24{,}416$ lbs. Determine the stress in each member.

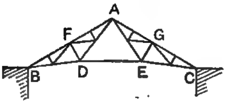


FIG. 167.

Also determine the stresses due to a wind pressure of 30 lbs. per square foot of roof surface acting normally to *AB*, when rollers are under (*a*) *C*, (*b*) *B*.

82. Determine the stresses in all the members of the truss loaded and of the dimensions shown in Fig. 168.

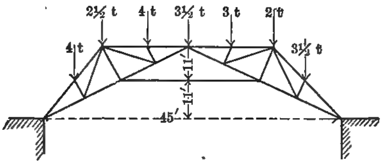


FIG. 168.

83. The horizontal boom *CD*, Fig. 169, is divided into eight segments, each 8 ft. long, by seven intermediate supports; the depth of the truss at each end $= 16$ ft.; a weight of 1 ton is concentrated at *C* and at *D*, and a weight of 2 tons at each of the points of division. Determine the stresses in the several members.

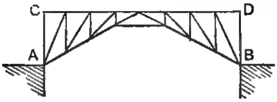


FIG. 169.

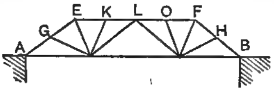


FIG. 170.

84. Fig. 170 is a skeleton diagram of a roof-truss of 72 ft. span and 12 ft. deep; *G*, *K*, *L*, *O*, *H* are respectively the middle points of *AE*, *EL*, *EF*, *LF*, *FB*; $AE = EL = LF = FB = 20$ ft.; the trusses are 12 ft. centre to centre; the dead weight of the roof $= 12$ lbs. per square foot; the normal wind pressure upon *AE* may be taken $= 30$ lbs. per square foot; the end *A* is fixed and *B* is on rollers. Draw a stress diagram. Show by dotted lines how the stress diagram is modified with rollers under *A*, *B* being fixed.

85. The platform of a bridge of 84 ft. span and 9 ft. deep is carried by a pair of trusses of the type shown in the figure. If the load borne by each truss is 300 lbs. per lineal foot, find the stresses in all the members.

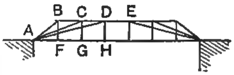


FIG. 171.

Ans. Stress in $AB=6000$; $AC=1200\sqrt{73}$; $AD=3600\sqrt{17}$; $BC=4$
$CD=14400$; $DE=28800$.
Stress in horizontal chord $=48000$; in each vertical $=3600$ lbs.

86. The inclined bars of the trapezoidal truss represented by Fig. make angles of 45° with the vertical; a load of 10 tons is applied at top joint of the left rafter in a direction of 45° with the vertical. Assum the reaction at the right to be vertical, find the stresses in all the pieces the frame.

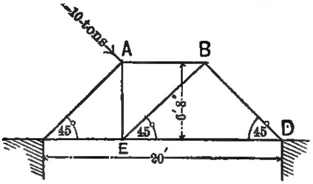


FIG. 172.

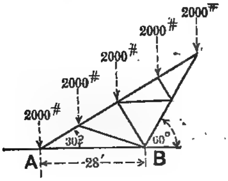


FIG. 173.

Ans. Vertical reaction at $D=\frac{10}{3}\sqrt{2}$; stress in $DE=\frac{10}{3}\sqrt{2}$; $DB=6\frac{2}{3}$;

$$BE=6\tfrac{2}{3};\ BA=\frac{20}{3}\sqrt{2};\ AE=\frac{10}{3}\sqrt{2};\ AC=\frac{10}{3};\ CE=\frac{20}{3}\sqrt{2}\text{ tons.}$$

87. An overhanging roof, Fig. 173, is supported on columns and load as shown. Draw the stress diagram and also determine the forces in vertical columns at A and B.

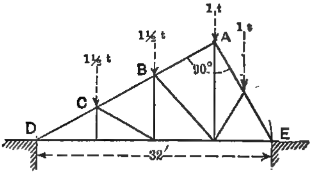


FIG. 174.

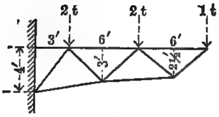


FIG. 175.

88. A factory-roof of 32 ft. span is loaded as shown in Fig. 174, and t angle *A* is 90°. Determine the stresses in all the members. Also draw t stress diagram when there is a normal wind pressure on *AB* equivalent ½ ton at *C* and *B*, and ¼ ton at *D* and *A*, rollers being placed, 1st, under and, 2d, under *E*.

89. A cantilever, Fig. 175, is loaded and of the dimemsions as show Determine the stresses in all the members.

90. Draw the stress diagram for the truss represented by Fig. 176, t load at each of the points *B* and *C* being 500 lbs.

Also, if the rafter AB is subjected to a normal wind pressure of 100 lbs. per lineal foot, introduce the member BF to prevent distortion and state in pounds the stress it should be designed to bear. Draw the stress diagram of the modified truss, assuming that A is fixed, but that D rests upon rollers.

($AB = 15' = AE$; $BC = 10'$; angle $BAD = 45°$; angle $EAD = 30°$.)

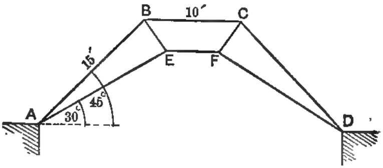


FIG. 176.

91. The frame Fig. 177 is loaded as indicated. Find the stresses in the members and the horizontal tension in the chord at the base, shown by the dotted line.

92. A crescent-shaped roof of the dimensions and loaded as shown in Fig. 178 is fixed at the right supports and rests upon rollers at the left support.

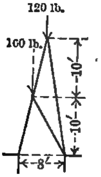


FIG. 177.

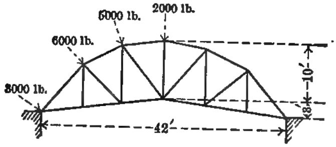


FIG. 178.

The joints of the upper members lie in the arc of a circle of 24 ft. radius, and each load acts towards the centre of this circle. Draw the stress diagram.

93. A station roof with trusses 15 ft. apart is supported on central columns (Fig. 179) and the weight of the roof-covering and truss may be taken at 30 lbs. per square foot of covered area. A horizontal wind pressure exerts an additional force of 30 lbs. per square foot of projected vertical area. Determine the stresses in the several members, assuming that the point of intersection of the total resultant force with the vertical post is a virtual hinge.

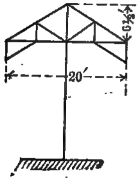


FIG. 179.

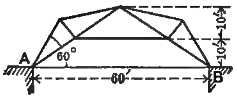


FIG. 180.

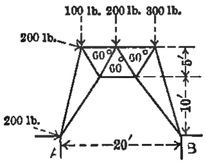


FIG. 181.

94. Draw the stress diagrams for the roof-truss Fig. 180 when subjected to a horizontal wind load of 30 lbs. per square foot of projected vertical area.

The roof-trusses are 20 ft. apart and the foot of a rafter is support rollers, *first* on the leeward and *second* on the windward side.

95. The frame of the dimensions and loaded as in Fig. 181 is fixed on the windward and rests upon rollers on the leeward side at B. Dra stress diagram and show by dotted lines how the stresses are modified the rollers are on the windward side.

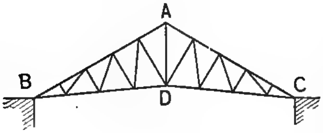


FIG. 182.

96. In the accompanying roof-truss $AB = AC = 30$ ft., and the are all normal to the rafters. Finc stresses in all the members, the loa each of the joints in the rafters bei tons (angle $ABC = 30°$ and angle $= 10°$). How will the stresses be fied if there is a force of 2 tons acti each of the points of support betwe and B at right angles to the rafter, and a force of 1 ton at A, assuming the end B is fixed and that C rests upon rollers?

97. The load upon a roof-truss of the type Fig. 183 is 1000 lbs. at joint; the span 100 ft.; the rise $= 25$ ft. Find the stresses in the diff members. How will the stresses be affected by an additional load o lbs. at each of the joints between the foot and ridge on one side?

Ans. Stress in $BD = 5500\sqrt{5}$; $DF = 5000\sqrt{5}$;
$FH = 4500\sqrt{5}$; $HL = 4000\sqrt{5}$; $LN = 3500\sqrt{}$
$NA = 3000\sqrt{5}$; $DE = 0$; $FG = 500$; $HK = 100$
$LM = 1500$; $NO = 2000$; $AP = 5000$;
$BE = 11000 = EG$; $GK = 10000$; $KM = 9000$;
$MO = 8000$; $OP = 7000$; $DG = 500\sqrt{5}$;
$FK = 1000\sqrt{2}$; $HM = 500\sqrt{13}$; $LO = 1000\sqrt{}$
$NP = 500\sqrt{29}$ lbs.

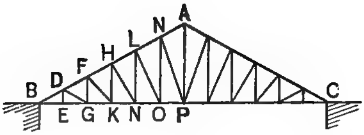


FIG. 183.

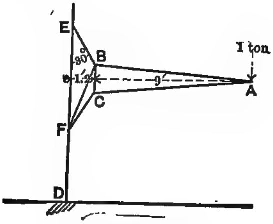


FIG. 184.

98. The frame Fig. 184 supports a load of 1 ton at A. Find the str in the several members, and also the overturning moment at the foot c post, which is 18 ft. in length. ($BC = 1'.6$.) Also show how the stresse

modified when the weight hangs by a chain in two falls, the chain passing over a pulley at A along a horizontal line from A to the post.

99. A braced semi-arch is 10 ft. deep at the wall and projects 40 ft. The upper flange is horizontal, is divided into *four* equal bays, and carries a uniformly distributed load of 40 tons. The lower flange forms the segment of a circle of 104 ft. radius. The bracing consists of a series of isosceles triangles, of which the bases are the equal bays of the upper flange. Determine the stresses in all the members.

100. A bowstring roof-truss, with vertical and diagonal bracing, of 50 ft. rise and five panels is to be designed to resist a wind blowing horizontally with a pressure of 40 lbs. per square foot. The depth of the truss at the centre is 10 ft. Determine *graphically* the stresses in the several members of the truss, assuming that the roof rests on rollers at the windward support.

101. The inner flange of a bent crane forms a quadrant of a circle of 20 ft. radius and is divided into *four* equal bays. The outer flange forms the segment of a circle of 23 ft. radius. The two flanges are 5 ft. apart at the foot, and are struck from centres in the same horizontal line. The bracing consists of a series of isosceles triangles, of which the bases are the equal bays of the inner flange. The crane is required to lift a weight of 10 tons. Determine the stresses in all the members.

102. The domed roof of a gas-holder for a clear span of 80 ft. is strengthened by secondary and primary trussing as in the figure. The points B and C are connected by the tie BPC passing beneath the central strut AP, which is 15 ft. long, and is also common to all the primary trusses; the rise of A above the horizontal is 5 ft.; the secondary truss $ABEF$ consists of the equal bays AH, HG, GB, the ties BE, EF, FA, of which BE is horizontal, and the struts GE, FH, which are each 2 ft. 6 in. long and are parallel to the radius to the centre of GH; the secondary truss $ACLK$ is similar to $ABEF$; when the holder is empty the weight supported by the truss is 36,000 lbs., which may be assumed to be concentrated at G, H, A, M, N, in the proportions 8000, 4000, 1000, 4000, and 8000 lbs., respectively. Determine the stresses in the different members of the truss.

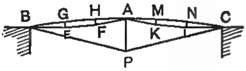


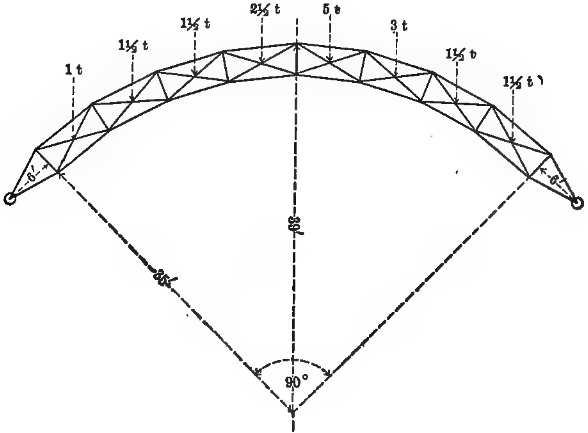
FIG. 185.

103. The top beam of a roof for a clear span of 96 ft. consists of six bars, AB, BC, CD, DE, EF, FG, equal in length and so placed that A, B, C, D, E, F, G are on circle of 80 ft. radius; the lower boom also consists of six equal rods, AH, HK, KL, LM, MN, NG, the points H, K, L, M, and N being on a circle of 148 ft. radius; B is connected with H, C with K, D with L, E with M, and F with N; the opposite corners of the bays are connected by cross-braces; the end A is fixed to its support, G being allowed to slide freely over a smooth bedplate. Determine graphically the stresses in the various mem-

bers when there is a normal wind pressure per lineal foot of 460 lbs. u *AB,* 340 lbs. upon *BC,* and 60 lbs. upon *CD.*

104. A lattice-work arch of the dimensions and loaded as in Fig. 18 pin-jointed at the supports. Find the line of resistance which will through the pin joints and also through the point where the greatest l intersects the upper chord of the arch. Also determine the horizontal th of the arch under the given loading.

FIG. 186.



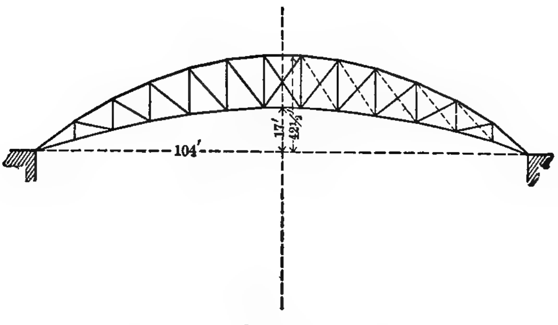


FIG. 187.

105. Fig. 187 represents the sickle (or bowstring) truss for a roof 204 ft. span. The joints of the upper and lower bows lie on the arcs of par olas, the rise of the lower bow being 17 ft. and of the upper bow 42½ Draw the stress diagram for the dead weight of the truss, which is very proximately equivalent to 1½ tons per panel, i.e., to 1½ tons concentrated each of the intermediate joints in the upper bow. The trusses are 24

apart, and the load on each truss, including snow and wind pressure, is assumed to be 40 lbs. per square foot of covered area. Draw the stress diagram when this load covers (*a*) one half of the truss, (*b*) three fourths of the truss. Also show how the diagrams are modified by the introduction of the dotted members.

106. Each rib for the support of a dome of 320 ft. diameter consists of two concentric booms 6.4 ft. apart and connected by triangular bracing which divides the rib into 15 bays of equal length. The rib is pinned at the centre and at the two supports (*A*). The dead load on each of the exterior joints 1 to 15 is 8000 lbs. Determine the horizontal thrust and draw a stress diagram for the members between *A* and *B*. Also determine the horizontal thrusts at *A* and *B* and draw the stress diagram (*a*) for a load of 32 tons at joint No. 7, and (*b*) for loads of 24, 28, 44, and 48 tons, at the joints Nos. 5, 6, 11, and 14, respectively.

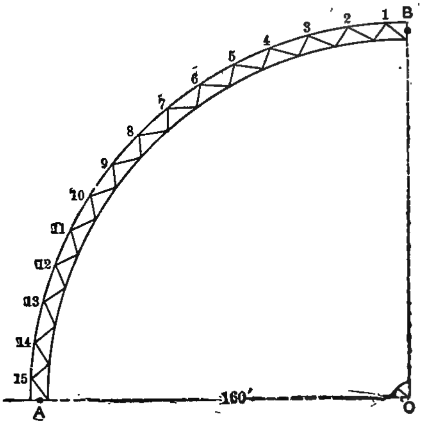


FIG. 188.

107. In the preceding example, draw the stress diagram for a normal load of 10 tons at the joint No. 8, passing through the centre *O* of the dome.

108. The figure represents a portion of a Warren girder cut off by the plane *MN* and supported upon the abutment at *A*. The reaction at $A = 20$ tons; the load concentrated at each of the points $B = 4$ tons. Find the stresses in each of the members met by *MN*.

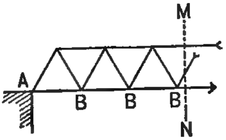


FIG. 189.

Ans. Stress in tension chord $= \frac{104}{3}\sqrt{3}$ tons; in compression chord $= 32\sqrt{3}$ tons; compression in diagonal $= 16\sqrt{3}$ tons.

109. The figure is a portion of a bridge-truss cut off by the plane M. and supported upon the abutment at A; $AC=C$ $=14\frac{7}{24}$ ft.; the depth $BC=DE=17\frac{1}{4}$ ft.; in the thi panel the compression in the upper chord is 64,6 lbs.; the tension in the lower chord is 53,800 lb Find the reaction at A, the equal weights supporte at C and E, and the diagonal stress T.

Ans. Reaction $=38943$ lbs.; weight at C and a $E=12954$ lbs.; $T=16578$ lbs.

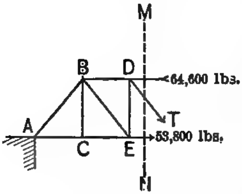


FIG. 190.

110. The figure represents a bowstring truss of 80 ft. span, cut off by th plane MN and supported at O. The upper flange $OCDE$ is an arc of a circle of 85 ft. radius; $OA=AB$ $=$etc.$=10$ ft.; the rise of the truss$=10$ ft.; a load of 15 tons is concentrated at each of the points A and B; the reaction at $O=45$ tons. Find the stresses in the members out by the plane MN.

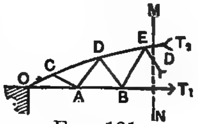


FIG. 191.

Ans. $T_1=94.827$ tons; $T_2=95.744$ tons; $D=.97$ tons.

111. The figure represents a portion of a roof-truss cut off by a plane M and supported at A. The strut DC is vertical $AD=23$ ft., and the distance of D from $AC=7$ ft.; the angle between AC and the horizonta $=\cos^{-1}\frac{4}{5}$; the vertical reaction at $A=7$ tons the horizontal reaction at $A=2\frac{1}{2}$ tons; at eac of the points B and C a weight of 4 tons is con centrated. Find the stresses in the member met by MN. (AD and T_1 make equal angle with the rafter and $DB=DC$.)

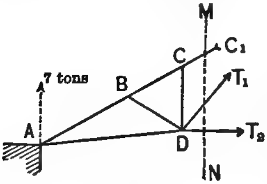


FIG. 192.

Ans. $C_1=13.2$ tons; $T_2=2.1$ tons; $T_1=10.8$ tons

112. Fig. 193 represents a portion of a 16-panel bridge-truss with one en supported at A. The load at each panel-point is W and the dimensions ar as indicated. Find the stresses in the members cut by a vertical sectio MN.

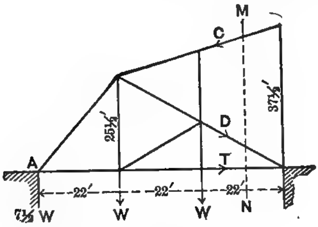


FIG. 193.

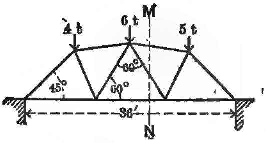


FIG. 194.

Ans. $C=11.86W$; $T=7\frac{1}{3}W$; $D=4.74W$.

113. Find, by the "method of sections" the stresses in the members of th truss Fig. 194 cut by a vertical section MN, the dimensions and loads bein as indicated.

115. The cantilever roof-truss Fig. 195 is supported at *C* and by the stay *AB*. Find the position of the load which will produce *no stress* in *sr*.

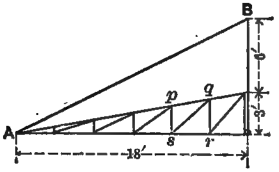


FIG. 195.

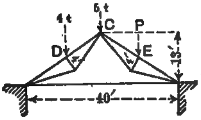


FIG. 196.

Hence also find the maximum tensile and compressive stresses in the members *pq*, *qr*, *rs*, *sq*.

116. The roof-truss Fig. 196, of 40 ft. span and 13 ft. rise, is loaded with 4 tons at *D* and 5 tons at *C*. Find the load *P* at *E* which will make the stress diagram close. The struts are each 4 ft. in length.

117. Fig. 197 is the truss for a factory roof of 72 ft. span and 18 ft. rise.

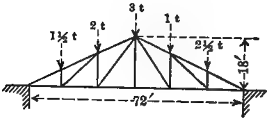


FIG. 197.

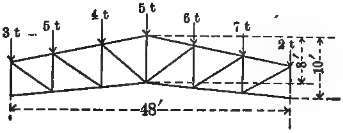


FIG. 198.

It is loaded as indicated. Draw the stress diagram.

118. Fig. 198 is a six-panel truss of 48 ft. span, the dimensions and loading being as indicated. Determine the stresses in all the members.

119. A ship's gangway is loaded and of the dimensions shown in Fig. 199.

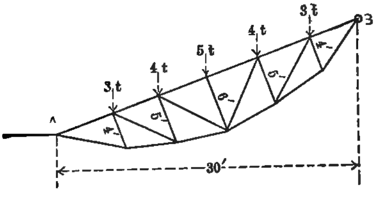


FIG. 199.

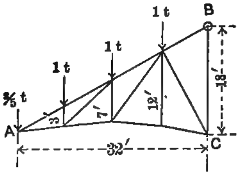


FIG. 200.

The end *A* rests upon rollers, and *B* is hinged to the wharf. When *B* is 10 ft. vertically above *A*, find the stresses in all the members.

120. The end *A* of the cantilever truss Fig. 200 rests upon rollers, *B* is hinged and the bearing at *C* produces a horizontal thrust. Draw the stress diagram for the given loads.

121. The cantilever span Fig. 201 is loaded with 5500 lbs. per foot

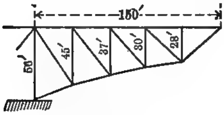


FIG. 201.

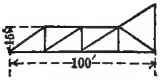


FIG. 202.

of *bridge.* Determine the stresses in all the members, and check the res
for the pier panel by the method of moments.

122. The dead weight of a draw-span having equal arms of 100 ft.,
202, is 1200 lbs. per lineal
of truss. Determine the stres
in all the members when
bridge is open.

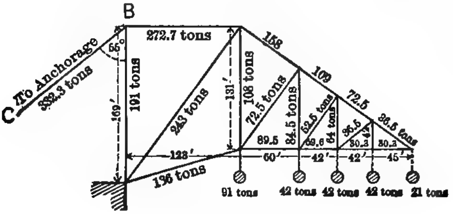


FIG. 203.

123. Fig. 203 is the skele
diagram of a cantilever for
viaduct in India. Determine
stresses in the various memb
under the indicated loading.

124. The figure represents the shore portion of one of the trusses fo
cantilever highway bridge. The depth of truss over pier = 51 ft.; the length of each panel = 17 ft.; the load at *A* (from weight of centre span) = 16,800 lbs.; the width of roadway = 15 ft.; the load per square foot of roadway = 80 lbs. Find the stresses in all the members, assuming the reaction at the pier *F* to be vertical.

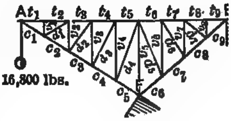


FIG. 204.

Ans. $t_1 = t_2 = 28000$; $t_3 = 36500$; $t_4 = 45000$; $t_5 = 53500$;
$t_6 = 55200$; $t_7 = 48400$; $t_8 = 41600 = t_9$; $c_1 = 5600\sqrt{34}$
$c_2 = 7300\sqrt{34}$; $v_1 = 10200$; $v_2 = 15300$; $v_3 = 20400$.

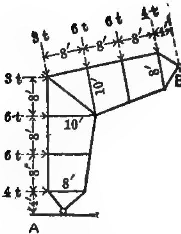


FIG. 205.

125. The truss Fig. 205 is hinged at *A* and *B*,
reaction at *B* being horizontal. The loads and dim
sions are as indicated. Draw the stress diagram.

126. The upper ends of the verticals of an
panel truss lie in the arc of a parabola whose para
eter is 128 ft., the depth at the centre being 32
Draw stress diagrams (*a*) for a load of 4 tons at e
panel-point; (*b*) for an additional load of 6 t
at the first, second, and third panel-points. (Thro
single-intersection type.)

Ans.

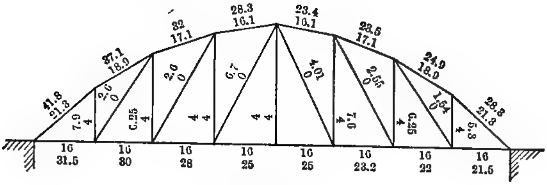


FIG. 206.

127. The loads upon the truss of a swing-bridge when open are as indicated in Fig. 207, *C* being the counterweight at *D* and *R* the reaction at each of the bearing points *A* and *B*. Find *R* and *C*, and draw a stress diagram for the truss under the given loading. The panels are each 6 ft. in length and the truss is 8 ft. deep at the 1st vertical and 12 ft. deep over the turntable.

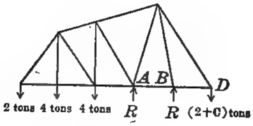


FIG. 207.

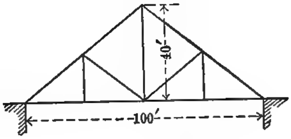


FIG. 208.

128. A *four*-panel *A* truss, Fig. 208, of 100 ft. span is 40 ft. deep. The panel dead load is 18,750 lbs. Determine the stresses in all the members. Also determine the stresses in the several members due to a panel live load of 53,250 lbs. concentrated at the 2d and 3d panel-points.

129. A Warren girder 80 ft. long is formed of five equilateral triangles. Weights of 2, 3, 4, 5 tons are concentrated, respectively, at the 1st, 2d, 3d, and 4th apex along the upper chord. Determine the stresses in all the members of the girder.

Ans. *Compression* chord: Stress in 1st bay $=2\sqrt{3}$; 2d $=5\frac{1}{3}\sqrt{3}$; 3d $=7\sqrt{3}$; 4th $=6\frac{1}{3}\sqrt{3}$; 5th $=2\frac{2}{3}\sqrt{3}$.

Tension chord: Stress in 1st bay $=4\sqrt{3}$; 2d $=6\frac{2}{3}\sqrt{3}$; 3d $=7\frac{1}{3}\sqrt{3}$; 4th $=5\frac{1}{3}\sqrt{3}$.

Diagonals: Stress in 1st and 2d $=4\sqrt{3}$; 3d and 4th $=2\frac{2}{3}\sqrt{3}$; 5th and 6th $=\frac{2}{3}\sqrt{3}$; 7th and 8th $=2\sqrt{3}$; 9th and 10th $=5\frac{1}{3}\sqrt{3}$.

130. A Warren girder of 60 ft. span, composed of six equilateral triangles, carries upon its lower chord a weight of 2 tons at the first and second joints, 15 tons at the centre joint, and $7\frac{1}{2}$ tons at the fourth and fifth joints. Find the stresses in all the members.

Ans. Stresses in tension chord: 1st bay $=\frac{19}{4}\sqrt{3}$; 2d $=\frac{163}{12}\sqrt{3}$; 3d $=\frac{258}{12}\surd$
4th $=\frac{275}{12}\sqrt{3}$; 5th $=\frac{69}{4}\sqrt{3}$; 6th $=\frac{79}{12}\sqrt{3}$.

Stresses in compr. chord: 1st bay $=\frac{19}{2}\sqrt{3}$; 2d $=\frac{53}{3}\sqrt{3}$; 3d $=\frac{49}{2}\surd$
4th $=\frac{64}{3}\sqrt{3}$; 5th $=\frac{79}{6}\sqrt{3}$.

Diag. stresses 1st and 2d bays $=\frac{19}{2}\sqrt{3}$; 3d and 4th $=\frac{49}{6}\sqrt{3}$; 5th a
6th $=\frac{41}{6}\sqrt{3}$; 7th and 8th $=\frac{19}{6}\sqrt{3}$; 9th and 10th $=\frac{49}{6}\sqrt{3}$; 11th a
12th $=\frac{79}{6}\sqrt{3}$.

131. Determine the stresses in the members of a Fink truss of 240 ft. sp and sixteen panels; depth of truss $=30$ ft.; uniformly distributed loa 64 tons.

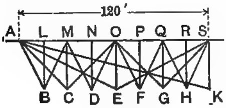


FIG. 209.

Ans. Stress in $BA=\sqrt{5}=BM=DM=DO=F$ $FQ=QH=HS$; in $CA=4\sqrt{2}=CO=GO=GS$; in $=8\sqrt{5}=ES$; in $AK=16\sqrt{17}$; in $BL\,4=DN=FP=H$ in $CM=8=GQ$; in $EO=16$; in $KS=32$; in $AM=8$ $MO=OQ=QS$.

132. Determine the stresses in the members of a Bollman truss 100 long and 12½ ft. deep, under a uniformly distributed load of 200 tons, toget with a single load of 25 tons concentrated at 25 ft. from one end.

Ans. Stress in $AB=\frac{175}{8}\sqrt{2}$; $BL=\frac{125}{8}\sqrt{2}$; $AD=\frac{75}{2}\sqrt{5}$; $DL=\frac{25}{2}\surd$
$AF=\frac{125}{8}\sqrt{10}$; $FL=\frac{75}{8}\sqrt{26}$; $AH=\frac{25}{2}\sqrt{17}=HL$; in $BC=25=F$
$HK=$etc.; $DE=50$ tons; compression along $AL=193\frac{3}{4}$ tons.

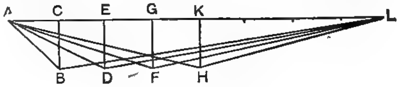


FIG. 210.

133. Determine the chord, vertical and diagonal stresses in a Howe tr of 80 ft. span, 8 ft. depth, and ten panels, due to a load of 40 tons (*a*) c centrated at the centre; (*b*) concentrated at the third panel-point; (*c*) u formly distributed; (*d*) distributed so that 5 tons is at first panel-point, tons at second, and 25 tons at third. *Ans.* Numbers on figures in tons

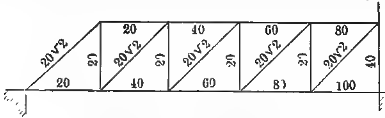


FIG. 211.—Case *a*.

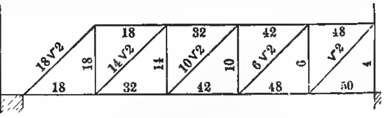


FIG. 211.—Case *c*.

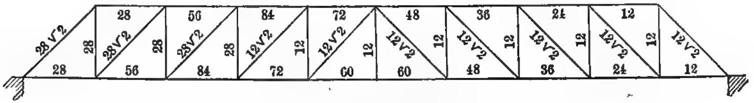


FIG. 211.—Case *b*.

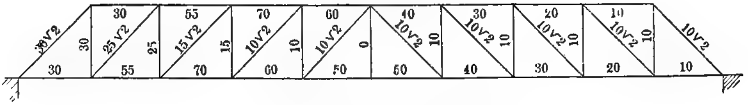


FIG. 211.—Case *d*.

134. Draw a stress diagram for the *A* truss, Fig. 212, (*a*) due to a dead load of 600 lbs. per lineal foot of truss, (*b*) due to a live load of 1200 lbs. per lineal foot of truss covering the first *two*, *three*, *four*, and *six* panels, respectively.

135. A Baltimore truss 300 ft. long, one half of which is shown in Fig. 213, carries a load of 80,000 lbs. at each panel-point of the lower chord and 40,000 lbs. at each panel-point of the top chord. Find the stresses in all the members.

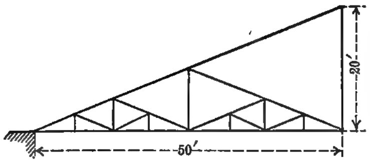


FIG. 212.

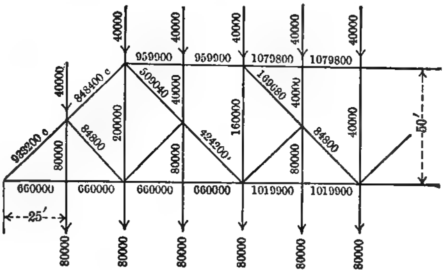


FIG. 213.

136. The lower lateral system of a cantilever arm, Fig. 214, carries at the outer end a horizontal wind load of 472,080 lbs. from the suspended span.

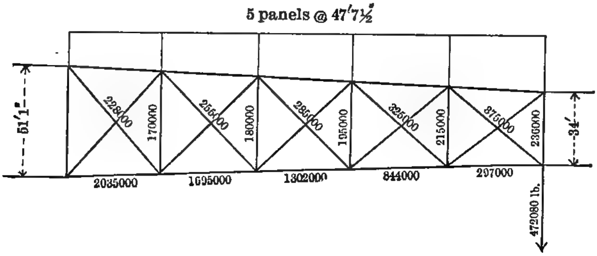


FIG. 214.

Find the stresses in all members, assuming that the diagonal stresses are equally divided between the two systems of diagonals.

137. The cantilever arm shown in Fig. 215 carries a vertical load of 700,000 lbs. at its outer extremity. Find the stresses in all members.

138. One arm of a swing span is loaded as shown in Fig. 216. Find all the stresses when the span is open, assuming that of each panel load 32,000 lbs. is concentrated at the top chord and the remainder at the bottom chord.

139. Find all the stresses in the truss, Fig. 217, for an upward reaction of 100,000 lbs. at the end of the span.

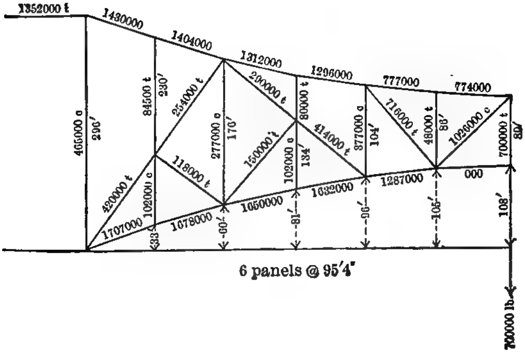


FIG. 215.

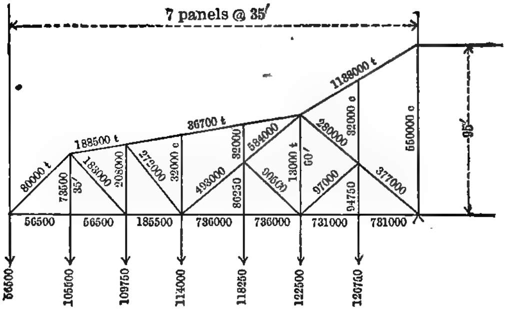


FIG. 216.

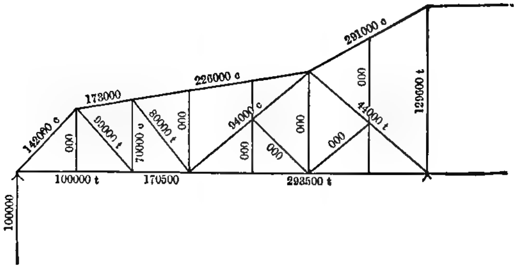


FIG. 217.

140. In the same truss, Fig. 218, a load of 100,000 lbs. is applied at the third panel-point from the left. Assuming that the reaction at the left support is 42,000, find the stresses in all members.

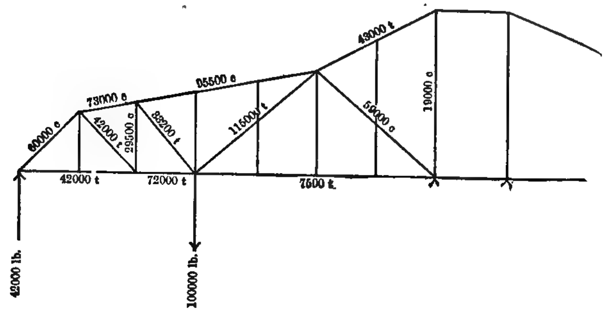


FIG. 218.

141. A three-pin steel arch truss of the dimensions and loaded as shown in Fig. 219 has the joints of the lower chord lying in the arc of a parabola. Draw the stress diagram. Verify your results by using the method of sections to determine the stresses in the members intersected by the cutting-planes *MN*, *M′N′*.

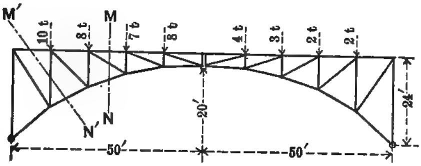


FIG. 219.

142. Draw the stress diagram for the 3-pin arch, Fig. 220, the left truss being loaded with 1200 lbs. per lineal foot.

143. Fig. 221 is the 3-hinged truss for a bridge hinged at *C* and at the two abutments *A* and *B*. The load at each of the points 1 and 2 is 5 tons and 10 tons are concentrated at each of the intermediate panel-points. The bracing consists of isosceles triangles with the equal bays of the horizontal chords as bases, and the joints in the lower chord lie in the arc of a circle of 90 ft. radius. The depth of the truss at the abutments is 18 ft. Determine the reactions at *A*, *B*, and *C*, and draw a stress diagram for the truss.

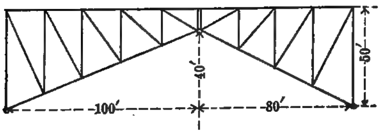


FIG. 220.

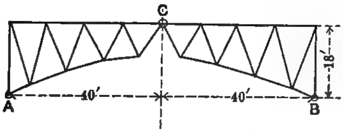


FIG. 221.

144. Fig. 222 represents one of the two symmetrical trusses for the central span of a suspension bridge. The joints in the upper chord lie on a parabolic

arc and consecutive joints are connected by straight links. The dead per lineal foot of truss is 750 lbs., and the two intermediate trusses are pi at the centre and at the piers. Determine the stresses in the members

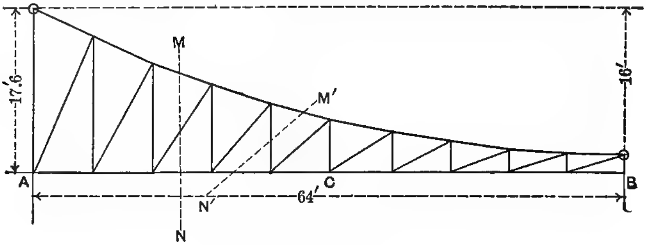


Fig. 222.

by the sections MN and $M'N'$. Also determine the stresses in the s. members when a live load of 1250 lbs. per lineal foot covers (*a*) the two panels, (*b*) the panels from A to C, (*c*) the panels from C to B.

145. Each shore arm in the suspension bridge of the preceding exan is identical with *one half* of the central span. Determine the stresses in corresponding members of the shore arms (*a*) for the dead load of 750 lbs. lineal foot, (*b*) when the live load of 1250 lbs. per lineal foot covers *first* shore arm and *second* one half of the central span.

146. A braced pier is of the dimensions and loaded as in Fig. 223. De mine the stresses in all the members and also find the forces in the vert anchorage-bars.

147. Draw a stress diagram for the braced pier Fig. 224, of the dim

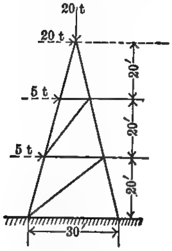


Fig. 223.

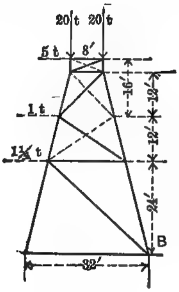


Fig. 224.

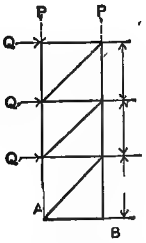


Fig. 225.

sions and loaded as shown. Also find the resultant vertical and horizor forces at A and B, specifying whether they are in tension or compressior

148. A trestle, Fig. 225, 30 ft. high and 10 ft. wide is loaded as sho Draw the stress diagram and find the character and magnitude of the res ant vertical and horizontal forces at A when (*a*) $P=2Q$, (*b*) $P=3Q$, (*c*) $P=$

149. The figure represents a pier, square in plan, supporting the ends of two deck-trusses, each 200 ft. long and 30 ft. deep. The height of the pier is 50 ft. and is made up of three panels, the upper and lower being each 17 ft. deep. Ten square feet of bridge surface and 10 square feet of train surface per lineal foot are subjected to a wind pressure of 40 lbs. per square foot. The centre of pressure for the bridge is 68 ft., and for the train 86 ft., above the pier's base. The wind also produces a horizontal pressure of 4000 lbs. at each of the intermediate panel-points on the windward side of the pier. Width of pier = 17 ft. at top and $33\frac{2}{3}$ ft. at bottom. The bridge load = 1600 lbs. per lineal foot, live load = 3000 lbs. per lineal foot. Determine:

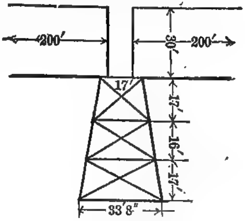


FIG. 226.

(*a*) The overturning moment. (3180 ft.-tons.)

(*b*) The horizontal force due to the wind at the top of the pier. (61.6 tons.)

(*c*) The tension in the vertical anchorage-ties at *S* and *T*. (*Nil.*)

(*d*) The vertical and horizontal reactions at *T*. (275 and 65.6 tons.)

Draw a diagram giving the wind stresses in all the members, and indicate which are in tension and which in compression.

Ascertain whether the wind pressure of 40 lbs. per square foot upon a train of empty cars weighing 900 lbs. per lineal foot will produce a tension anywhere in the inclined posts. What will be the tension in the anchorage-ties? (20.75 tons.)

150. The figure represents one half of one of the piers of the Bouble Viaduct. The spans are crossed by two lattice girders 14′ 9″ deep and having a deck platform. The height of the pier is 183′ 9″ and is made up of eleven panels of equal depth. Width of pier at top = 13′ $1\frac{1}{2}$″, at bottom = 67′ 7″. With wind pressure at 55.3 lbs. per square foot, the total pressure on the girder, train, and pier have been calculated to be 20, 16.2, and 20 tons, acting at points 196.2, 210.3, and 92.85 ft., respectively, above the base. The dead weight upon each half pier is $222\frac{1}{2}$ tons, of which 60 tons is weight of half span, 120 tons the weight of the half pier, and $42\frac{1}{2}$ tons the weight of the train. Assuming that the wind pressure on the pier is a horizontal force of 2 tons at each panel-point on the windward side, and that the weight of the pier may be considered as a weight of 6 tons at each panel-point, determine:

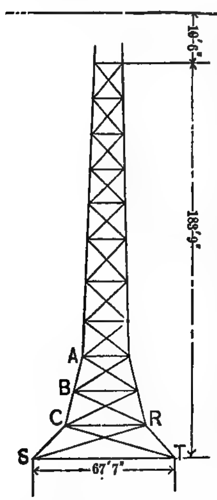


FIG. 227.

(*a*) The overturning moment.

(*b*) The total horizontal force at the top of the pier due to the wind.

(*c*) The tension in each of the vertical anchorage-ties at S and T due the wind pressure.

(*d*) The vertical and horizontal reactions at T.

Show that the greatest compressive stress occurs in the member RT, a that it amounts to 422 tons.

Draw a stress diagram giving the stresses in all the members, indicati which are in tension and which in compression. Width of pier at $A=20$ f at $B=23\frac{1}{2}$ ft., at $C=36\frac{1}{2}$ ft.

151. Draw the stress diagram for a trestle loaded and of the dimensio as shown in Fig. 228.

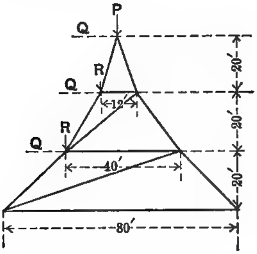


FIG. 228.

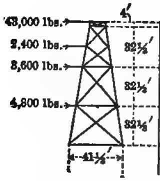


FIG. 229.

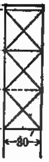


FIG. 230.

152. Determine the stresses in the several members of a steel trestl shown by Fig. 229 when subjected to horizontal wind loads of 43,00 2400, 3600 and 4800 lbs. at A, C, D, and E, respectively. AB is equ to 4 ft., and the vertical distance bewteen B and D, D and E, E and is $33\frac{1}{3}$ ft. The width of the trestle at the top is 8 ft. and at the botto $41\frac{1}{3}$ ft.

CHAPTER II.

SHEARING FORCES AND BENDING MOMENTS.

Note.—In this chapter it is assumed that all forces act in one and the same plane, and that the deformations are so small as to make no sensible alteration either in the forces or in their relative positions.

1. Equilibrium of Beams.

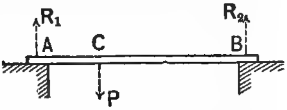


FIG. 231.

CASE I. *AB* is a beam resting upon two supports in the same horizontal plane. The reactions R_1 and R_2 at the points of support are vertical and the resultant P of the remaining external forces must also act vertically in an opposite direction at some point C. According to the principle of the lever,

$$R_1 = P\frac{BC}{AB}, \quad R_2 = P\frac{AC}{AB}, \quad \text{and} \quad R_1 + R_2 = P.$$

CASE II. *AB* is a beam supported or *fixed* at one end, and such a beam is often called a *cantilever* or *semi-girder*. The fixture at *A* tends to prevent any deviation from the straight in that portion of the beam, and the less the deviation the more perfect is the fixture.

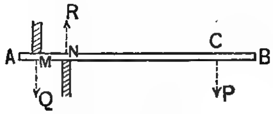


FIG. 232.

The ends may be fixed by means of two props (Fig. 232), or by allowing it to rest upon one prop and preventing upward motion by a ledge (Fig. 233), or by building it into a wall (Fig. 234).

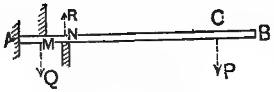


FIG. 233.

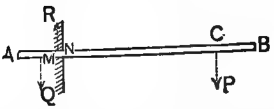


FIG. 234.

In any case it may be assumed that the effect of the fixture, whether perfect or imperfect, is to develop two unequal forces, Q and R, acting in opposite directions at points M and N.

These two forces are equivalent to a left-handed couple (Q, – the moment of which is $Q \cdot MN$, and to a single force $R-Q$ at Hence $R-Q$ must $=P$

CASE III. AB is an inclined beam supported at A and rest upon a smooth vertical surface at B.

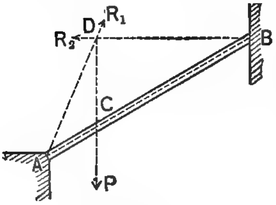


FIG. 235.

The vertical weight P, acting at point C, is the resultant load upon Let the direction of P meet the horizon line of reaction at B in the point D.

The beam is kept in equilibrium the weight P, the reaction R_1 at A, the reaction R_2 at B. Now the two for R_2 and P meet at D, so that the force R_1 must also pass through

Hence $R_1 = P \sec ADC$ and $R_2 = P \tan ADC$.

Note.—The same principles hold if the beam in Cases I and is inclined, and also in Case III whatever may be the directi of the forces P and R_2.

CASE IV. *In general*, let the beam AB be in equilibrium un the action of any number of forces P_1, P_2, P_3, . . . , Q_1, Q_2, Q_3, . of which the magnitudes and points of application are given, a

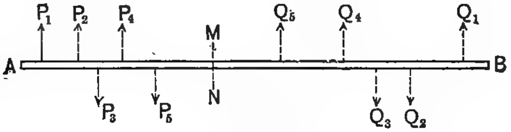


FIG. 236.

which act at right angles to the length of the beam. Suppose beam to be divided into two segments by an imaginary plane M Since the whole beam is in equilibrium, each of the segments m also be in equilibrium. Consider the segment AMN.

It is kept in equilibrium by the forces P_1, P_2, P_3, . . . and the reaction of the segment BMN upon the segment AMN at plane MN; call this reaction E_1. The forces P_1, P_2, P_3, . . . equivalent to a single resultant R_1 acting at a point distant r_1 fr MN. Also, without affecting the equilibrium, two forces, e equal and parallel to R_1, but opposite to one another in directi may be applied to the segment AMN at the plane MN, and three equal forces are then equivalent to a single force R_1 at MN, and a couple $(R_1, -R_1)$ of which the moment is $R_1 r_1$.

Thus the external forces upon AMN are reducible to a single force R_1 at MN, and a couple $(R_1, -R_1)$. These must be balanced by E_1, and therefore E_1 is equivalent to a single force $-R_1$ at MN and a couple $(-R_1, R_1)$.

In the same manner the external forces upon the segment BMN are reducible to a single force R_2 at MN, and a couple $(R_2, -R_2)$ of which the moment is $R_2 r_2$. These again must be balanced by E_2, the reaction of the segment AMN upon the segment BMN.

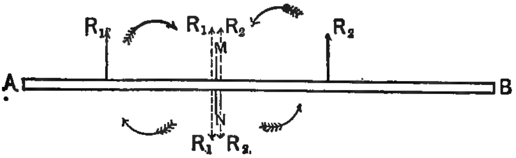


FIG. 237.

Now E_1 and E_2 evidently neutralize each other, so that the force R_1 and the couple $(R_1, -R_1)$ must neutralize the force R_2 and the couple $(R_2, -R_2)$. Hence the force R_1 and the couple $(R_1, -R_1)$ are respectively equal but opposite in effect to the force R_2 and the couple $(R_2, -R_2)$; i.e.,

$$R_1 = R_2, \quad R_1 r_1 = R_2 r_2, \quad \text{and} \quad r_1 = r_2.$$

The force R_1 tends to make the segment AMN slide over the segment BMN at the plane MN, and is called the *Shearing Force* with respect to that plane. It is equal to the algebraic sum of the forces on the *left* of MN,

$$= P_1 + P_2 - P_3 + \ldots = \Sigma(P).$$

So $R_2 = Q_1 - Q_2 - Q_3 + \ldots = \Sigma(Q)$ is the algebraic sum of the forces on the *right* of MN, and is the force which tends to make the segment BMN slide over the segment AMN at the plane MN. R_2 is therefore the *Shearing Force* with respect to MN, and is equal to R_1 in magnitude, but acts in an opposite direction.

Again, let $p_1, p_2, p_3, \ldots, q_1, q_2, q_3, \ldots,$ be respectively the distances of the points of application of P_1, P_2, P_3, . . . , Q_1, Q_2, Q_3, . . . from MN.

Then R_1r_1 = the algebraic sum of the moments about MN of all the forces on the *left* of MN,

$$= P_1p_1 + P_2p_2 - P_3p_3 + \ldots = \Sigma(Pp),$$

is the moment of the couple $(R_1, -R_1)$.

This couple tends to bend the beam at the plane MN, and its moment is called the *Bending Moment* with respect to MN of all he forces on the *left* of MN.

So R_2r_2 = the algebraic sum of the moments about MN of all the forces on the *right* of MN,

$$= Q_1q_1 - Q_2q_2 - \ldots = \Sigma(Qq),$$

is the *Bending Moment*, with respect to MN, of all the forces on the *right* of MN, and is equal but opposite in effect to R_1r_1.

It is seen that the Shearing Force and Bending Moment *change sign* on passing from one side of MN to the other, so that to define them *absolutely* it is necessary to specify the segment under consideration.

If the segment CB is removed, AC may be kept in equilibrium by applying to the face of the section at C, forces of precisely the same magnitude and character as those which acted when ACB was a continuous beam. These forces may be resolved into horizontal and vertical components. The algebraic sum of the latter, i.e., the *shearing force* at C, must necessarily be equal to the algebraic sum of the vertical forces acting externally upon AC, i.e., to R_1. Again, there are no horizontal forces acting externally upon the beam, and therefore the algebraic sum of the horizontal components of the forces at C must be nil. Hence these horizontal components must be equivalent to a couple which will neutralize the couple due to the external forces acting upon AC, of which the moment is R_1r_1. The moment of the horizontal components at C is usually called the *moment of resistance*, but is sometimes also known as the *elastic moment*, as it is due to the elastic deformation of the section under consideration.

Again, Fig. 238 represents a horizontal beam of depth resting upon supports at A and B. The vertical reaction at A is R, and the resultant load upon AD (including the weight of AD) is W at a distance x from D.

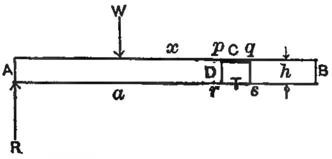


FIG. 238.

Let a rectangular portion $pqrs$ of the beam be removed and replaced by two members, pq capable of bearing a pressure only, and rs capable of bearing a tension only, as, e.g., a chain. These members alone are not sufficient to maintain equilibrium, and it can be easily shown experimentally that an upward shearing force, S, must be applied at r and that

$$S = R - W.$$

Also, the bending moment at D due to R and W

$$= Ra - Wx,$$

and this must be neutralized by the moment of the compression C in pq with respect to r *or* by the moment of the tension T in rs with respect to p.

Therefore $\qquad Ra - Wx = Ch = Th.$

Hence also $\qquad C = T.$

2. Examples of Shearing Force and Bending Moment.—In general the terms shearing force and bending moment will be designated by the abbreviations S.F. and B.M., the shearing force and bending moment at any point distant x from the origin by S_x, M_x, and the shearing forces and bending moments at any points A, B . . . by S_a, S_b . . . and M_a, M_b . . . respectively.

The beams, unless otherwise specified, are horizontal.

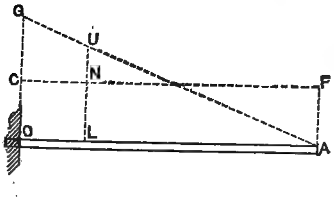


FIG. 239.

EX. *a. A cantilever OA, Fig. 239, is fixed at O and carries a weight P at the free end A.*

$$S_x = P,$$

and the S.F. is the same at every point between O and A. Describe the rectangle $OCFA$, taking $OC = P = AF$. The line CF is the S.F. diagram and the S.F. at any point L is the vertical distance LN between that point and the line CF.

Again,

$$M_x = P(l - x),$$

which is Pl when $x = 0$, i.e., at O, and is 0 when $x = l$, i.e., at A.

Take the vertical line $OG=Pl$ and join GA. The line GA is the B. diagram and the B.M. at any point L is the vertical distance LU betwe that point and the line GA. The B.M. is evidently greatest at O, so that

the max. B.M. = Pl.

Also, $\quad LU=M_x=P(l-x)=\text{area } ALNF$

$= \textit{the total } S.F. \text{ between } L \text{ and } A.$

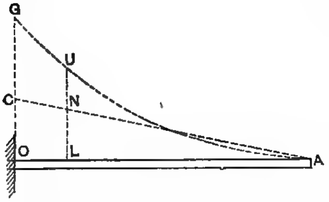


Fig. 240.

Ex. b. *A cantilever OA, Fig. 240, is fi at O and carries a uniformly distributed lo of intensity w.* Then

$$S_x=w(l-x),$$

which is wl when $x=0$, i.e, at O, and is when $x=l$, i.e., at A.

Take the vertical line $OC=wl$ and join C The line CA is the S.F. diagram and the S at any point L is the vertical distance LN between that point and the line C

Again, $$M_x=\frac{w}{2}(l-x)^2,$$

the equation to a parabola with its vertex at A.

The B.M. is $\frac{wl^2}{2}$ when $x=0$, i.e.. at O, and is 0 when $x=l$, i.e., at A.

Take the vertical line $OG=\frac{wl^2}{2}$ and trace the parabola AG. The cur GA is the B.M. diagram and the B.M. at any point L is the vertical distar LU between that point and the parabolic arc AG. The B.M. is eviden greatest at O, so that

the max. B.M. $= \frac{\mathbf{wl^2}}{\mathbf{2}}$.

Also, $\quad LU=M_x=\frac{w}{2}(l-x)^2=\text{area } ALN$

$=\text{total S.F. between } L \text{ and } A.$

Ex. c. *A cantilever OA, Fig. 241, is fixed at O and carries a weight P at together with a uniformly distributed load of intensity w.*

$$S_x=P+w(l-x),$$

the equation to a straight line.

The S.F. is $P+wl$, when $x=0$, i.e., at O, and is P when $x=l$, i.e., at

Take the vertical lines $OC=P+wl$ and $AF=P$ and join CF. The CF is the S.F. diagram and the S.F. at any point L is the vertical dista LN between that point and the line CF.

Again, $$M_x=P(l-x)+\frac{w}{2}(l-x)^2,$$

which may be written in the form

$$M_x+\frac{P^2}{2w}=\frac{w}{2}\left(\frac{P}{w}+l-x\right)^2,$$

the equation to a parabola with its vertex at H, where $AT=\frac{P}{w}$ and $HT=\frac{P^2}{2w}$.

The B.M. is $Pl+\frac{wl^2}{2}$ when $x=0$, i.e., at O, and is 0 when $x=l$, i.e., at A.

Take the vertical line $OG=Pl+\frac{wl^2}{2}$ and trace the parabola GAH. The curve GA is the B.M. diagram and the B.M. at any point L is the vertical distance LU between that point and the parabolic arc GA.

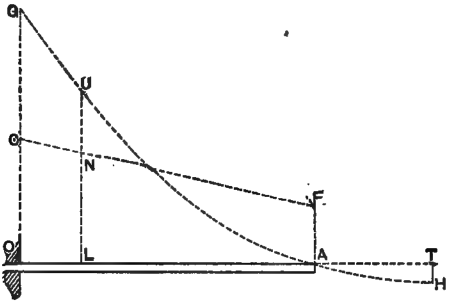


FIG. 241.

The B.M. is evidently greatest at O, so that

$$\text{the max. B.M.}=Pl+\frac{wl^2}{2}.$$

Also, $$LU=M_x=P(l-x)+\frac{w}{2}(l-x)^2=\text{area } LNFA$$

$$=\textit{total S.F.}\text{ between } L \text{ and } A.$$

Ex. *d. The beam OA, Fig.* 242, *resting upon supports at O and A, carries a weight P concentrated at a point B dividing the beam into the segments OB=a, BA=b.* Then

$$a+b=l.$$

Let R_1, R_2 be the reactions at O and A respectively. Taking moments about A and O,

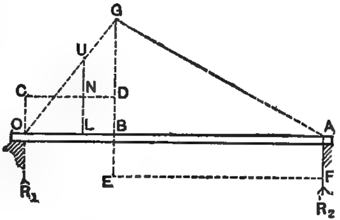


FIG. 242.

$$R_1=\frac{Pb}{l} \quad\text{and}\quad R_2=\frac{Pa}{l}.$$

Then, *between O and B,*

$$S_x=R_1=P\frac{b}{l}$$

and *between B and A*,

$$S_x = R_1 - P = -R_2 = -P\frac{a}{l},$$

so that on passing the point B, the S.F. changes sign.

Take the vertical lines $OC = R_1 - BD$ and $BE = R_2 = AF$; join CD FE. The broken line $CDEF$ is the S.F. diagram, and the S.F. at any p L is the vertical distance LN between that point and the broken line CD

Again, *between O and B*,

$$M_x = R_1 x,$$

the equation to a straight line.

The B.M. is 0 when $x = 0$, i.e., at O,

and is $$R_1 a = P\frac{ab}{l} \text{ when } x = a, \text{ i.e., at } B.$$

So, between B and A,

$$M_x = R_1 x - P(x - a),$$

the equation to a straight line.

The B.M. is $R_1 a = P\frac{ab}{l}$ when $x = a$, i.e., at B,

and is $$R_1 l - P(l - a) = 0, \text{ when } x = l, \text{ i.e., at } A.$$

Take the vertical line $BG = P\frac{ab}{l}$ and join GO and GA. The lines OG, are the B.M. diagram, and the B.M. at any point L is the vertical distance between that point and the lines OG, GA.

The B.M. is evidently greatest at B, so that

$$\textbf{the max. B.M.} = \mathbf{P}\frac{\mathbf{ab}}{\mathbf{l}}.$$

If $a = b = \frac{l}{2}$, the weight is at the centre and

$$\textbf{the max. B.M.} = \frac{\mathbf{Pl}}{\mathbf{4}}.$$

Also, *between O and B*,

$$LU = M_x = R_1 x = \text{area } OCNL$$
$$= \text{total S.F. between } O \text{ and } L.$$

Between B and A,

$$LU = M_x = R_1 x - P(x - a) = R_1 a + (R_1 - P)(x - a)$$
$$= \textit{algebraic} \text{ sum of the areas } OCDB \text{ and } BLNE$$
$$= \text{total S.F. between } O \text{ and } L.$$

Ex. *e. The beam OA, Fig.* 243, *resting upon supports at O and A, carri uniformly distributed load of intensity w.*

Then $$R_1 = \frac{wl}{2} = R_2,$$

$$S_x = R_1 - wx = w\left(\frac{l}{2} - x\right),$$

which is $\frac{wl}{2}$ when $x = 0$, i.e., at O,

is 0 when $x = \frac{l}{2}$, i.e., at the middle of the beam,

and is $-\frac{wl}{2}$ when $x = l$, i.e., at A.

The S.F. therefore changes sign on passing the middle point B of the beam and increases uniformly from its mimimum value zero at B to its maximum value $\frac{wl}{2}$ *at O and at A.*

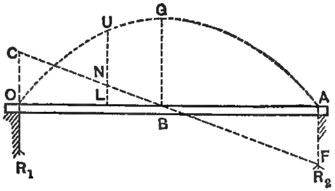


FIG. 243.

Take the vertical lines $OC = \frac{wl}{2} = AF$ and join CF. The line CF is the S.F. diagram, and the S.F. at any point L is the vertical distance LN between that point and the line CF.

Again, $$M_x = R_1x - \frac{wx^2}{2} = \frac{w}{2}(lx - x^2),$$
which may be written in the form

$$M_x - \frac{wl^2}{8} = -\frac{w}{2}\left(\frac{l}{2} - x\right)^2,$$

the equation to a parabola OGA of parameter $\frac{2}{w}$ and with its vertex at a point defined by $OB = \frac{l}{2}$ and $BG = \frac{wl^2}{8}$. The parabola OGA is the B.M. diagram, and the B.M. at any point L is the vertical distance LU between that point and the parabola.

The B.M. is least at the supports and gradually increases towards the centre of the beam where it is greatest, so that the

$$\text{max. B.M.} = \frac{wl^2}{8}.$$

Also, $$LU = M_x = \frac{wl^2}{8} - \frac{w}{2}\left(\frac{l}{2} - x\right)^2$$
$$= \text{area } BOC - \text{area } BLN$$
$$= \text{area } OCNL$$
$$= \text{total S.F. between } O \text{ and } L.$$

Ex. *f. The beam OA, Fig. 244, resting upon supports at O and A, carries a uniformly distributed load of intensity w together with a weight P concentrated at the point B dividing the beam into the segments OB = a and BA = b.*

Then $$R_1 = \frac{wl}{2} + P\frac{b}{l}, \quad R_2 = \frac{wl}{2} + P\frac{a}{l}.$$

Between O and B

$$S_x = R_1 - wx = P\frac{b}{l} + w\left(\frac{l}{2} - x\right),$$

the equation to a straight line.

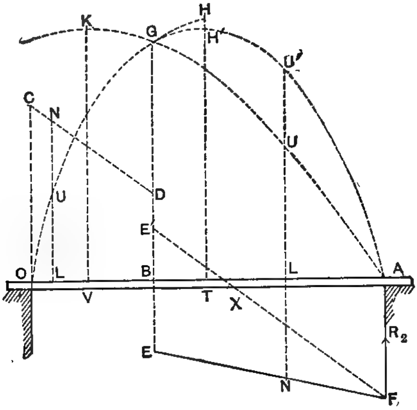


FIG. 244.

The S.F. is $P\frac{b}{l} + \frac{wl}{2}$ when $x=0$, i.e., at O,

and is $P\frac{b}{l} + w\left(\frac{l}{2} - a\right)$ when $x=a$, i.e., at B.

Between B and A

$$S_x = R_1 - wx - P = -w\left(\frac{l}{2} - x\right) - P\frac{a}{l},$$

the equation to a straight line.

The S.F. is $w\left(\frac{l}{2} - a\right) - P\frac{a}{l}$ when $x=a$, i.e., at B,

and is $-\frac{wl}{2} - P\frac{a}{l}$ when $x=l$, i.e., at A.

Take the vertical lines

$$OC = P\frac{b}{l} + \frac{wl}{2},\ BD = P\frac{b}{l} + w\left(\frac{l}{2} - a\right),$$

$$BE = w\left(\frac{l}{2} - a\right) - P\frac{a}{l},\ AF = -\frac{wl}{2} - P\frac{a}{l},$$

and join CD and EF. The broken line $CDEF$ is the S.F. diagram and S.F. at any point L is the vertical distance LN between that point and broken line $CDEF$.

Immediately on the right of B

$$\text{the S.F.} = w\left(\frac{l}{2} - a\right) + P\frac{b}{l} - P = w\left(\frac{l}{2} - a\right) - P\frac{a}{l},$$

and the point E will lie above or below B according as

$$w\left(\frac{l}{2}-a\right) > \text{ or } < P\frac{a}{l}.$$

In the latter case the S.F. changes sign at B, and the B.M. is found to have its greatest value at this point.

If E falls above B, then EF cuts OA at a point X at which the S.F. is nil, and the B.M. is then greatest at X.

Again, *between O and B,*

$$M_x = R_1x - \frac{wx^2}{2} = x\left(P\frac{b}{l}+\frac{wl}{2}\right) - \frac{wx^2}{2},$$

which may be written in the form

$$M_x - \frac{1}{2w}\left(P\frac{b}{l}+\frac{wl}{2}\right)^2 = -\frac{w}{2}\left(P\frac{b}{l}+\frac{wl}{2}-x\right),$$

the equation to a parabola OGH of parameter $\frac{2}{w}$, and having its vertex at a point defined by

$$OT = P\frac{b}{l}+\frac{wl}{2} \quad\text{and}\quad TH = \frac{1}{2w}\left(P\frac{b}{l}+\frac{wl}{2}\right)^2.$$

The parabolic arc OG is the B.M. diagram *between O and B*, and the B.M. at any point L is the vertical distance LU between that point and the arc OG.

Between B and A

$$M_x = R_1x - \frac{wx^2}{2} - P(x-a),$$

which may be written in the form

$$M_x - Pa - \frac{1}{2w}\left(\frac{wl}{2}-P\frac{a}{l}\right)^2 = -\frac{w}{2}\left(\frac{wl}{2}-P\frac{a}{l}-x\right)^2,$$

the equation to a parabola AGK of parameter $\frac{2}{w}$ and having its vertex at a point K defined by

$$OV = \frac{wl}{2}-P\frac{a}{l} \quad\text{and}\quad KV = Pa + \frac{1}{2w}\left(\frac{wl}{2}-P\frac{a}{l}\right)^2.$$

The parabola AGV is the B.M. diagram between B and A, and the B.M. at any point L is the vertical distance LU between that point and the parabola.

If P is at the middle point, $a = b = \frac{l}{2}$,

and the B.M. at the centre $= P\frac{1}{4}+\frac{wl^2}{8}$.

Again, between O and B,

$$LU = M_x = R_1 x - \frac{wx^2}{2} = \text{area } OCNL$$

$$= \text{total S.F. between } O \text{ and } B.$$

Between B and A,

$$LU = M_x = R_1 x - \frac{wx^2}{2} - P(x-a)$$

= *algebraic* sum of the areas $OCDB$ and $BENL$

= total S.F. from O to L.

An examination of Exs. *a* to *f* shows that the S.F. and B.M. ordinates in Exs. *c* and *f* are the algebraic sums of the corresponding ordinates in Exs. *a* and *b* and Exs. *d* and *e*. Hence it follows that the effects of different loads upon beams may be determined separately and the resultant effects are then found by superposing the corresponding results thus obtained.

Ex. 1. *A cantilever OA, Fig.* 245, 12 *ft. long carries a load of* 200 *lbs. at A and a uniformly distributed load of* 600 *lbs. Draw to scale the S.F. and B.M. diagrams and determine the S.F. and B.M. at* 6 *ft. from O. For the S.F. diagram* take a vertical scale of measurement so that 1 in. = 1600 lbs.

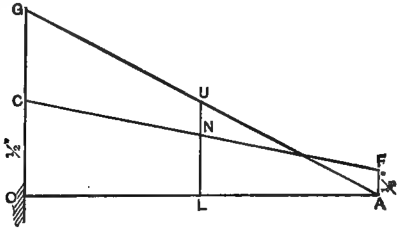


FIG. 245.

The S.F. at $O = 600 + 200 = 800$ lbs. $= \frac{1}{2}$ in.

and at $A = 200$ lbs. $= \frac{1}{8}$ in.

Take the vertical lines $OC = \frac{1}{2}$ in., $AF = \frac{1}{8}$ in., and join CF. At 6 ft. from O

$LN = \frac{5}{16}$ in. by measurement = 500 lbs.

= S.F. at L.

For the B.M. diagram take a vertical scale of measurement so that 1 in. = 6000 ft.-lbs.

The B.M. at $O = 600 \times 6 + 200 \times 12 = 6000$ ft.-lbs. = 1 in.

and at $A = 0$.

Take the vertical line $OG = 1$ in. and join GA. At 6 ft. from O

$LU = \frac{1}{2}$ in. by measurement = 3000 ft.-lbs. = B.M. at L.

Ex. 2. *A beam OA, Fig. 246, resting upon supports at O and A, 30 ft. apart, carries a uniformly distributed load of 6000 lbs., and a single weight of 600 lbs., at a point B dividing the beam into segments OB = 10 ft. and BA = 20 ft.*

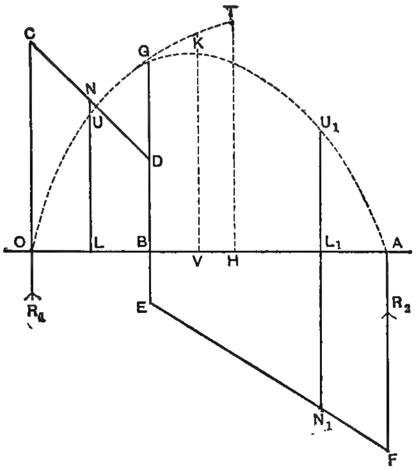


FIG. 246.

Draw to scale the S.F. and B.M. diagrams, and obtain by measurement the S.F. and B.M. at points 5 ft. from O and from A.

Then, $R_1 \times 30 = 600 \times 20 + 6000 \times 15 = 102000$

and $R_2 \times 30 = 600 \times 10 + 6000 \times 15 = 96000.$

Therefore $R_1 = 3400$ lbs. and $R_2 = 3200$ lbs.

For the S.F. take a vertical scale of measurement so that 1 in. = 3000 lbs.

The S.F. at $O = R_1 = 3400$ lbs. $= 1\frac{2}{15}$ in.;

immediately on left of $B = R_1 - 10 \times 200 = 1400$ lbs. $= \frac{7}{15}$ in.;

immediately on right of $B = R_1 - 10 \times 200 - 600 = 800$ lbs. $= \frac{4}{15}$ in.

at $A = R_1 - 30 \times 200 - 600 = -3200$ lbs. $= -1\frac{1}{15}$ in.

Take the vertical lines

$OC = 1\frac{2}{15}$ in., $BD = \frac{7}{15}$ in., $BE = \frac{4}{15}$ in., $AF = 1\frac{2}{15}$ in., and join CD and EF.

By measurement LN at 5 ft. from $O = \frac{4}{5}$ in. = 2400 lbs.
= S.F. at L.

By measurement $L'N'$ at 25 ft. from $O = \frac{11}{15}$ in. = 2200 lbs.
= S.F. at L'

and is of course negative.

Again, *between O and B,*

$$M_x = 3400x - 100x^2$$

or

$$M_x - 28900 = -100(17-x)^2,$$

a parabola with a vertex at a point 17 ft. measured horizontally and 28, ft.-lbs. measured vertically from *O*.

Also, the B.M. at $O = 0$

and at $B = 24000$ ft.-lbs.

Between B and A,

$$M_x = 3400x - 100x^2 - 600(x-10)$$

or

$$M_x - 25600 = -100(14-x)^2,$$

a parabola with its vertex at a point 14 ft. measured horizontally and 13, ft.-lbs. measured vertically from *O*.

Also, the B.M. at $B = 24000$ ft.-lbs.

and at $A = 0$.

Trace the parabolas *OGH* and *AGK*, taking a vertical scale of measu ment so that 1 in. = 24,000 ft.-lbs. Then

$$BG = 1 \text{ in.}, \quad HT = 1\tfrac{49}{240} \text{ in.}, \quad KV = 1\tfrac{1}{15} \text{ in.}$$

By measurement *LU* at 5 ft. from $O = \frac{29}{48}$ in. = 14500 ft.-lbs.
= B.M. at *L*.

By measurement *L′U′* at 25 ft. from $O = \frac{9}{16}$ in. = 13500 ft.-lbs.
= B.M. at *L*.

3. Relation Between Shearing Force and Bending Moment.—

the examples of the preceding article it has been shown that at a

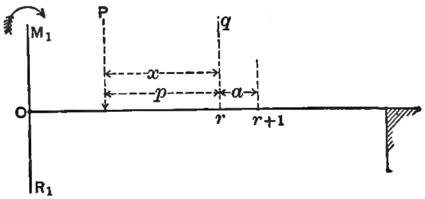


FIG. 247.

point of the beams under consideration the B.M. at any point is t *total S.F.* between that point and a support. The following is general proof of this statement.

Let the beam *OA*, Fig. 247, supported at *O* and *A* be acted up by a number of weights concentrated at different points along the beam;

Let a be the distance between two such consecutive points r and $r+1$, and take $Or=x$.

Let q be the weight at r and let p be the distance between r and the line of action of the resultant P of the weight between O and r.

Let M_1, R_1 be the B.M. and vertical reaction at O.

Then $$R_1(x+a)-P(p+a)-qa+M_1=M_{r+1}$$

and $$R_1x-Pp+M_1=M_r.$$

Therefore $$\begin{aligned}(R_1-P-q)a=Sa=M_{r+1}-M_r&\\ &=\text{increment of B.M. between } r \text{ and } r+1\\ &=\Delta M,\end{aligned}$$

S being the S.F. between r and $r+1$.

Now S is *nil* if $M_{r+1}=M_r$, and every point of the beam between r and $r+1$ is subjected to the same constant B.M., the case being one of simple bending without shear, as in a car-axle.

Again, the weights on the beam may become infinite in number and so, in the limit, form a continuous load. Then

$$dM=Sdx.$$

From the equations

$$\frac{\Delta M}{a}=S \text{ for concentrated weights,}$$

and $$\frac{dM}{dx}=S \text{ for a continuous load,}$$

it is at once evident that the S.F. at any point is measured by the tangent of the slope of the B.M. polygon or curve at that point.

Also, designating *algebraic sums* by the symbol Σ,

the $$\begin{aligned}\text{B.M.}&=\Sigma(\Delta M)=\Sigma(Sa)\\ &=\text{the } \textit{total S.F.} \text{ for the concentrated weights,}\end{aligned}$$

and for a continuous load

$$M=\int Sdx=\textit{the total } \text{S.F.}$$

between O and the point under consideration.

Again, ΔM and dM change sign with S and the B.M. is a *m* (or a *min.* in certain special cases) at the point where the chan occurs.

For a continuous load this point is defined by the condition

$$\frac{dM}{dx}=0.$$

If w is the intensity of the continuous load at x from O,

$$S=R_1-\int w dx,$$

and therefore
$$-w=\frac{dS}{dx}=\frac{d^2M}{dx^2},$$

equations connecting the intensity w, the S.F. and the B.M. at a point in the beam.

4. Live Loads.

Ex. *a. A weight P travels from A to B over a horizontal girder of lengt resting upon supports at A and B.*

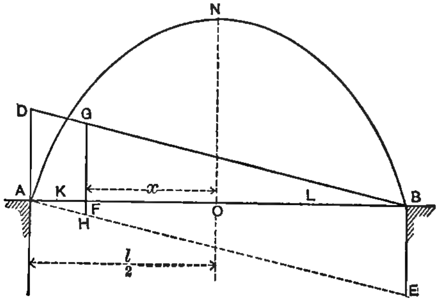


FIG. 248.

When P is at F distant x from O, the middle point of the girder, the action R_1 at $A=\frac{P}{l}\left(\frac{l}{2}+x\right)$.

Between A and F,

$$S_x=R_1,$$

and the *positive* S.F. at any point F is *greatest at the instant after P passes t point.*

Between F and B,

$$S_x=R_1-P=-\frac{P}{l}\left(\frac{l}{2}-x\right),$$

and the *negative* S.F. at any point F is *greatest at the instant before P passes that point.*

Also, the *positive* S.F. is P when $x=\frac{l}{2}$, i.e., at A,

$$\text{and is } 0 \text{ when } x=-\frac{l}{2}, \text{ i.e., at } B;$$

$$\text{the } \textit{negative} \text{ S.F. is } 0 \text{ when } x=\frac{l}{2}, \text{ i.e., at } A,$$

$$\text{and is} -P \text{ when } x=-\frac{l}{2}, \text{ i.e., at } B.$$

Take the vertical lines $AD=BE=P$ and join DB and AE.

Then DB is the *positive* and AE the *negative* S.F. diagram, and the S.F. at any point F is the *positive* vertical distance FG or the *negative vertical* distance FH between DB and AB or between AE and AB respectively.

Again, the B.M. at any point K between A and F distance y from O is

$$M_y=R_1\left(\frac{l}{2}-y\right),$$

which is the greatest when y has its least value, i.e., when $y=x$ and K coincides with F.

So, *between F and B* the B.M. at any point L distant y from F

$$=R_1\left(\frac{l}{2}-x+y\right)-Py$$

$$=R_1\left(\frac{l}{2}-x\right)-y(P-R_1),$$

which is greatest when y has its least value, i.e., when $y=0$ and L coincides with F.

Hence, *the B.M. at any point is greatest at the instant P passes that point,* and the maximum B.M. diagram is therefore given by

$$M_x=R_1\left(\frac{l}{2}-x\right)=\frac{P}{l}\left(\frac{l^2}{4}-x^2\right),$$

the equation to a parabola ANB, passing through the points A and B and having its vertex at N vertically above O the centre of the girder and at the distance $ON=\frac{Pl}{4}$.

In addition to the live load, let a single weight Q be concentrated at the point T distant a from O. The corresponding vertical reaction at A

$$=\frac{Q}{l}\left(\frac{l}{2}+a\right).$$

At x from O *between A and T*

$$\text{the maximum } S_x=\frac{P}{l}\left(\frac{l}{2}+x\right)+\frac{Q}{l}\left(\frac{l}{2}+a\right).$$

At x from O *between T and O*

$$\text{the maximum } S_x=\frac{P}{l}\left(\frac{l}{2}+x\right)+\frac{Q}{l}\left(\frac{l}{2}+a\right)-Q$$

$$=\frac{P}{l}\left(\frac{l}{2}+x\right)-\frac{Q}{l}\left(\frac{l}{2}-a\right).$$

At x from O between O and B

$$\text{the maximum } S_x=\frac{P}{l}\left(\frac{l}{2}-x\right)-\frac{Q}{l}\left(\frac{l}{2}-a\right).$$

The S.F. diagram for the live load is DB, and that for the concentrat load the broken line $dgef$.

The total S.F. at any point is the *algebraic* sum of the vertical distan between that point and the two diagrams. If $AdgT$ and $TefB$ are invert

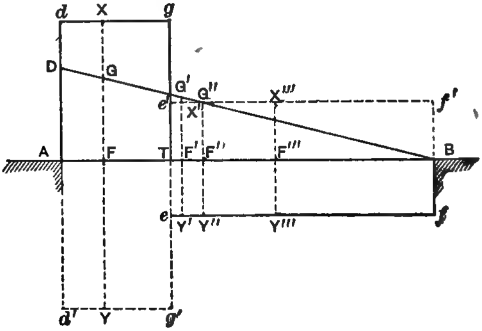


FIG. 249.

so as to take the positions $Ad'g'T$ and $Te'f'B$, the S.F. at $F=GY$; at $F'=G'-X'F'=G'X'$; at $F''=0$; at $F'''=G'''F'''-X'''F'''=-G'''X'''$.

Again, *between A and T*

$$\text{the maximum B.M. at } x \text{ from } O=\frac{P}{l}\left(\frac{l^2}{4}-x^2\right)+\frac{Q}{l}\left(\frac{l}{2}+a\right)\left(\frac{l}{2}-x\right);$$

between T and O

$$\text{the maximum B.M. at } x \text{ from } O=\frac{P}{l}\left(\frac{l^2}{4}-x^2\right)+\frac{Q}{l}\left(\frac{l}{2}+a\right)\left(\frac{l}{2}-x\right)-Q(a-x)$$

$$=\frac{P}{l}\left(\frac{l^2}{4}-x^2\right)+\frac{Q}{l}\left(\frac{l}{2}-a\right)\left(\frac{l}{2}+x\right);$$

between O and B

$$\text{the maximum B.M. at } x \text{ from } O=\frac{P}{l}\left(\frac{l^2}{4}-x^2\right)+\frac{Q}{l}\left(\frac{l}{2}-a\right)\left(\frac{l}{2}-x\right).$$

In Fig. 250, ANB is the maximum B.M. diagram for the live load and the B.M. diagram for the concentrated load consists of the two lines AY', BY'.

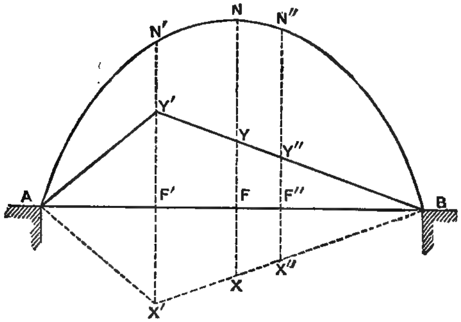


FIG. 250.

The total maximum B.M. at any point is the algebraic sum of the vertical distances between that point and the two diagrams. If $AY'B$ is inverted so as to take the position $AX'B$,

the total maximum B.M. at $F' = F'N' + F'Y' = X'N'$;
" " " " " $F = FN + FY = XN$;
" " " " " $F'' = F''N'' + F''Y'' = X''N''$.

Ex. *b. Let a continuous load weighing w per unit of length travel over the girder from O towards A.*

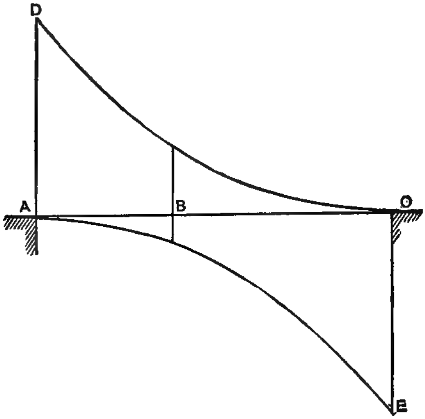


FIG. 251.

When the load covers a length $OB = x$, the reaction at $A = \frac{wx^2}{2l}$.

This is evidently the S.F. at all points *in front of the advancing load.*

Between B and O, at any point distant y from B,

$$\text{the S.F.} = \frac{wx^2}{2l} - wy,$$

which is greatest when $y=0$.

Hence, the maximum S.F. at any point due to the live load is in f
of the advancing load, when the load covers the longer segment OB, an
given by

$$S_x = \frac{wx^2}{2l},$$

the equation to a parabola OD with its vertex at O and intersecting
vertical at A in the point D where $AD = \frac{wl}{2}$.

As the load travels off the girder, the S.F. *behind* the load is found in
cisely the same manner, the S.F. diagram being the parabola AE with
vertex at A and intersecting the vertical OE in the point E where $OE = \frac{wl}{2}$

Let the girder also carry a uniformly distributed dead load of intensity
Considering one half of the girder, DH is the S.F. for the live load, and

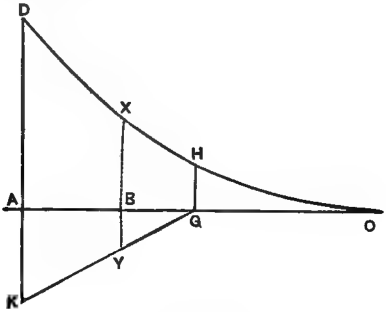


FIG. 252.

the S.F. for the dead load, may be traced *below* OA. The total maxim
S.F. at any point B is then XY, the vertical distance at B between DH
KG.

Also, $$DK = \frac{wl}{2} + \frac{w'l}{2}$$

and $$GH = \frac{wl}{8}.$$

It will be subsequently shown that the design of the web of a girder p
tically depends upon the S.F.

Again, the B.M. at B when the live load covers OB is

$$\frac{wx^2}{2l}(l-x).$$

Between A and B, at y from A,

$$\text{the B.M.} = \frac{wx^2}{2l}y,$$

which increases until $y = l - x$.

If the load advances a distance z beyond B,

$$\text{the reaction at } A = \frac{w}{2l}(z+x)^2,$$

$$\text{and the B.M. at } B = \frac{w}{2l}(z+x)^2(l-x) - \frac{wz^2}{2},$$

which is a maximum when

$$0 = \frac{w}{l}(z+x)(l-x) - wz,$$

or

$$z = l - x,$$

i.e., when the live load covers the whole girder.

It will be subsequently shown that the design of the flanges of a girder depends essentially upon the B.M., and the maximum flange stresses are therefore to be found when the live load is uniformly distributed over the total length OA.

In the results deduced above it is assumed that the length of the live load is not less than that of the girder.

Ex. 3. *The two main girders of a single-track bridge of* 80 *ft. span carry a dead load of* 2500 *lbs. per lineal foot. Determine the maximum S.F.* and *B.M. at* 10 *ft. from a support when a live load of* 3000 *lbs. per lineal foot travels over the bridge.*

The dead load on each girder

$$= 80 \times \frac{2500}{2} = 100000 \text{ lbs.}$$

The reaction at each support due to the dead load

$$= 50000 \text{ lbs.}$$

The S.F. due to the dead load at 10 ft. from a support

$$= 50000 - 10 \times \frac{2500}{2} = 37500 \text{ lbs.}$$

The S.F. due to the live load at 10 ft. from the support is greatest when the live load covers the 70 ft. segment.

The corresponding reaction at the unloaded end

$$= \frac{70}{80} \times \frac{3000}{2} \times 35 = 45937\tfrac{1}{2} \text{ lbs.}$$

Therefore the total maximum S.F. at 10 ft. from a support

$$= 37500 + 45937\tfrac{1}{2} = 83437\tfrac{1}{2} \text{ lbs.}$$

The B.M. at any point is a maximum when the live load covers the whole girder.

The total load on the girder is then

$$=100000+80\times\frac{3000}{2}=220000,$$

and the maximum B.M. at 10 ft. from a support

$$=110000\times10-\frac{2500+3000}{2}\cdot10.5$$

$$=962500\text{ ft.-lbs.}$$

5. Moments of Forces with Respect to a Given Point Q.—*First*, consider a single force P_1.

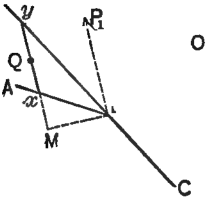


FIG. 253.

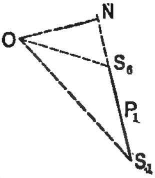


FIG. 254.

Describe the force and funicular polygons, i.e., the line S_1S_6 and the lines AB, BC.

Through the point Q draw a line parallel to S_1S_6, cutting the lines AB and CB produced in x and y.

Drop the perpendiculars BM and ON upon yx and S_1S_6 produced.

Then $$\frac{xy}{BM}=\frac{S_1S_6}{ON}=\frac{P_1}{ON}$$

and $$P_1\cdot BM=xy\cdot ON.$$

But BM is equal to the length of the perpendicular from Q to the line of action of P_1, and the product $xy\cdot ON$ is, therefore, equal to the moment of P_1 with respect to Q. Hence, if a scale is so chosen that ON = unity, this moment becomes equal to xy; i.e., *it is the intercept cut off by the two sides of the funicular polygon on a line drawn through the given point parallel to the given force.*

Next, let there be two forces, P_1, P_2.

Describe the force and funicular polygons $S_2S_1S_6$ and $ABCD$.

Let the first and last sides (AB and DC) be produced to meet in G, and let a line through the given point Q parallel to the line S_2S_6 intersect these lines in x and y.

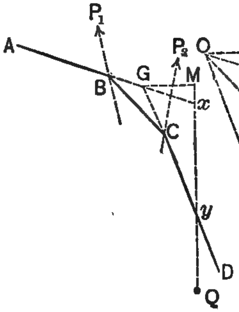


FIG. 255.

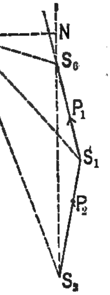


FIG. 256.

Draw GM perpendicular to xy and ON perpendicular to S_2S_6.

Then
$$\frac{xy}{GM}=\frac{S_2S_6}{ON}=\frac{\text{resultant of } P_1 \text{ and } P_2}{ON},$$

and hence (the resultant of P_1 and P_2) $\times GM = xy \cdot ON$.

But GM is equal to the length of the perpendicular from Q upon the resultant of P_1 and P_2, which is parallel to S_2S_6 and must necessarily pass through G. Hence, if a scale is so chosen that $ON =$ unity, xy is equal to the moment of the forces with respect to Q; i.e., *it is the intercept cut off by the first and last sides of the funicular polygon on a line drawn through the given point parallel to the resultant force.*

A *third* force P_3 may be compounded with P_1 and P_2, and the proof may be extended to three, four, or any number of forces.

The result is precisely the same if the forces are parallel.

The force polygon of the n parallel forces $P_1, P_2, \ldots P_n$ becomes the straight line $S_6S_1S_2 \ldots S_n$. Let the first and last sides of the funicular polygon meet in G. Drop the perpendiculars GM, ON upon xy and S_6S_n, xy as before, being the intercept cut off on a line through the given point Q parallel to S_6S_n. Then

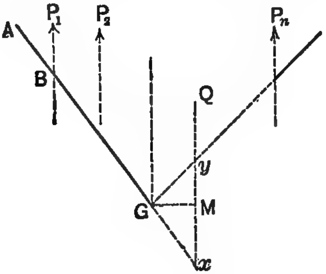


FIG. 257.

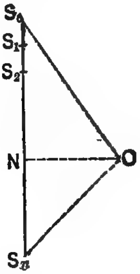


FIG. 258.

$$xy \cdot ON = GM \cdot S_6S_n. \quad \text{Hence, etc.}$$

Thus the moment of any number of forces in one and the same plane with respect to a given point may be represented by the intercept cut off by the first and last sides of the funicular polygon on a line drawn through the given point parallel to the resultant of the given forces.

6. Bending Moments. Stationary Loads. — Let a horizon beam AB supported at A and B carry a number of weights P_2, P_3, . . . at the points N_1, N_2, N_3, . . .

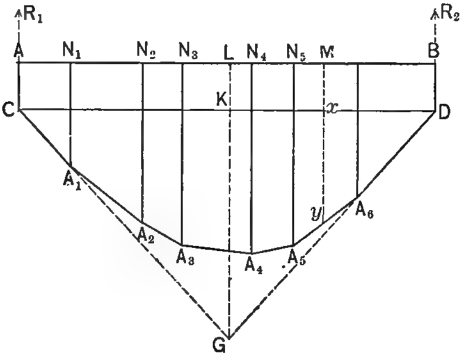


FIG. 259.

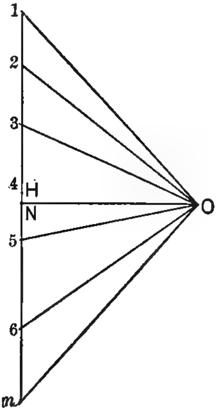


FIG. 260.

The force polygon is a vertical line $1234 \ldots n$, where $12 =$ $23 = P_2$, etc.

Take any pole O and describe the funicular polygon $A_1A_2A_3$.

Let the *first* and *last* sides of this polygon be produced to me in G and to cut the verticals through A and B in the points C and

Join CD.

Let the vertical through G cut AB in L and CD in K; LG the line of action of the resultant.

Draw OH parallel to CD.

From the similar triangles $O1H$ and GCK,

$$\frac{1H}{OH} = \frac{GK}{CK}.$$

From the similar triangles OnH and GDK,

$$\frac{nH}{OH} = \frac{GK}{DK}.$$

Therefore $$\frac{1H}{nH} = \frac{DK}{CK} = \frac{BL}{AL} = \frac{R_1}{R_2},$$

R_1, R_2 being the reactions at A and B respectively.

But $$1H + nH = 1n = P_1 + P_2 + \ldots = R_1 + R_2.$$

Hence $$1H = R_1 \quad \text{and} \quad nH = R_2.$$

Thus *the line drawn through the pole parallel to the closing line CD divides the line of loads into two segments, of which the one is equal to the reaction at A and the other to that at B.*

Let it now be required to find the *bending moment* at any point M of the beam, i.e., the moment of all the forces on one side of M with respect to M.

In the figure these forces are R_1, P_1, P_2, P_3, P_4, P_5, and the corresponding *force polygon* is $H123456$. The first and last sides of the *funicular* polygon of the forces are CD parallel to OH, and A_5A_6 parallel to $O6$. If the vertical through M meet these sides in x and y, then, as shown in Art. 5, the moment of the forces R_1, P_1, P_2, P_3, P_4, P_5 with respect to M, i.e., the bending moment at M, $= ON \cdot xy$, ON being the perpendicular from O upon $1H$ produced.

Hence, if a scale is chosen so that the polar distance ON is *unity, the bending moment at any point of the beam is the intercept on the vertical through that point cut off by the closing line CD and the opposite bounding line of the funicular polygon.*

7. **Moving Loads.**—Beams are often subjected to the action of moving loads, as, e.g., in the case of the main girders of a railway bridge, and it becomes a matter of importance to determine the bending moments for different positions of the loads. It may be assumed that the loads are concentrated on wheels which travel across the bridge at invariable distances apart.

At any given moment, let the figure represent a beam 11 under the loads P_1, P_2, $P_3 \ldots$ Describe the corresponding funicular polygon $CC'C'' \ldots D$, the closing line being CD.

Let the loads now travel from *right to left.* The result will be precisely the same if the loads remain *stationary* and if the supports 11 are made to travel from *left to right.*

Thus, if the loads successively move through the distances 12, 23, 34, . . . to the *left,* the result will be the same if the loads are kept stationary and if the supports are successively moved to the *right* into the positions 22, 33, 44, . . . The new funicular polygons are evidently $C'C'' \dots D'$, $C''C''' \dots D''$, $C'''C'''' \dots D'''$, the new closing lines being $C'D'$, $C''D''$, $C'''D'''$, . . .

The *bending moment* at any point M is measured by xy fo first distribution, $x'y'$ for the second, $x''y''$ for the third, etc.,

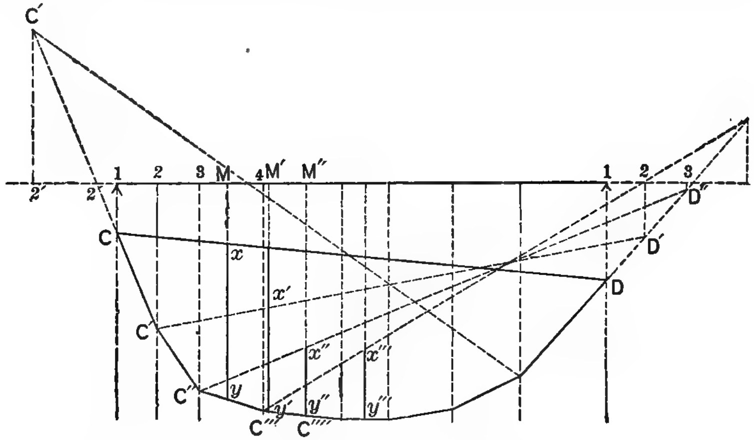


FIG. 261.

position of M for the successive distributions being defined $MM' = 12$, $M'M'' = 23$, $M''M''' = 34$, . . .

Similarly, if the loads move from *left to right*, the result wil the same if the loads are kept stationary and if the supports made to move from *right to left*.

It is evident that the envelope for the closing line CD fo distributions of the loads is a certain curve, called the *envelop moments*. The intercept on the vertical through any point of beam cut off by this curve and the opposite boundary of the fun lar polygon is the greatest possible bending moment at that p to which the girder can be subjected.

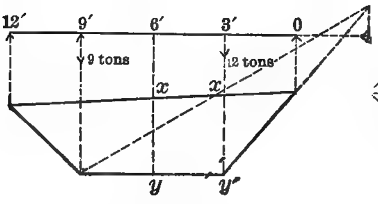


FIG. 262*a*.

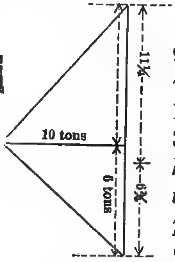


FIG. 262*b*.

Ex. 4. *Loads of* 12 9 *tons are concentr upon a horizontal bean* 12 *ft. span at distance* 3 *and* 9 *ft. from the ri hand support. Find the B.M. at the mi point of the beam, and* (*b*) *the maximum B produced at the same p when the loads travel over the beam at the fixed distances of* 6 *ft. apart.*

Scales for lengths, $\frac{1}{8}$ in. = 1 ft.; for forces, $\frac{1}{16}$ in. = 1 ton.
Take polar distance = $\frac{5}{8}$ in. = 10 tons.
Case a. B.M. = $xy \times 10 = 3.15 \times 10$ tons = $31\frac{1}{2}$ ton-ft.
Case b. B.M. = $x'y' \times 10 = 3.6 \times 10$ tons = 36 ton-ft.

8. Analytical Method of Determining the Maximum Shear and Bending Moment at any Point of an Arbitrarily Loaded Girder AB.— At any given moment let the load consist of a number of weights $w_1, w_2, \ldots w_n$, concentrated at points distant $a_1, a_2, \ldots a_n$, respectively, from B.

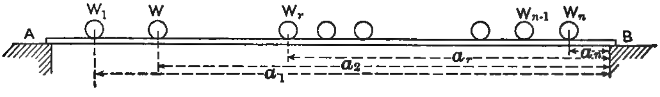


FIG. 263.

The corresponding reaction R_1 at A is given by

$$R_1 l = w_1 a_1 + w_2 a_2 + \ldots + w_n a_n,$$

l being the length of the girder.

Let $W_n = w_1 + w_2 + \ldots + w_n$, the sum of the n weights;

Let $W_r = w_1 + w_2 + \ldots + w_r$, the sum of the first r weights.

The *shear* at a point P between the rth and the $(r+1)$th weights is

$$S_1 = R_1 - w_1 - w_2 - \ldots w_r = R_1 - W_r.$$

Let all the weights now move towards A through a distance x, and let p of the weights move off the girder, q of the weights be transferred from one side of P to the other, and s new weights, viz., $w_{n+1}, w_{n+2}, \ldots w_{n+s}$, advance upon the girder, their distances from B being $a_{n+1}, a_{n+2}, \ldots a_{n+s}$, respectively;

Let $L = w_1 + w_2 + \ldots + w_p$, the total weight leaving the girder;

Let $T = w_{r+1} + w_{r+2} + \ldots + w_{r+q}$, the total weight transferred from one side of P to the other;

Let $R_p l = w_1 a_1 + w_2 a_2 + \ldots + w_p a_p$;

Let $R_q l = w_{r+1} a_{r+1} + w_{r+2} a_{r+2} + \ldots + w_{r+q} a_{r+q}$;

Let $R_s l = w_{n+1} a_{n+1} + w_{n+2} a_{n+2} + \ldots + w_{n+s} a_{n+s}$.

Thus R_p, R_q, R_s are the reactions at A due respectively to the weight which leaves the girder, the weight which is transferred, and the new weight which advances upon the girder.

The reaction R_2 at A with the new distribution of the loads given by

$$\begin{aligned}R_2l=w_{p+1}(a_{p+1}+x)+w_{p+2}(a_{p+2}+x)+\ \ldots\ +w_r(a\ +x)\\ +w_{r+1}(a_{r+1}+x)+\ \ldots\ +w_n(a_n+x)+w_{n+1}a_{n+1}+\ \ldots\\ +w_{n+s}a_{n+s}=R_1l-R_pl+x(W_n-L)+R_sl,\end{aligned}$$

and hence

$$(R_2-R_1)l=(R_s-R_p)l+x(W_n-L).$$

Also, the corresponding *shear* at P is

$$\begin{aligned}S_2&=R_2-(w_{p+1}+w_{p+2}+\ \ldots\ +w_r+w_{r+1}+\ \ldots\ +w_{r+q})\\ &=R_2-(W_r-L+T).\end{aligned}$$

Hence the *shear* at P with the first distribution of weights greater or less than the *shear* at the same point with the seco distribution according as

$$S_1 \gtrless S_2,$$

or
$$R_1-W_r \gtrless R_2-W_r+L-T,$$

or
$$T-L \gtrless R_2-R_1,$$

or
$$T-L \gtrless R_s-R_p+\frac{x}{l}(W_n-L). \quad . \quad . \quad . \quad . \quad ($$

When no weights leave or advance upon the girder, R_s, R_p, an are severally nil, and hence

$$S_1 \gtrless S_2,$$

according as
$$\frac{T}{x} \gtrless \frac{W_n}{l};$$

i.e., according *as the weight transferred divided by the distance thro which it is transferred is greater or less than the total weight on the gir divided by the span.*

Again, let z be the distance of P from B, and let

$$R_rl=w_1a_1+w_2a_2+\ \ldots\ +w_ra_r.$$

The *bending moment* at P with the *first* distribution of weights

$$\begin{aligned}M_1&=R_1(l-z)-w_1(a_1-z)-w_2(a_2-z)-\ \ldots\ -w_r(a_r-z)\\ &=R_1(l-z)-R_rl+zW_r.\end{aligned}$$

The *bending moment* at the same point with the second distribution is

$$\begin{aligned} M_2 &= R_2(l-z) - w_{p+1}(a_{p+1}+x-z) - w_{p+2}(a_{p+2}+x-z) - \dots \\ &\qquad - w_r(a_r+x-z) - \dots - w_{r+q}(a_{r+q}+x-z) \\ &= R_2(l-z) - (R_r l - R_p l + R_q l) - (x-z)(W_r - L + T). \end{aligned}$$

Hence the bending moment at P with the *first* distribution of weights is greater or less than the bending moment at the same point with the second distribution according as

$$M_1 \gtrless M_2,$$

or

$$R_1(l-z) - R_r l + zW_r \gtrless R_2(l-z) - (R_r - R_p + R_q)l - (x-z)(W_r - L + T),$$

or $$zW_r - (R_p - R_q)l + (x-z)(W_r - L + T) \gtrless (R_2 - R_1)(l-z)$$

or

$$z(L - T + R_s - R_p) + l(R_q - R_s) + x(W_r - L + T) \gtrless \frac{x}{l}(l-z)(W_n - L). \quad \text{(B)}$$

Note.—If no weights leave or advance upon the girder R_s, R_p, and L are severally nil, and

$$M_1 \gtrless M_2,$$

according as

$$-zT + lR_q + x(W_r + T) \gtrless \frac{x}{l}(l-z)W_n.$$

If also the point P coincide with the rth weight, and the distance of transfer, $x_1 = a_r - a_{r+1}$, then

$$R_q l = w_{r+1}a_{r+1}, \quad T = w_{r+1}, \quad \text{and} \quad z = a_r.$$

Hence $M_1 \gtrless M_2$, according as

$$-w_{r+1}a_r + w_{r+1}a_{\ +1} + (a_r - a_{r+1})(W_r + w_{r+1}) \gtreqless \frac{a_r - a_{r+1}}{l}(l - a_r) \ W_n,$$

or $$\frac{\mathbf{W_r}}{\mathbf{1 - a_r}} \gtrless \frac{\mathbf{W_n}}{\mathbf{1}};$$

i.e., according as *the sum of the first r weights divided by the length* *of the corresponding segment is greater or less than the total weight upon the girder divided by the span.*

If the weights are concentrated at the panel-points of a truss, the last relation may be expressed in the form

$$\frac{\text{first }(r)\text{ weights}}{r\text{ panels}} \gtrless \frac{\text{total weight}}{\text{total number of panels}}.$$

9. Graphical Determination of the Maximum B.M. at any Point of an Arbitrarily Loaded Girder AB.—When all the weights remain on the girder the last result in the preceding article indicates a simple graphical method for the determination of the max. B.M. at any point.

Erect a vertical line AC at one of the supports A, and scale off

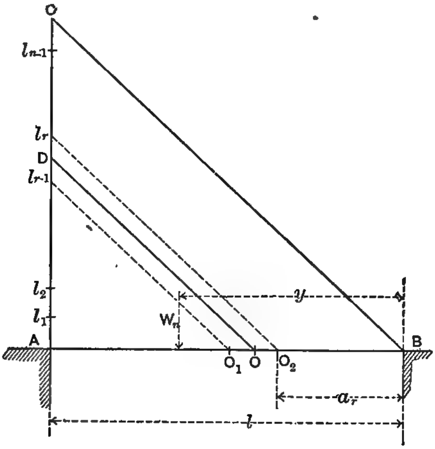


FIG. 264.

lengths Al_1, $l_1l_2, \dots l_{r-1}l_r, \dots, l_{n-1}C$ to represent the several weights $w_1, w_2, \dots, w_r, \dots w_n$, taken in order.

Join BC, and from any point D between l_r and l_{r-1} drawn DO parallel to CB intersecting AB in O.

The B.M. at O is a max. when the weight w_r is concentrated at that point.

If the line parallel to CB passes through the division point between two adjacent load lengths the B.M. at the point of intersection with AB will be a max. when *either* of the two loads represented by the adjacent load lengths is concentrated at this point. For example, the B.M. at O_1 is a max. when either w_{r-1} or w_r is concentrated at O_1, while the B.M. at O_2 is a max. when either w_r or w_{r+1} is concentrated at O_2.

Again, the S.F. between the rth and $\overline{r+1}$th loads $=R_1-W_r$, and this is zero if the B.M. is a maximum.

Therefore $$R_1=W_r.$$

Let y be the distance of the C. of G. of the n weights from B.

Then $$W_n\cdot y=R_1\cdot l=W_r\cdot l.$$

Therefore $$\frac{y}{l}=\frac{W_r}{W_n}=\frac{Al_n}{AC}=\frac{AO_2}{AB}=\frac{l-a_r}{l}$$

and $$y=l-a_r.$$

Hence the rth load and the C. of G. of all the loads are equally distant from the ends (or the centre) of the span.

Ex. 5. *A series of loads of* 3000, 23,600, 20,100, 21,700, 22,900, 18,550, 18,000, 18,000, *and* 18,000 *lbs. travel, in order, over a truss of* 240 *ft. span and ten panels.*

Let $Ap_1p_2\ldots B$ be the truss, p_1, p_2, p_3, . . . being the panel-points. Let the loads travel from B towards A, and compare the shear in the panel p_4p_5 when the weight of 3000 lbs. has reached p_5 with the shear in the same panel when the weights have advanced another 24 ft.

$$R_s=\frac{1}{10}\times 18550=1855\text{ lbs.},\quad R_p=0,\quad \frac{x}{l}=\frac{1}{10};$$

$$W_n=91300\text{ lbs.},\quad L=0,\quad T=3000\text{ lbs.}$$

Hence $S_1 \gtrless S_2$, according as (see A)

$$3000-0 \gtrless 1855+\frac{1}{10}(91300-0) \gtrless 10985;$$

and, therefore, $$S_1<S_2.$$

Let the weights again advance 24 ft.

$$R_s=\frac{1}{10}\times 18000=1800\text{ lbs.},\quad R_p=0,\quad \frac{x}{l}=\frac{1}{10};$$

$$W_n=109300\text{ lbs.},\quad L=0,\quad T=23600\text{ lbs}$$

Hence $S_1 \gtrless S_2$, according as (see A, Art. 8)

$$23600-0 \gtrless 1800-0+\frac{1}{10}(109300-0), \quad \text{or} \quad 23600 \gtrless 12730,$$

and therefore $$S_1 > S_2.$$

Hence the shear in the panel p_4p_5 is a maximum when the weight of 30 lbs. is at p_4.

Again, let the 3000 lbs. be at p_2, and compare the bending moment at with the bending moment at the same point when the weights have advance first 24 ft. and then 48 ft. towards A.

First. $z=120$ ft., $L=0$, $T=22900$ lbs., $R_sl=18000\times24$, $R_p=0$, R_ql 22900×96, $x=24$ ft., $W_r=68400$ lbs., $W_n=145850$ lbs.

Hence $M_1 \gtrless M_2$, according as (see B, Art. 8)

$$120(0-22900+1800-0)+22900\times96-18000\times24$$
$$+24(68400-0+22900) \gtrless \frac{24}{240}(240-120)(145850-0$$

or $$1425600 \gtrless 1750200,$$

and therefore $$M_1 \gtrless M_2.$$

Second. $z=120$ ft., $L=3000$ lbs., $T=18550$, $R_s=0$, $R_pl=3000\times21$ $R_ql=18550\times96$, $x=24$ ft., $W_r=91300$ lbs., $W_n=163850$ lbs.

Hence $M_1 \gtrless M_2$, according as (see B)

$$120\left(3000-18550+0-3000\cdot\frac{216}{240}\right)+240\left(18550\cdot\frac{96}{240}-0\right)$$
$$+24(91300-3000+18550) \gtrless \frac{24}{240}(240-120)(163850-3000),$$

or $$2155200 \gtrless 1930200,$$

and therefore $$M_1 > M_2.$$

Hence the bending moment at p_5 is a maximum when the weight of 30 lbs. is at p_1, i.e., when all the panel-points are loaded.

Ex. 6. *Three equal wheels, each loaded with one ton, roll at the same co stant rate over a girder AB of* 12 *ft. span. The* 1*st and* 2*d wheels are* 4 *ft. ar the* 2*d and* 3*d wheels* 2 *ft. apart. Draw the maximum S.F.* (*Fig.* 265) *and B.* (*Fig.* 266) *diagrams.*

Let the wheels roll from A to B;

Let S_a, S_b, S_c be the S.F. behind the 1st, 2d, and 3d wheels, respectively

Let M_a, M_b, M_c be the B.M. at the 1st, 2d, and 3d wheels;

Let R be the reaction at A.

For the 1*st wheel:*

Let x be its distance from A.

From $x=1'$ to $x=4'$ the 1st wheel only is on the girder.

Therefore $$R=\frac{12-x}{12}=1-\frac{x}{12}$$

and $$S_a=R=1-\frac{x}{12},$$

which is 1 when $x=0$ and is $\frac{2}{3}$ when $x=4'$.

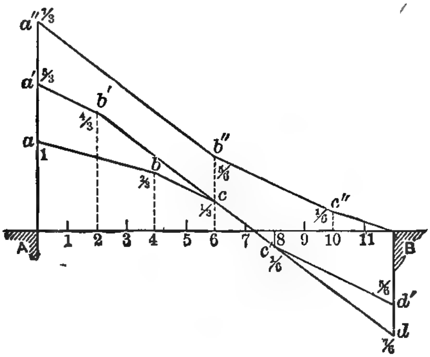


Fig. 265.

In this case ab is the S.F. diagram from A to 4.

Also, $$M_x=Rx=x-\frac{x^2}{12},$$

a parabola with its vertex at D, 6′ horizontally and 3 ft.-tons vertically from A.

The B.M. is 0 when $x=0$ and is $\frac{8}{3}$ ft.-tons when $x=4'$.

AC is the corresponding B.M. diagram.

From $x=4'$ to $x=6'$ the 1st and 2d wheels are on the girder. Then

$$R=\frac{12-x}{12}+\frac{16-x}{12}=\frac{7}{3}-\frac{x}{6},$$

and $$S_a=R-1=\frac{4}{3}-\frac{x}{6},$$

which is $\frac{2}{3}$ when $x=4'$ and is $\frac{1}{3}$ when $x=6'$.

In this case bc is the S.F. diagram from 4 to 6.

Also, $$M_x=Rx-1\times 4=\frac{7}{3}x-\frac{x^2}{6}-4,$$

a parabola with its vertex at F, 7′ horizontally and $4\frac{1}{6}$ ft.-tons vertica from A.

The B.M. is $\frac{8}{3}$ when $x=4'$ and is 4 when $x=6'$.

CE is the corresponding B.M. diagram.

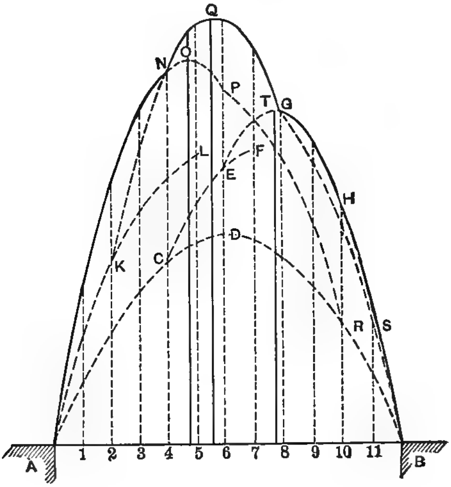


FIG. 266.

From $x=6'$ to $x=8'$ the three wheels are on the girder. Then

$$R=\frac{12-x}{12}+\frac{16-x}{12}+\frac{18-x}{12}=\frac{23}{6}-\frac{x}{4}$$

and
$$S_a=R-2=\frac{11}{6}-\frac{x}{4},$$

which is $\frac{1}{3}$ when $x=6'$ and is $-\frac{7}{6}$ when $x=12'$.

In this case cd is the S.F. diagram from 6 to 12.

Also,
$$M_x=Rx-1\times4-1\times6=\frac{23}{6}x-\frac{x^2}{4}-10,$$

a parabola with its vertex at T, $7\frac{2}{3}'$ horizontally and $4\frac{25}{36}$ ft.-tons vertica from A.

The B.M. is 4 when $x=6'$ and is O when $x=12'$.

The B.M. is $4\frac{2}{3}$ when $x=8'$.

EGHB is the corresponding B.M. diagram.

For the 2d *wheel:*

Let x be now the distance of this wheel from A.

From $x=0$ to $x=2'$ the first two wheels only are on the girder. Tl

$$R=\frac{12-x}{12}+\frac{8-x}{12}=\frac{5}{3}-\frac{x}{6}$$

and $$S_b = R = \frac{5}{3} - \frac{x}{6},$$

which is $\frac{5}{3}$ when $x=0$ and is $\frac{4}{3}$ when $x=2'$.

In this case the S.F. diagram is $a'b'$ from A to 2.

Also, $$M_x = R_x = \frac{5}{3}x - \frac{x^2}{6},$$

a parabola with its vertex at L, 5′ horizontally and $4\frac{1}{6}$ ft.-tons vertically from A.

The B.M. is 0 when $x=0$ and is $\frac{8}{3}$ when $x=2'$.

AK is the corresponding B.M. diagram.

From $x=2'$ to $x=8'$ the three wheels are on the girder. Then

$$R = \frac{14-x}{12} + \frac{12-x}{12} + \frac{8-x}{12} = \frac{17}{6} - \frac{x}{4}$$

and $$S_b = R - 1 = \frac{11}{6} - \frac{x}{4},$$

which is $\frac{4}{3}$ when $x=2'$ and is $-\frac{4}{6}$ when $x=8'$.

In this case the S.F. diagram is $b'c'$ from 2 to 8.

Also, $$M_x = Rx - 1.2 = \frac{17}{6}x - \frac{x^2}{4} - 2,$$

a parabola with its vertex at Q, $5\frac{2}{3}'$ horizontally and $6\frac{1}{36}$ ft.-tons vertically from A.

The B.M. is $\frac{8}{3}$ when when $x=2'$, is $\frac{14}{3}$ $x=8'$, and is $\frac{16}{3}$ when $x=4'$.

$KNQG$ is the corresponding B.M. diagram.

From $x=8'$ to $x=12'$ the 2d and 3d wheels only are on the girder. Then

$$R = \frac{12-x}{12} + \frac{14-x}{12} = \frac{13}{6} - \frac{x}{6}$$

and $$S_b = R - 1 = \frac{7}{6} - \frac{x}{6},$$

which is $-\frac{1}{6}$ when $x=8'$ and is $-\frac{5}{6}$ when $x=12'$.

In this case the S.F. diagram is $c'd'$ from 8′ to 12′.

Also, $$M_x = Rx - 1 \times 2 = \frac{13}{6}x - \frac{x^2}{6} - 2,$$

a parabola with its vertex at $6\frac{1}{2}'$ horizontally and $5\frac{1}{24}$ ft.-tons vertically from A.

The B.M. is $\frac{14}{3}$ when $x=8'$, is 0 when $x=12'$, and is 3 when $x=10'$.

GSB is the corresponding B.M. diagram.

For the 3d wheel:

Let x be now the distance of the 3d wheel from A.

From $x=0$ to $x=6$ the three wheels are on the girder. Then

$$R=\frac{12-x}{12}+\frac{10-x}{12}+\frac{6-x}{12}=\frac{7}{3}-\frac{x}{4}$$

and $$S_c=R=\frac{7}{3}-\frac{x}{4},$$

which is $\frac{7}{3}$ when $x=0$ and is $\frac{5}{6}$ when $x=6'$.

In this case $a''b''$ is the S.F. diagram from A to 6.

Also, $$M_x=Rx=\frac{7}{3}x-\frac{x^2}{4},$$

a parabola with its vertex at O, $4\frac{2}{3}'$ horizontally and $5\frac{4}{9}$ ft.-tons vertic from A.

The B.M. is 0 when $x=0$, is 5 when $x=6'$, and is $\frac{16}{3}$ when x

AN is the corresponding B.M. diagram.

From $x=6'$ to $x=10'$ the 2d and 3d wheels only are on the girder. T

$$R=\frac{12-x}{12}+\frac{10-x}{12}=\frac{11}{6}-\frac{x}{6}$$

and $$S_c=R=\frac{11}{6}-\frac{x}{6},$$

which is $\frac{5}{6}$ when $x=6'$ and is $\frac{1}{6}$ when $x=10'$.

In this case the S.F. diagram is $b''c''$ from $6'$ to $10'$.

Also, $$M_x=Rx=\frac{11}{6}x-\frac{x^2}{6},$$

a parabola with its vertex at P, $5\frac{1}{2}'$ horizontally and $5\frac{3}{24}$ ft.-tons vertic from A.

The B.M. is 5 when $x=6'$ and is $\frac{5}{3}$ when $x=10'$.

PR is the corresponding B.M. diagram.

From $x=10'$ to $x=12'$ the 3d wheel only is on the girder. Then

$$R=\frac{12-x}{12}=1-\frac{x}{12}$$

and $$S_c=R=1-\frac{x}{12},$$

which is $\frac{1}{6}$ when $x=10'$ and is 0 when $x=12'$.

In this case $c''b$ is the S.F. diagram from 10 to 12.

Also, $$M_x = R_x = x - \frac{x^2}{12},$$

a parabola with is vertex at D, 6′ horizontally and 3 ft.-tons vertically from A.

The B.M. is $\frac{5}{3}$ when $x = 10'$ and is 0 when $x = 12'$.

DRB is the corresponding B.M. diagram.

The maximum B.M. diagram is therefore made up of the parabolic arcs *AN*, *NQG*, *GHB*, and the absolute maximum B.M. is $6\frac{1}{36}$ ft.-tons at $5\frac{2}{3}$ ft. from A.

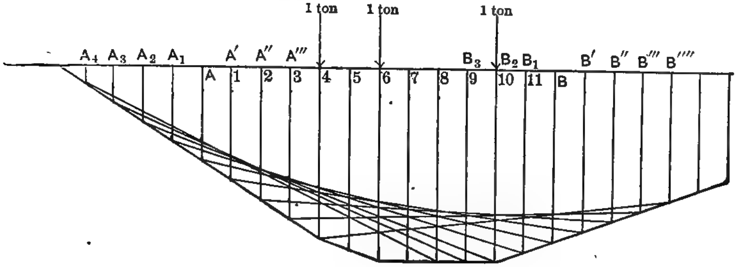


FIG. 267.

The B.M. may also be obtained in the manner explained in Art. 7. Take as the scale for lengths $\frac{5}{32}$ in. $=1$ ft., and as the scale for forces $\frac{5}{16}$ in. $=1$ ton. Also take the polar distance $=\frac{15}{16}$ in. $=3$ tons.

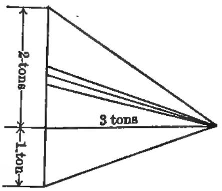


FIG. 268.

Draw the funicular polygon when the wheels are concentrated at 4, 6, and 10. Move the supports right and left, 1 ft. at a time, closing the funicular polygon after each operation. The "envelope of moments," Fig. 267, is thus obtained and the maximum B.M. at any point is the intercept on the vertical through that point cut off by this envelope and the opposite boundary of the funicular polygon. This intercept can be easily scaled and the B.M. in foot-tons is 3×the intercept in feet.

10. Hinged Girders.—Any point of a girder at which the B.M. is *nil* is called a point of *contrary flexure*, and on passing such a point the B.M. necessarily changes sign. A hinge (or pin) may therefore be introduced at this point and, if it is strong enough to bear the shear, the equilibrium of the girder is not affected. Consider a horizontal girder resting upon four supports at A, B, C, and D and *hinged* at the points E and F in the side spans.

Let $AE = a$, $EB = b$, $BC = c$, $CF = e$, $DF = d$;

Let W_1, W_2, W_3, W_4, W_5 be the loads upon *AE*, *EB*, *BC*, *I* *FC*, respectively, and let x_1, x_2, x_3, x_4, x_5 be the several distances the corresponding centres of gravity from the points *E*, *B*, *C*, *F*,

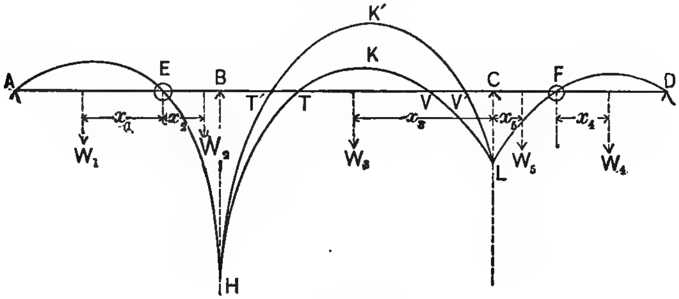


FIG. 269.

Since the B.M. is *nil* at *A* and *E* and also at *D* and *F*, the t portions *AE* and *DF* are in precisely the same condition as t independent girders carrying the same loads and resting upon sı ports at the ends.

The portion *EF* may also be treated as an independent gir supported at *B* and *C* and carrying in addition to the weights *I* W_3, W_5,

$$\text{a weight } W_1\left(1-\frac{x_1}{a}\right) \text{ at the cantilever end } E$$

$$\text{and} \qquad \text{“} \quad \text{“} \quad W_4\left(1-\frac{x_4}{d}\right) \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad F,$$

these two weights being equal to the reactions at *E* and *F* respectiv on the assumption that *AE* and *DF* are independent girders.

Let R_1, R_2, R_3, R_4 be the reactions at *A*, *B*, *C*, *D*, respective

Then $$R_1a = W_1x_1$$

and $$R_4d = W_4x_4.$$

R_1 and R_4 are therefore always positive and there is no tendency the part of the girder to rise from off the supports at *A* and *D*, a consequently no anchorage is needed at these points.

Take moments about *C* and *B*. Then

$$-(W_1-R_1)(b+c)-W_2(x_2+c)+R_2c-W_3x_3+W_5x_5+(W_4-R_4)e=($$

and

$$-(W_1-R_1)b-W_2x_2+W_3(c-x_3)-R_3c \\ +W_5(x_5+c)+(W_4-R_4)(c+e)=0,$$

two equations giving R_2 and R_3, since R_1 and R_4 have been already determined.

The *pier moments* P_1 at B and P_2 at C are

$$P_1=-(W_1-R_1)b-W_2x_2=-W_1\frac{b}{a}(a-x_1)-W_2x_2$$

and
$$P_2=-(W_4-R_4)e-W_5x_5=-W_4\frac{e}{d}(d-x_4)-W_5x_5,$$

their values depending solely upon the loads on the spans containing the hinges.

The bending moment at any point in BC distant x from B

$$=R_2x-(W_1-R_1)(b+x)-W_2(x_2+x)-M$$
$$=P_1+x(R_1+R_2-W_1-W_2)-M,$$

M being the bending moment due to the load upon the length x.

The shearing-force and bending-moment diagrams for the whole girder can now be easily drawn.

For any given loads upon the side spans, let AEH and DFL be the bending-moment curves for the portions AB, CD; BH and CL representing the pier moments at B and C respectively. The bending moments for the least and greatest loads upon BC will be represented by two curves, HKL, $HK'L$, and the distances TT', VV', through which the points of contrary flexure must move, indicate those portions of the girder which are to be designed to resist bending actions of opposite signs.

Again, let the two hinges be in the intermediate span.

Let $AB=a$, $BE=b$, $EF=c$, $FC=e$, $CD=d$;

Let W_1, W_2, W_3, W_4, W_5 be the loads upon AB, BE, EF, CD, CF, respectively, and let x_1, x_2, x_3, x_4, x_5 be the several distances of the corresponding centres of gravity from the points B, B, F, C, C.

EF evidently may be treated as an independent girder supported at the two ends and carrying a load W_3.

AE and DF may be treated as independent girders carrying the loads W_1, W_2 and W_4, W_5, respectively, and also loaded at the cantilever ends E and F with weights equal to the reacti

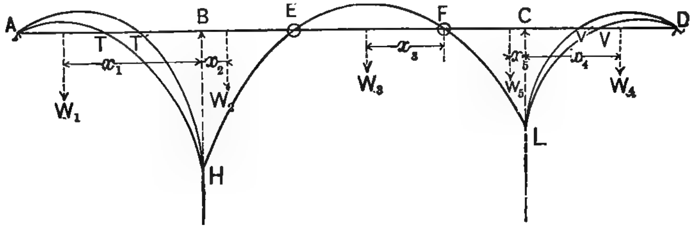


FIG. 270.

at E and F due to the load W_3 upon girder EF, which is assum to be independent. Thus

$$\text{the load at } E = W_3\frac{x_3}{c};$$

$$\text{" " " } F = W_3\left(1-\frac{x_3}{c}\right).$$

The pier moments P_1 at B and P_2 at C are

$$P_1 = W_2x_2 + W_3x_3\frac{b}{c}$$

and
$$P_2 = W_5x_5 + W_3\left(1-\frac{x_3}{c}\right)e,$$

their values depending solely upon the loads on the span contain the hinges.

Let R_1, R_2, R_3, R_4 be the reactions at A, B, C, D, respective and take moments about the points B, A, D, C. Then

$$R_1a - W_1x_1 + \left(W_2x_2 + W_3x_3\frac{b}{c}\right) = 0 = R_1a - W_1x_1 + P_1;$$

$$-R_2a + W_1(a-x_1) + W_2(a+x_2) + W_3x_3\frac{a+b}{c} = 0;$$

$$R_3d - W_3\left(1-\frac{x_3}{c}\right)(e+d) - W_5(x_5+d) - W_4(d-x_4) = 0;$$

$$-R_4d - W_3\left(1-\frac{x_3}{c}\right)e - W_5x_5 + W_4x_4 = 0 = -R_4d + W_4x_4 - P_2.$$

R_2 and R_3 are always *positive,*

R_1 is *positive* or *negative* according as $W_1x_1 \gtrless P_1$, and

R_4 " " " " " " $W_4x_4 \gtrless P_2$.

Thus there will be a downward pressure or an upward pull at each end according as the moment of the load upon the adjoining span is greater or less than the corresponding pier moment. The ends must therefore be anchored down or they will rise off their supports.

The shearing-force and bending-moment diagrams for the whole girder can now be easily drawn.

Let *HEFL* be the bending-moment curve for any given load upon the span *BC*, *BH* and *CL* being the pier moments at *B* and *C* respectively.

The bending-moment curves for the least and greatest loads on the side spans may be represented by curves *ATH*, *AT'H* and *DVL*, *DV'L*, and the distances *TT'*, *VV'* through which the points of contrary flexure move indicate those portions of the girder which are to be designed to resist bending actions of opposite signs.

Reverse strains may, however, be entirely avoided by making the length of *EF* sufficiently great as compared with the lengths of the side spans.

The preceding examples serve to illustrate the mechanical principles governing the stresses in cantilever bridges.

EXAMPLES.

1. A beam 20 ft. long and weighing 20 lbs. per lineal foot is placed upon a support dividing it into segments of 16 and 4 ft., and is kept horizontal by a downward force *P* at the middle point of the smaller segment. Find the value of *P* and the reaction at the support.

Show that the required force *P* will be doubled if a single weight of 150 lbs. is suspended from the end of the longer segment. Draw shearing-force and bending-moment diagrams in both cases. *Ans.* 1200 lbs.; 1600 lbs.

2. A man and eight boys carry a stick of timber, the man at the end and the eight boys at a common point. Find the position of this point, if the man is to carry twice as much as each boy.

Ans. Distance between supports = $\frac{5}{8}$ length of beam.

3. A timber beam is supported at the end and at one other point; the reaction at the latter is double that at the end. Find its position.

Ans. Distance between supports = $\frac{3}{4}$ length of beam.

4. Two beams ABC, BCD are bolted at B and C so as to act as one beam supported at A and D; $AB=12$ ft., $BC=4$ ft., $CD=16$ ft.; each of the bolts will bear a bending moment of 100 lb.-ft. Find the greatest weight which can be concentrated on the portion BC.

Also find the greatest load which can be uniformly distributed from A to D.

Draw the corresponding S.F. and B.M. diagrams in each case.

Ans. $14\frac{7}{12}$ lbs.; 25 lbs.

5. A beam AB of 30′ span rests upon a support at A, is fixed in a wall at B, and is hinged at a point C dividing AB into the two segments $AC=20'$ and $CB=10'$. Draw the S.F. and B.M. diagrams (*a*) for a uniformly distributed load of 30 tons, (*b*) for an additional load of 4 tons concentrated at 10′ and at 25′ from A.

In each case give the maximum S.F. and maximum B.M. on AC and on CB. *Ans.* (*a*) 10 tons, 50 ft.-tons; 20 tons, 175 ft.-tons.
(*b*) 2 tons, 20 ft.-tons; 6 tons, 40 ft.-tons.

6. A horizontal girder of length $2l$ rests upon supports and carries N weights, each equal to W. If $2a$ is the distance between consecutive weights, place the weights so as to throw a maximum bending moment on the girder. Find this moment.

Ans. If N is even, maximum B.M. $=\dfrac{WN}{8l}(4l^2+a^2-2Nal)$ when 1st weight is $l-a(N-\frac{3}{2})$ from support.

If N is odd, maximum B.M. $=\dfrac{WNl}{2}-\dfrac{Wa}{4}(N^2-1)$ when 1st weight is $l-a(N-1)$ from support.

7. Two weights P and Q ($<P$) are carried by a horizontal girder of length l resting upon supports at the ends, the distance between the weights being a Place the weights so as to throw a maximum bending moment on the girder and find the value of this moment.

Ans. Maximum B.M. $=\dfrac{(Wl-Qa)^2}{4wl}$ when P is $\dfrac{Wl-Qa}{2w}$ from support, W being $P+Q$.

8. Three loads of P, Q, and R tons, spaced 6′ and 4′ apart, are carried by a girder of 20′ span. If the B.M. at the middle point of the girder is the same when either P or Q is concentrated there, show that $P:Q:R::5:3:2$. Also find in terms of P the maximum B.M. at 5′, 10′, and 15′ from the end.

Ans. 5.6 P ft.-tons; 6.2 P ft.-tons; 4.4 P ft.-tons.

9. Three loads of 2, 4, and 3 tons, spaced 4′ apart, are to be carried by a girder of 18′ span. Place the loads so as to throw a maximum B.M. on the beam at 4′, 6′, 9′, 12′, and 15′ from the end, and find the values of the several bending moments.

Ans. $19\frac{1}{9}$ ft.-tons; $26\frac{2}{3}$ ft.-tons; $30\frac{1}{2}$ ft.-tons; $25\frac{1}{3}$ ft.-tons; $17\frac{1}{6}$ ft.-tons.

10. Fig. 271 is a step-ladder in which $AB=9'$, $AC=10'$, $AE=8'$, and $BC=6'$. A man weighing 160 lbs. stands at a point 3′ from A. Find the tension in DE, and draw S.F. and B.M. diagrams for each leg, assuming the floor to be smooth. *Ans.* 288.8 lbs.

11. A man of weight W ascends a ladder of length l which rests against a smooth wall and the ground and is inclined to the vertical at an angle α. The ladder has n rounds. Find the bending moment at the rth round from the foot when the man is on the pth round from the foot. (Neglect weight of ladder.)

Ans. $Wpl\dfrac{n-r+1}{(n+1)^2}\sin\alpha.$

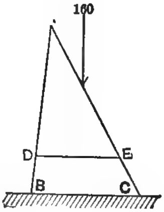


FIG. 271.

12. A regular prism of weight W and length a is laid upon a beam of length $2l(>a)$. If the prism is so stiff as to bear at its ends only, show that the bending action on the beam is less than if the bearing were continuous from end to end of prism.

Ans. 1st. Max. B.M. $=\dfrac{W}{2l}\left(l-\dfrac{a}{4}\right)^2$;

2d. " " $=\dfrac{W}{2}\left(l-\dfrac{a}{4}\right)$.

13. A railway girder 50 ft. in the clear and 6 ft. deep carries a uniformly distributed load of 50 tons. Find the maximum shearing stress at 20 ft. from one end when a train weighing $1\frac{1}{4}$ tons per lineal foot crosses the girder.

Also find the minimum theoretic thickness of the web at a support, 4 tons being the safe shearing inch-stress of the metal. *Ans.* $16\frac{1}{4}$ tons; .195 in.

14. A beam is supported at one end and at a second point dividing its length into segments m and n. Find the two reactions. Also find the ratio of m to n which will make the maximum positive moment equal to the maximum negative moment. *Ans.* $\dfrac{w}{2m}(m^2-n^2)$, $\dfrac{w}{2m}(m+n)^2$, $m:n::1+\sqrt{2}:1$.

15. One of the supports of a horizontal uniformly loaded beam is at the end. Find the position of the other support so that the straining of the beam may be a minimum. *Ans.* Distance from end support $=\dfrac{\text{length.}}{\sqrt{2}}$.

16. A steel plate girder of 80 ft. span carries a uniformly distributed load of 80 tons and also loads ot 4, 6, and 8 tons concentrated at points on the girder 10, 40, and 60 ft., respectively, from one end. Draw the S.F. and B.M. diagrams. State the B.M. at each of the concentrated loads and find the position and amount of the maximum B.M.

Ans. 435; 1020; 790 ft.-tons; at the centre, 1020 ft.-tons.

17. A beam of 80 ft. span carries weights of 4, 6, 6, 6, 6, and 5 tons at points 10, 20, 30, 40, 50, and 60 ft., respectively, from one end. Determine the supporting forces at the two ends and draw the S.F. and B.M. diagrams. Also state the B.M. at the centre. *Ans.* $18\frac{3}{4}$, $14\frac{3}{4}$ tons; 430 ft.-tons.

18. A beam of 80 ft. span carries a uniformly distributed load of 80 tons and also a concentrated load. Find the amount and position of the latter, the supporting forces at the ends being 55 and 45 tons. Also find the B.M. and S.F. at the concentrated load and at the centre.

Ans. 20 tons at 20 ft. from support; 900 ft.-tons; 35 and 15 tons; 1000 ft.-tons; -5 tons.

19. The total load on the axle of a truck is 6 tons. The wheels are 6 ft. apart and the two axle-boxes 5 ft. apart. Draw the curve of bending moment on the axle and state what it is in the centre.

20. In the truss (Fig. 272) the points of trisection D and E of the horizontal member BC are connected with the middle points F and G of the rafters by the members DF and EG. Weights of 2, 3, and 4 tons are concentrated at the points F, A, and G, respectively. Find the supporting forces at B and C and draw the curves of S.F. and B.M. for the member BC. The span is 60 ft. and the rise 30 ft. *Ans.* 4 tons; 5 tons.

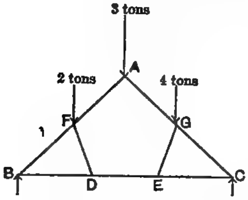


Fig. 272.

21. The post OB, 14 ft. in length, is pivoted at O and is acted upon by a horizontal force P at G, where $AG = GO = 4\frac{1}{2}$. The member AF is pivoted at A and is loaded with 5 tons at F. The joints of the members BC, CD, DE, and EF lie on the arc of a circle with its centre at O, BF subtending an angle 60° at O. AF is connected with $BCDEF$ by means of parallel ties inclined at 30° to the vertical and spaced at equal distances apart along AF. Determine the stresses in these ties and draw S.F. and B.M. diagrams for the post OB and the member AF.

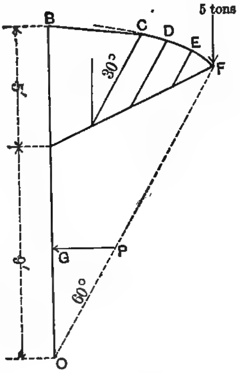


Fig. 273.

22. A beam 20 ft. long is loaded at four points equidistant from each other and the ends, with equal weights of 3 tons. Find the bending moment at each of these points and draw the curve of shearing force.

23. A uniform beam $20\sqrt{3}$ ft. in length rests with one end on the ground and the other against a smooth vertical wall; the beam is inclined at 60° to the vertical and has a joint in the middle which can bear a bending moment of 30,000 lb.-ft. Find the greatest load which may be uniformly distributed over the beam. Also find how far the foot of the beam should be moved towards the wall in order that an additional 2000 lbs. may be concentrated at the joint.

Draw curves of shearing force and bending moment in each case.

Ans. 8000 lbs.; distance = 10 ft.

24. A girder AB, Fig. 274, 30 ft. long, carries a brick wall 16 ft. high by 1 ft. thick and weighing 120 lbs. per cubic foot. A doorway 8×6 ft. wide is cut in the wall with its centre line 10 ft. from the end support. Draw S.F. and B.M. diagrams, stating the scales used.

25. A girder 60 ft. long, Fig. 275, carries a brick wall 1 ft. thick and weighing 120 lbs. per cubic foot. The remaining dimensions of the wall are shown in the sketch. Draw S.F. and B.M. diagrams, stating the scales used.

26. A beam carries a load of 2 tons per foot run over the central half of its length, and a load of 1 ton per foot run over the remaining portions. It has to be supported by two props so that the greatest stress induced in the beam is the least possible. Assuming the beam of uniform section, find the proper position of the props.

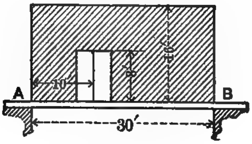


Fig. 274.

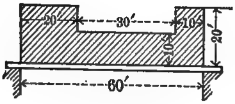


Fig. 275.

27. In a beam *ABCDE* the length (*AE*) of 24 feet is divided into four equal panels of 6 ft. each by the points *B*, *C*, *D*. Draw the diagram of moments for the following conditions of loading, writing their values at each panel-point: (I) Beam supported at *A* and *E*, loaded at *D* with a weight of 10 tons; (II) beam supported at *B* and *D*, loaded with 10 tons at *C*, and with a weight of 2 tons at each end *A* and *E*; (III) beam encastré from *A* to *B*, loaded with a weight of 2 tons at each of the points *C*, *D*, and *E*.

28. A beam *ABCD*, 28 ft. in length, overhangs its two supports at *B* and *C* by the lengths $AB=8$ ft. and $CD=4$ ft. The length of *BC* is 16 ft. and *O* is its middle point. A load of 100 lbs. is suspended from *A*, and loads of 800 and 200 lbs. are uniformly distributed over *BO* and *CD* respectively. Find the position and amount of the maximum positive B.M. on the beam and draw the S.F. and B.M. diagrams.

Ans. $1153\frac{3}{8}$ ft.-tons at $6\frac{1}{4}$ ft. from B.

S.F. in tons: 100 at *A* and *B*, $+625$ at *B*, -175 at *B* and *C*; $+200$ at *C*, 0 at *D*.

B.M. in ft.-tons: 0 at *A*, -800 at *B*, $+1000$ at *O*, -400 at *C*, 0 at *D*.

29. A girder *AD* of 18 ft. span is to carry three weights of 2, 4, and 3 tons, taken in order and spaced 4 ft. apart. The points *B* and *C* divide the span into the three segments $AB=4$ ft., $BC=8$ ft., and $CD=6$ ft. Show that the B.M. at any point in *AB*, *BC*, or *CD* is a maximum when the weight of 2, 4, or 3 tons weight, respectively, is concentrated at that point. Also find the two points at which the B.M. is a maximum for *two* distributions of the load. Determine the corresponding bending moments.

Ans. $19\frac{1}{9}$ ft.-tons at 4 ft. from *A*; $25\frac{1}{3}$ ft.-tons at 12 ft. from *A*.

30. A uniformly loaded beam rests upon two supports. Place the supports so that the straining of the beam may be a minimum.

Ans. Distance of each support from centre $=l\left(1-\frac{1}{\sqrt{2}}\right)$.

31. Two bars *AC*, *CB* in the same horizontal line are jointed at *C* and supported upon two props, the one at *A*, the other at some point in *CB* distant *x* from *C*. The joint *C* will safely bear *n* lb.-ft.; the bars are each *l* ft. in length and w lbs. in weight. Find the limits within which x must lie.

Ans. $l\dfrac{wl \pm 2n}{3wl \mp 2n}$.

32. A uniform load PQ moves along a horizontal beam resting upon supports at its ends A and B. Prove that the bending moment at a given point O is a maximum when PQ occupies such a position that $OP:OQ::OA:OB$.

Draw curves of maximum shearing force and bending moment for all points of the beam.

33. A beam is supported at the ends and loaded with two weights mW and nW at points distant a, b, respectively, from the consecutive supports. Show that the bending action is greatest at mW or nW according as $\dfrac{m}{n} \gtrless \dfrac{b}{a}$.

34. Find the maximum B.M. on a horizontal beam of length l supported at the two ends and carrying a load which varies in intensity from w at one end to $w+pl$ at the other.

Ans. Maximum B.M. $=\dfrac{x}{3}\left(2R-\dfrac{wx}{2}\right)$, where $x(px+2w)=2R$, R being the reaction at the end.

35. A wheel supporting 10 tons rolls over a beam of 20 ft. span. Place the wheel in such a position as to give the maximum bending moment, and find its value. *Ans.* At the centre; 50 ton-ft.

36. Two wheels a ft. apart support, the one mW tons, the other nW tons, m being $>n$, and roll over a beam of l ft. span. Show that the bending moment is an *absolute* maximum at the centre or at a point whose distance from the nearest support is $\dfrac{l}{2}-\dfrac{na}{2(m+n)}$ according as $l \lessgtr a\left(1+\sqrt{\dfrac{m}{m+n}}\right)$, and find its value in each case. $(l<2a.)$

Ans. $\dfrac{mWl}{4}$ ton-ft.; $\dfrac{m+n}{4l}W\left\{l-\dfrac{na}{m+n}\right\}^2$ ton-ft.

37. Two equal wheels 4 ft. apart and loaded, the one with 4 and the other with 6 tons, roll in this order at a constant rate from left to right across a girder of 9 ft. span. Find the maximum B.M. at 5 ft. and at 8 ft. from the left support, and also find the position and amount of the absolute maximum B.M. Draw the maximum S.F. and B.M. diagrams.

Ans. $13\frac{1}{3}$ ft.-tons, $5\frac{1}{3}$ ft.-tons; absolute maximum B.M. $=14\frac{25}{36}$ ft.-tons at $3\frac{5}{6}$ ft. from left support.

38. Two wheels, each supporting 7 tons and spaced 4 ft. apart, roll over a girder of $7\frac{1}{2}$ ft. span. Find the maximum B.M. at the centre and the absolute maximum B.M. for the whole span. Also show that there are two distributions of the loads which will give the same maximum B.M. at two points. Find the positions of these points and the corresponding B.M.

Ans. $13\frac{1}{8}$ ft.-tons; $14\frac{7}{60}$ ft.-tons at 1 ft. from the centre; $13\frac{1}{15}$ ft.-tons at 3 ins. from the centre.

39. Two wheels supporting, the one 11 tons, the other 7 tons, and spaced 6 ft. apart, roll over a girder of $12\frac{1}{2}$ ft. span. Draw the B.M. and S.F. diagrams, and find the *absolute* maximum B.M. for the whole span. Also show that there are two points at each of which two distributions of the loads give the same maximum B.M.s. Find these points and the corresponding B.M.s.

Ans. 37.21 ft.-tons at $1\frac{1}{6}$ ft. from the centre, 34.32 ft.-tons at 3 ins. from the centre, and 26.66 ft.-tons at 9.212 ft. from the centre.

40. In the preceding example show that the maximum *negative* shear at $4\frac{3}{22}$ ft. from a support when the 7-ton wheel only is on the beam, is the same as the maximum negative shear at the same point when both of the wheels are on the beam, and find its value. Also show that the maximum *negative* shear at $9\frac{3}{7}$ ft. from a support is the same when only the 11-ton wheel is on the beam as when the two wheels are on the beam, and find its value.

Ans. $\frac{637}{275}$ tons; $\frac{1452}{175}$ tons.

41. Solve Ex. 39 when the girder carries an additional load of $\frac{5}{8}$ ton per lineal foot.

Ans. 49.1205 ft.-tons at .959 ft. from the centre; 46.5075 ft.-tons at 3 ins. and 34.535 ft.-tons at $3\frac{5}{28}$ ft. from the centre.

42. A rolled joist weighing 150 lbs. per lineal foot and 20 ft. long carries a uniformly distributed load of 6000 lbs., and two wheels 5 ft. apart, the one bearing 5000 lbs. and the other 3000 lbs., roll over the joist. Find the maximum shears at the supports, at the centre, and at 5 ft. from each end.

Ans. 10,250 lbs.; 9750 lbs.; 3250 lbs.; 6750 lbs.; 6250 lbs.

43. A rolled joist 17 ft. long is supported at one end and at a point 13 ft. distant from that end. Two wagon-wheels 5 ft. apart and each carrying a load of 1300 lbs. pass over the joist. Find the maximum positive and negative moments due to these weights, and also the corresponding reactions.

Ans. Maximum *positive* B.M. $=5512\frac{1}{2}$ lb.-ft.;
reactions $=1550$ and 1050 lbs.
Maximum *negative* B.M. $=5200$ lb.-ft.;
reactions $=1700$ lbs. and -400 lbs.
or $=2900$ lbs. and -300 lbs.

The maximum B.M. diagram for each half of the 13-ft. span is given by

$$M_x=100(21-2x)x,$$

x being measured from a support.

44. Two wheels loaded, the one with 6 and the other with 4 tons, and spaced 4 ft. apart, roll from left to right across a girder of 18 ft. span. Draw the S.F. and B.M. diagrams. Determine the position and amount of the absolute maximum B.M., and also the maximum B.M. at 6 ft. from the left support. In the latter case show that the maximum B.M. is the same for two different distributions of the loads.

Ans. $23\frac{4}{27}$ ft.-tons at $9\frac{2}{3}$ ft. from left support; $18\frac{2}{3}$ ft.-tons.

45. Three wheels, each loaded with a weight W and spaced 5 ft. apart, roll over a beam of 18 ft. span. Place the wheels in such a position as to give the maximum bending moment, and find its value.. Also place (*a*) the wheels so that B.M. at any point between the two hindmost wheels may be constant, and find its value. Also (*b*) determine all the positions of the

wheels which will give the same bending moment at 6 and 12 ft. from one end, and find its value.

Ans. $8\frac{1}{2}W$ when middle weight is at centre of beam.
(*a*) 1st wheel at 1 ft. from support; B.M. $= 7W$.
(*b*) When distance between end wheel and support is $\geqq$ 2 ft. and $\leqq$ 6 ft. B.M. $= 7W$.

46. Three wheels loaded with 8, 9, and 10 tons and spaced 5 ft. apart, are placed upon a beam of 15 ft. span, the 8-ton wheel being 3 ft. from the left abutment. Determine graphically the B.M. at 6 ft. from the left abutment. Also find the greatest B.M. at the same point when the weights travel over the beam, and the *absolute maximum* bending moment to which the beam is subjected.

Ans. $47\frac{3}{5}$ ton-ft.; $53\frac{1}{5}$ ton-ft.; absolute maximum B.M. $= 56\frac{101}{324}$ ton-ft. at 2d wheel when 1st is $2\frac{17}{54}$ ft. from support.

47. Three loads of 5, 3, and 4 tons, spaced 4 ft. apart, travel in order over a girder of 12 ft. span. Draw maximum S.F. and B.M. diagrams. Show that the B.M. is an absolute maximum and equal to $18\frac{3}{8}$ ton-ft. at $5\frac{1}{4}$ ft. from one end when the 5-ton load is concentrated at that point. Also show that the absolute maximum B.M.s at 6 and 8 ft. from one end are 18 and $14\frac{2}{3}$ ton-ft. respectively, and that they are each produced by two different distributions of the loads.

48. Three loads of 3, 5, and 4 tons, spaced 4 ft. apart, travel in order from left to right over a girder of 12 ft. span. Find the position and amount of the *absolute* maximum B.M. Also show that at 4 ft. and at 8 ft. from the left support there are two distributions of the loads which will give the same maximum B.M. Find the amounts of these bending moments at the points in question.

Ans. $22\frac{1}{3}$ ft.-tons at $5\frac{5}{6}$ ft. from left support; $18\frac{3}{8}$ ft.-tons; $17\frac{1}{3}$ ft.-tons.

49. Three loads of 3, 4, and 5 tons, spaced 4 ft. apart, travel in order from left to right over a girder of 12 ft. span. Find the position and amount of the absolute maximum B.M. Show that 4 ft. from the left support there are two distributions of the loads for which the B.M. has the same maximum value, and find its amount. Also show that at some point between 6 ft. and 7 ft. from the left support the maximum B.M. is the same for two distributions of the load. Find the position of this point and the corresponding B.M.

Ans. $21\frac{1}{9}$ ft.-tons at $5\frac{2}{3}$ ft. from left support; $17\frac{1}{3}$ ft.-tons; 19.46 ft.-tons at 6.472 ft. from left support.

50. Four wheels, each carrying 5 tons, travel over a girder of 24 ft. clear span at equal distances 4 ft. apart. Determine graphically the maximum B.M. at 8 ft. from a support, and also the absolute maximum B.M. on the girder.

Ans. $\frac{220}{3}$ ton-ft.; $80\frac{5}{6}$ ton-ft.

51. Four wheels each loaded with a weight W and spaced 5 ft. apart roll over a beam of 18 ft. span. Place the wheels in such a position as to give the maximum bending moment, and find its value.

Ans. One wheel off the beam and middle wheel of remaining three at the centre; maximum B.M. $= 8\frac{1}{2}W$. If all wheels are on beam. maximum B.M. $= 8\frac{35}{72}W$.

52. All the wheels in the preceding example being on the beam, the B.M. at the centre for a certain range of travel is constant and equal to that for a particular distribution of the wheels when only three are on the beam. Find the range, the B.M., and the position of the three wheels.

Ans. While the end wheel travels 3 ft. from the support; $8W$; the first wheel 5 ft. from the support.

53. If the load on each of the wheels in Ex. 51 is 5 tons, and if the beam also carries a uniformly distributed load of 20 tons, and two loads of 2 and 3 tons concentrated at points distant 5 and 9 ft., respectively, from one end, find the maximum shearing force (both positive and negative) and the maximum bending moment for the whole span; also find the loci for the maximum shearing force and bending moment at each point.

Ans. Denoting the distance from support by x, the *maximum positive* shearing-force diagram is given by equations $18S_x = 443 - 40x$ from $x = 0$ to 3, $= 428 - 35x$ from $x = 3$ to 5, $= 392 - 35x$ from $x = 5$ to 8, $= 352 - 30x$ from $x = 8$ to 9, $= 298 - 30x$ from $x = 9$ to 13, $= 233 - 25x$ from $x = 13$ to 18; and the *maximum negative* shearing-force diagram by equations $18S_x = 233 - 25x$ from $x = 0$ to 5, $= 172 - 30x$ from $x = 5$ to 9, $= 118 - 30x$ from $x = 9$ to 10, $= 68 - 35x$ from $x = 10$ to 13, $= 68 - 35x$ from $x = 13$ to 15, $= -7 - 40x$ from $x = 15$ to 18.

Maximum positive shear $= \frac{443}{18}$ tons; maximum negative shear $= \frac{727}{18}$ tons; *maximum bending moment* curve is given by $M_x = \frac{623}{18}x - \frac{20}{9}x^2$ from $x = 0$ to $x = 3$; $M_x = \frac{304}{9}x - \frac{85}{18}x^2$ from $x = 3$ to $x = 5$; $M_x = \frac{229}{6}x - \frac{20}{9}x^2 - 15$ from $x = 5$ to $x = 8$; $M_x = x\frac{575}{18} - \frac{85}{18}x^2 + 12$ from $x = 8$ to $x = 9$; abs. max. B.M. = 142 ton-ft.

54. Four wheels loaded with 4, 4, 8, and 8 tons are placed upon a girder of 24 ft. span at distances of 3 in., $6\frac{1}{2}$ ft., $8\frac{3}{4}$ ft., and 9 ft. from the left support. Find by scale measurement the bending moment at the centre of the girder. If the wheels travel over the girder at the given distances apart, find the maximum B.M. to which the girder is subjected.

Ans. Max. B.M. = 122.2296 ton-ft.

55. Four wheels loaded with 5, 2, 4, and 3 tons travel in order from left to right over a girder of 12 ft. span at distances 4, 3, and 2 ft. apart. Draw the max. S.F. and B.M. diagrams. Give the maximum positive and negative shearing forces at 0, 3, 7, 9, and 12 ft. from the left support. Find the position and amount of the *abs.* max. B.M. Show that there are two distributions of the wheels for which the B.M. at 5 ft. from the left support is a maximum, and find its value; also show that there is a point between 2 and 3 ft. and one between 8 and 9 ft. from the left support, at which the maximum B.M. is the same for two distributions of the wheels. Find the positions and amounts of these maximum bending moments.

Ans. $+8\frac{3}{4}$ and $-5\frac{3}{4}$ tons; $+5\frac{1}{4}$ and $-2\frac{1}{4}$ tons; $+3\frac{5}{12}$ and $-\frac{7}{12}$ tons; $+5\frac{1}{4}$ and $-2\frac{1}{2}$ tons; $+8\frac{3}{4}$ and $-4\frac{3}{4}$ tons; 21 ft.-tons at the centre; $20\frac{1}{4}$ ft.-tons; $14\frac{11}{14}$ ft.-tons at $2\frac{4}{7}$ ft. and 17.061 ft.-tons at 8.292 ft. from the left support.

56. A rolled joist weighing 450 lbs. per lineal foot and 20 ft. long carries the four wheels of a locomotive at 3, 8, 13, and 18 ft. from one end. Find the maximum bending moment and the maximum shears, both positive and negative, the load on each wheel being 10,000 lbs.

Ans. Max. B.M. = 123,600 lb.-ft.; max. shears = 23,500 lbs. and 25,500 lbs.

57. Solve the preceding example when a live load of $2\frac{2}{9}$ tons per lineal foot is substituted for the four concentrated weights on the wheels.

Ans. Max. B.M. $=$ 2,202,500 ton-ft.

58. The loads on the wheels of a locomotive and tender passing over a beam of 60 ft. span are 14,180, 14,180, 21,260, 21,260, 21,260, 21,260, 16,900, 16,900, 16,900, 16,900 lbs., counting in order from the front, the intervals being 5, $5\frac{1}{4}$, 5, 5, 5, $8\frac{3}{4}$, 5, 4, 5 ft. Place the wheels in such a position as to give the maximum bending moment, and find its value. Also find the maximum bending moments for spans of 30, 20, and 16 ft.

Ans. For 60 ft. span, max. B.M. is at 5th wheel and $=$ 1,559,925.4 lb.-ft. when 1st wheel is 7.85 ft. from support.

For 30 ft. span, max. B.M. at 5th wheel when 2d wheel is .596 ft. from support and $=$ 436,761.4 lb.-ft.

For 20 ft. span, max. B.M. at centre when 3d wheel is $2\frac{1}{2}$ ft. from support and $=$ 212,600 lb.-ft. $=$ max. B.M. at same point when 4th wheel is 5 ft. from support.

For 16 ft. span, max. B.M. is at 5th wheel and $=$ 132,875 lb.-ft. when 4th wheel is 5 ft. from support.

59. If the 60 ft. beam in the preceding example also carries a uniformly distributed load of 60,000 lbs., find the curves of maximum shearing force and bending moment at each point.

60. A span of l ft. is crossed by a beam in two half-lengths, supported at the centre by a pier whose width may be neglected. The successive weights on the wheels of a locomotive and tender passing over the beam are 14,000, 22,000, 22,000, 22,000, 22,000, 14,000, 14,000, 14,000, 14,000 lbs., the intervals being $7\frac{1}{2}$, $4\frac{1}{2}$, $4\frac{1}{2}$, $4\frac{1}{2}$, $10\frac{1}{2}$, 5, 5, 5 ft. Place the wheels in such a position as to throw the greatest possible weight upon the centre pier, and find the magnitude of this weight for spans of (1) 50 ft.; (2) 25 ft.; (3) 20 ft.; (4) 18 ft.

Ans. (1) 85,320 lbs.; (2) 56,880 lbs.; (3) 48,400 lbs.; (4) 44,000 lbs.

61. Loads of $3\frac{3}{4}$, 6, 6, 6, and 6 tons follow each other in order over a ten-panel truss at distances of 8, $5\frac{3}{4}$, $4\frac{1}{2}$, and $4\frac{1}{2}$ ft. apart. Apply the results of Art. 8 to determine the position of the loads which will give the maximum diagonal and flange stresses in the third and fourth panels.

62. A truss of 240 ft. span and ten panels has loads of $12\frac{1}{2}$, 10, 12, 11, 9, 9, 9, 9, and 9 tons concentrated at the panel-points. Find by *scale measurement* the bending moments at the four panel-points which are the most heavily loaded, and determine by Art. 8 whether these are the greatest bending moments to which the truss is subjected as the weights travel over the truss at the panel distances apart.

Ans. 1146, 1992, 2598, 2916 ton-ft.

63. Loads of $7\frac{1}{2}$, 12, 12, 12, 12 tons are concentrated upon a horizontal beam of 25 ft. span at distances of 18, 108, 164, 216, and 272 in., respectively, from the left support. Find graphically the bending moment at the centre, of the span. If the loads travel over the truss at the given distances apart, find the maximum B.M. at the same section.

Ans. 2319 ft.-tons.

64. A span of l ft. is crossed by two cantilevers fixed at the ends and hinged at the centre. Draw diagrams of shearing force and bending moment

(1) for a single weight W at the hinge, (2) for a uniformly distributed load of intensity w.

Ans. Taking hinge for origin, the shearing-force and bending-moment diagrams are given by

$$(1)\qquad S_x = \frac{W}{2};\quad M_x = -\frac{Wx}{2}.$$

$$(2)\qquad S_x = wx;\quad M_x = -\frac{wx^2}{2}.$$

65. A beam for a span of 100 ft. is fixed at the ends. Hinges are introduced at points 30 ft. from each end. Draw curves of shearing force and bending moment (1) when a weight of 5 tons is concentrated on each hinge; (2) when a uniformly distributed load of $\frac{1}{8}$ ton per lineal foot covers (*a*) the centre length, (*b*) the two side lengths, (*c*) the whole span.

Ans. (1) *Side span:* max. B.M. = − 150 ft.-tons; max. S.F. = 5 tons.
Centre span: " " = 0 ; " " = 0
(2) (*a*) *Side span:* " " = − 75 " ; " " = $2\frac{1}{2}$ "
Centre span: " " = + 25 " ; " " = $2\frac{1}{2}$ "
(*b*) *Side span:* " " = − $56\frac{1}{4}$ " ; " " = $3\frac{3}{4}$ "
min. " = 0
Centre span: " " = 0 ; " " = 0
(*c*) *Side span:* " " = − 75 " ; max. " = $4\frac{3}{8}$ "
min. " = $\frac{5}{8}$ "
Centre span: " " = + 25 " ; max. " = $2\frac{1}{2}$ "
min. " = 0

66. A beam $ABCD$ is supported at four points A, B, C, and D, and the intermediate span BC is hinged at the two points E and F. The load upon the beam consists of 15 tons uniformly distributed over AB, 10 tons uniformly distributed over BE, 5 tons uniformly distributed over FC, 30 tons uniformly distributed over CD, and a single weight of 5 tons at the middle point of EF. $AB = 15$ ft.; $BE = 5$ ft.; $EF = 15$ ft.; $FC = 10$ ft.; $CD = 25$ ft. Draw curves of B.M. and S.F., and find the points of inflexion.

Ans. *A to B.* S.F. in tons: +5 at A, −10 at B.
B to C. " " " : $+12\frac{1}{2}$ at B, $+2\frac{1}{2}$ at E, $\pm 2\frac{1}{2}$ at H, $-2\frac{1}{2}$ at F.
C to D. " " " : +17 at C, −13 at D.
A to B. Max. B.M. in ft.-tons: $+12\frac{1}{2}$ at 5 ft. from A, $-37\frac{1}{2}$ at B.
B to C. " " " " : $+18\frac{3}{4}$ at H, −50 at C.
C to D. " " " " : $+70\frac{5}{12}$ at $10\frac{5}{6}$ ft. from D.
Points of inflexion are 10 ft. from A in AB and $21\frac{2}{3}$ ft. from D in CD.

67. Solve example 69 when the girder carries an additional load of 130 tons uniformly distributed over 130 ft. from A, and also a concentrated load of 20 tons at the middle point of CD.

Ans. S.F. in tons: at $A = -87\frac{1}{2}$, at $B = \pm 167\frac{1}{2}$, at $H = +87\frac{1}{2}$, at $C = -62\frac{1}{2}$ and $+91\frac{1}{4}$, at 20-ton load $= 51\frac{1}{4}$, at $D = -8\frac{3}{4}$.
B.M. in ft.-tons: at $B = -5100$, at $43\frac{3}{4}$ ft. from $H = +1914\frac{1}{16}$, at centre of $BC = 1875$, at $C = -3300$, at 20-ton load $= -450$, at $8\frac{3}{4}$ ft. from $D = +38\frac{9}{32}$.
Pts. of contrary flexure: in AB none; in CD at $17\frac{1}{2}$ ft. from D
Required length of $CD = 71.854$ ft.

68. A girder AC, 80 ft. long, carries a uniformly distributed load of 80 tons and is supported at the three points A, B, and C. A hinge is introduced at H, and the distances are shown in Fig. 276. Draw to scale the S.F. and B.M. diagrams. Compare the results with those which would be obtained if the portion BC is hinged at K, Fig. 277, and a fourth support is introduced at D, the distances being as shown.

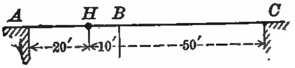


FIG. 276.

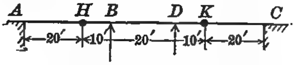


FIG. 277.

Ans. Fig. 1. Reactions $=10$ at A, $=48$ at B, $=22$ at C, in tons.
Total B.M. area $= +6965\frac{1}{3}$.
Pt. of contraflexure in BC at 44 ft. from C.
Fig. 2. Reactions $=10$ at A and $C=30$ at B and D, in tons.
Total B.M. area $= -1333\frac{1}{3}$.
Pt. of contraflexure in BD none, in DC at 20 ft. from C.

69. A span of 300 ft. is crossed by a girder $ABCD$ carrying a uniformly distributed load of 300 tons and resting upon supports at A, B, C, and D. The length of $AB=40$ ft., of $BC=180$ ft., and of $CD=80$ ft. The portion BC is divided by hinges at H and K into two equal cantilevers BH and CK, and a suspended span HK of 100 ft. Draw *to scale* the curves of S.F. and B.M., and determine the points of contrary flexure in the side spans. What must be the length of CD to make the reaction at D equal to *nil?*

Ans. Max. S.F. in tons: at $A=-50$, at $B=\pm 90$, at $H=50$, at $K=-50$, at $C=-90$ and $+75$, at $D=-5$.
Max. B.M. in ft.-tons: at $A=0$, at $B=-2800$, at centre of $BC=+1250$, at $C=-2800$, at $D=0$.
Pts. of contrary flexure: in AB none; in CD 10 ft. from D.
Required length of $CD=74.834$ ft.

CHAPTER III.

MOMENTUM. ENERGY. BALANCING.

1. Velocity—Acceleration.—The idea of velocity involves both speed and direction. The velocity of a body is its *rate of change of position*, and, if *constant*, it is measured by the number of units of length described in a unit of time.

The units of length ordinarily used are a foot, a metre, or a centimetre, and the unit of time is a second. Thus, if s feet are described in t seconds, the constant velocity v is given by

$$v=\frac{s}{t} \quad \text{or} \quad s=vt.$$

The velocity may not be constant, but may be always changing during an interval of time, however short, and then $\frac{s}{t}$ is the *average velocity*, the *actual* velocity at any point being the average velocity over a very small distance including that point.

The *average* velocity, for example, of a train travelling at the rate of 45 miles per hour is $\frac{45\times 5280}{60\times 60}=66$ ft. per second, but its *actual* velocity at any point may be very different from this amount.

At sea the speed is always measured in *knots*, a knot being 6080 ft. per hour, or an average velocity of very nearly $\frac{6080}{3600}=1.69$ ft. per second.

When a body has passed over s ft. in t secs., the average velocity in doing an additional distance of Δs ft. in Δt secs. is $\frac{\Delta s}{\Delta t}$, and if Δt is diminished indefinitely, the average velocity becomes the actual velocity at the instant t and is indicated by the relation

$$v=\frac{ds}{dt}.$$

The *acceleration* of a body is its *rate of change of velocity,* and if it is *constant,* it is measured by the change of velocity in a unit of time.

Thus, if the velocity changes from V f/s to v f/s in t secs., the acceleration a is given by

$$a=\frac{v-V}{t}, \quad \text{or} \quad v=V+at.$$

In Montreal, e.g., the velocity of a body falling freely from rest is

32.1765	f/s	or	980.73	cm/s	after	1 sec.
64.353	"	"	1961.46	"	"	2 "
96.5295	"	"	2942.19	"	"	3 "

There is, therefore, an increase in the velocity of 32.1765 f/s or 980.73 cm/s every second. The acceleration in this case is the same, and is called a *uniform acceleration.*

Again, let U be the average velocity. Then

$$\frac{s}{t}=U=\frac{v+V}{2}=V+\tfrac{1}{2}at,$$

and therefore

$$s=Vt+\tfrac{1}{2}at^2$$

and

$$as=\tfrac{1}{2}v^2-\tfrac{1}{2}V^2.$$

The acceleration may not be constant, but may be always changing in any interval of time, however small that interval may be, and in this case $(v-V)/t$ is the *average* acceleration. Let the velocity change from v after t secs. to $v+\Delta v$ after $(t+\Delta t)$ secs., so that Δv is the change of velocity in an additional period of Δt secs. Then $\frac{\Delta v}{\Delta t}$ is the average acceleration per second, and this is true, however small Δt may be, so that in the limit it becomes the actual acceleration at the instant t and is indicated by the relation

$$a=\frac{dv}{dt}=\frac{d^2s}{dt^2}.$$

The weights of various materials are given in the tables at the end of the chapter, and the specific gravity (sp. g.) of a substance is its weight as compared with the weight of an equal bulk of water.

The weight, for example, of a cubic foot of water is *approximately* 1000 ounces, and the weight of a cubic foot of cast iron is 7200 ounces; its sp. g. is therefore $7200/1000 = 7 \cdot 2$.

The *unit of force* is generally assumed to be the attraction of the earth on a pound weight and is called the *force of a pound*. Thus if a is the acceleration of a weight of W lbs. under the action of a force of P lbs.,

$$\frac{a}{g}=\frac{P}{W}.$$

Now the value of g is different in different parts of the world, being, for example,

32.1765	f/s	or	980.73	cm/s	in	Montreal,
32.19078	“	“	981.17	“	“	Greenwich,
32.182	“	“	980.9	“	“	London,
32.152	“	“	980	“	“	Baltimore,
			981	“	“	Paris,
32.088	“	“	978.04	“		at the Equator.

In order to avoid the use of the variable g a new unit of force called the *poundal* and one gth of that just defined has been introduced. Thus if $Q = Pg$, a force of Q poundals is a force of $\frac{Q}{g}(=P)$ pounds.

2. **Work.**—Work must be done to overcome a resistance. Thus bodies, or systems of bodies, which have their parts suitably arranged to overcome resistances are capable of doing work and are said to possess energy. This energy is termed *kinetic* or *potential* according as it is due to *motion* or to *position*. A pile-driver falling from a height upon the head of a pile drives the pile into the soil, doing work in virtue of its motion. Examples of potential energy, or *energy at rest*, are afforded by a *bent spring*, which does work when allowed to resume its natural form; a *raised weight*, which can do work by falling to a lower level; *gunpowder* and *dynamite*, which do work by exploding; a *Leyden jar* charged with electricity, which does work by being

discharged; *coal, storage batteries,* a *head of water,* etc. It is a evident that this potential energy must be converted into kine energy before work can be done. A familiar example of this tra formation may be seen in the action of a common pendulum. At end of the swing it is at rest for a moment and all its energy is pot tial. When, under the action of gravity, it has reached the low point, it can do no more work in virtue of its position. It has acquir however, a certain velocity, and in virtue of this velocity it does w which enables it to rise on the other side of the swing. At int mediate points its energy is partly kinetic and partly potential.

A measure of energy, or of the capacity for doing work, is *work done.*

The energy is exactly equivalent to the actual work done in following cases:

(*a*) If the effort exerted and the resistance have a common po of application.

(*b*) If the points of application are different but are rigidly c nected.

(*c*) If the energy is transmitted from member to member, p vided the members do not change form under stress, and that energy is absorbed by frictional resistance or restraint at the conn tions.

Generally speaking, work is of two kinds, viz., *internal work,* work done against the mutual forces exerted between the molecu of a body or system of bodies, and *external work,* or work done by against the external forces to which the body or bodies are subject In cases (*a*), (*b*), (*c*), above, the internal work is necessarily *nil.*

As a matter of fact, every body yields to some extent un stress, and work must be done to produce the deformation. F tional resistances tend to oppose the relative motions of memb and must also absorb energy. If, however, the work of deformati and the work absorbed by frictional resistance are included in term *work done,* the relation still holds that

$$\text{Energy} = \text{work done.}$$

A measure of work done is the *product of the resistance by distance through which it is overcome.* When a man raises a weight one pound one foot against the action of gravity he does a cert amount of work. To raise it two feet he must do twice as much work, and ten times as much to raise it ten feet. The amount of work must therefore be proportional to the number of feet through which the weight is raised. Again, to raise two pounds one foot requires twice as much work as to raise one pound through the same distance; while five times as much work would be required to raise five pounds, and ten times as much to raise ten pounds. Thus the amount of work must also be proportional to the weight raised. Hence a measure of the work done is the product of the number of pounds by the number of feet through which they are raised, the resulting number being designated *foot-pounds*. Any other units, e.g., a pound and an inch, a ton and an inch, a kilogramme and a metre, etc., may be chosen, and the work done represented in inch-pounds, inch-tons, kilogram-metres, etc. This standard of measurement is applicable to all classes of machinery, since every machine might be worked by means of a pulley driven by a falling weight.

3. **Oblique Resistance.**—Let a body move against a resistance R inclined at an angle θ to the direction of motion (Fig. 278). No work is done against the normal component $R \sin \theta$, as there is no movement of the point of application at right angles to the direction of motion. This component is, therefore, merely a pressure. The work done against the tangential component $R \cos \theta$ between two consecutive points M and N of the path of the body is $R \cos \theta \cdot MN$. Hence the total work done between any two points A and B of the path

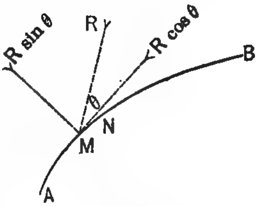


FIG. 278.

$$= \Sigma(R \cos \theta \cdot MN) = \int_0^s R \cos \theta ds,$$

s being the length of AB.

If AB is a straight line (Fig. 279), and if R is constant in direction and magnitude,

$$\text{the total work} = R \cos \theta \cdot AB = R \cdot AC,$$

AC being the projection of the displacement upon the line of action of the resistance. Let the path be the arc of a circle (Fig. 280) subtending an angle α at the centre. If R and θ remain consta the work done from A to B.

$$= R\cos\theta \times \text{arc } AB = R\cos\theta\cdot OA\cdot\alpha = R\cdot OM\cos\theta\cdot\alpha = Rp\alpha = M\alpha$$

p being the perpendicular from O upon the direction of R, a $M = Rp$ being the moment of resistance to rotation.

If there are more resistances than one, they may be trea

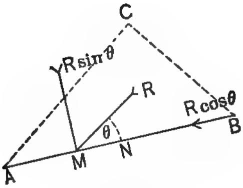


FIG. 279.

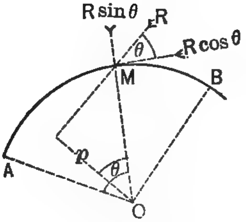


FIG. 280.

separately and their several effects superposed. In such ca M will be the total moment of resistance and will be equal to algebraic sum of the separate moments.

The normal component $R\sin\theta$ produces a pressure.

4. **Graphical Method.**—Let a body describe a path AB (Fig. 2 against a variable resistance of such character that its magnitude in the dir tion of motion may be represented at a point M by an ordinate MN to the curve C Let the path AB be subdivided into number of parts, each part MP being small that the resistance from M to P m be considered uniform. The mean va of this resistance $= \dfrac{MN+PQ}{2}$, and the w done in overcoming it $= \dfrac{MN+PQ}{2}MP =$ the area $MNQP$ in limit. Hence the total work done from A to $B =$ the area bound by the curves AB, CD, and the ordinates AC, BD.

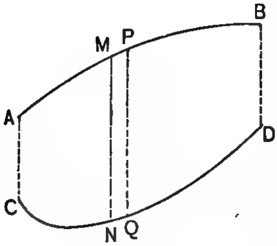


FIG. 281.

5. **Energy. Impulse. Momentum.**—The *work done* in overcomi a resistance of R lbs. through a distance of s ft. is Rs ft.-lbs.

The *power* required to overcome a resistance of R lbs. at a velocity of v f./s. is Rv in ft.-lbs. per second.

A *horse-power*, usually written H.P., is an arbitrary unit which was defined by Watt as the *power* which accomplished 33,000 ft.-lbs. per minute or 550 ft.-lbs. per second. Thus the H.P. required to overcome the resistance of R lbs. at a velocity of v f./s. is $\frac{Rv}{550}$.

One *foot-pound* = 1.356 *joules*.
One *British thermal unit* = 1058 *joules*.
One *watt* = one joule per second, and is the work done by a current of 1 ampere at 1 volt.

In electricity, however, power is usually reckoned in *kilowatts*, each of 1000 watts.

One H.P. $= 550 \times 1.356 = 746$ watts, so that
One kilowatt $= 1000 \div 746 = 1.34$ H.P.

An average of 14 watts per candle-power is required for an incandescent electric lamp, and a lamp of 2000 candle-power must therefore absorb 8 kilowatts, or about 11 H.P.

By the preceding article, when the body starts from rest, i.e., when $V=0$,

$$\frac{v^2}{2} = as = \frac{Pg}{W}s$$

and

$$v = at = \frac{Pg}{W}t.$$

Therefore

$$Ps = \frac{W}{g}\frac{v^2}{2}$$

and

$$Pt = \frac{W}{g}v.$$

Thus the terms Ps and $\frac{W}{g}\frac{v^2}{2}$ are convertible and so also are the terms Pt and $\frac{W}{g}v$.

But Ps, the product of a force of P pounds acting *in its own* *direction* through a distance of s ft., is the *work done* in ft.-lbs. Therefore $\frac{W}{g}\frac{v^2}{2}$ is also measured in ft.-lbs. and is called the *kinetic energy* or stored-up work of a body of W lbs. moving with a velocity of v f/s.

Again, Pt, the product of a force of P lbs. acting for t seconds, is its *impulse* and is expressed in *second-pounds*. Hence, too, the product $\frac{W}{g}v$, which is the mechanical equivalent of Pt, is expressed in second-pounds and is called the *momentum* or *quantity of motion* of a body of W lbs. moving with a velocity of v f/s.

The formulæ obtained may now be tabulated as follows:

When the body starts from rest, i.e. when $V=0$,

$$Ps=\frac{1}{2}\frac{W}{g}v^2 \text{ in foot-pounds,}$$

$$Qs=\tfrac{1}{2}Wv^2 \text{ in foot-poundals,}$$

$$Pt=\frac{W}{g}v \text{ in second-pounds,}$$

$$Qt=Wv \text{ in second-poundals.}$$

When the body starts with a velocity of V f./s.:

$$as=\tfrac{1}{2}v^2-\tfrac{1}{2}V^2 \quad \text{and} \quad at=v-V.$$

Therefore

$$Ps=\frac{1}{2}\frac{W}{g}v^2-\frac{1}{2}\frac{W}{g}V^2 \text{ in foot-pounds,}$$

$$Qs=\tfrac{1}{2}Wv^2-\tfrac{1}{2}WV^2 \text{ in foot-poundals,}$$

$$Pt=\frac{W}{g}v-\frac{W}{g}V \text{ in second-pounds,}$$

$$Qt=Wv-WV \text{ in second-poundals.}$$

The relation
$$\frac{W}{g}v=Pt$$

is the analytical statement of Newton's Second Law of Motion, which has been expressed by Clerk Maxwell in the following form: "This *change of momentum* is numerically *equal to the impulse which produces it, and is in the same direction.*"

This result is also true for two or more bodies or systems of bodies severally acted upon by extraneous forces, and the equation may be written

$$\Sigma mv = \Sigma Ft.$$

Hence, *the total change of momentum in any assigned direction is equal to the algebraic sum of the impulses in the same direction.* Therefore, also, if there are no extraneous forces, *the total momentum in any assigned direction is constant,* which is the principle of the conservation of linear momentum.

The relation

$$Ps = \frac{1}{2}\left(\frac{W}{g}v^2 - \frac{W}{g}V^2\right)$$

is the analytical expression of the statement that Ps, the *work done,* is equal to the *change of kinetic energy* in a given interval.

If the body is a material particle of a connected system, a similar result holds for every other particle of the system, and denoting *algebraic sum* by the symbol Σ,

$$\Sigma Ps = \frac{1}{2}\left(\Sigma\frac{W}{g}v^2 - \Sigma\frac{W}{g}V^2\right),$$

so that the sum of the work done by the several forces is equal to the total change of kinetic energy. This is a particular case of the *principle of conservation of energy* which asserts that *energy is indestructible.*

This principle, like Newton's laws of motion, admits of no general proof, but every experiment verifies its truth.

A part of the work ΣPs may be expended in doing what is called (*a*) *effective* or *useful work,* as, e.g., in overcoming an external resistance, and (*b*) *wasted work,* as, e.g., in overcoming frictional resistance.

Denoting by T_e the total effective work and by T_m the total available or *motive* work,

$$T_m - T_e = \Sigma Ps = \frac{1}{2}\left(\Sigma\frac{W}{g}v^2 - \Sigma\frac{W}{g}V^2\right)$$

$$= \text{the total change of kinetic energy.}$$

If it requires an expenditure of T_m ft.-lbs. of work to drive a machine giving T_e ft.-lbs. of *useful work*, the *efficiency* of the machine is defined to be the ratio of the useful to the total work or $\frac{T_e}{T_m}$.

In the case of a machine working at a normal speed, the velocities of the different parts are periodic, being the same at the beginning and end of any period or number of periods. For any such interval $v = V$, and therefore

$$T_m = T_e,$$

so that there is an equivalence between the motive and effective work.

It requires an expenditure of Wh ft.-lbs. of work to raise a weight of W pounds from rest to rest through a vertical distance of h feet. The weight will then possess an equivalent amount of *potential energy*, and if it is allowed to fall freely through the vertical distance of h feet it acquires a velocity of v f/s. given by

$$\frac{1}{2}\frac{v^2}{g} = h,$$

and therefore

$$\frac{1}{2}\frac{W}{g}v^2 = Wh.$$

This shows the equivalence between the *work done* in raising the weight, the *potential energy* of the weight at its highest point, and the work given out in the form of *kinetic energy* in falling freely.

Suppose that the weight moving with a velocity of v f/s strikes a second body, and that the point of application moves in the direction of the blow, through a small distance x against a mean resistance R'. Then

$$\begin{aligned} R'x &= \text{work required to overcome } R', \\ &= \text{kinetic energy of } W, \\ &= \frac{1}{2}\frac{W}{g}v^2 = W(h+x). \end{aligned}$$

Within a certain limit, called the *limit of elasticity*, the actual resistance is directly proportional to the distance through which the point of application moves, and therefore varies *uniformly* from *nil* to a maximum resistance R. Then

$$R' = \frac{R}{2}$$

and therefore

$$Rx = 2W(h+x).$$

Hence, if W is *suddenly applied from rest,*

$$h = 0$$

and $$R = 2W,$$

so that the effect of the *sudden application* is to develope a resistance equal to *twice* the weight.

6. Triangle and Parallelogram of Velocity and Acceleration.—So far the motion of a body on a straight line has only been considered. Let a steamer in t secs. move from O to A with a velocity of v_1 f/s., and in the same time let a body move across the deck in a direction parallel to OB with a velocity of v_2 f/s, Fig. 282. In t secs. the body will be at a point C defined by $OA = v_1t$ and $AC = v_2t$, AC being parallel to OB. The ratio $\frac{AC}{OA} = \frac{v_2t}{v_1t} = \frac{v_2}{v_1}$ is constant, and therefore C must describe the straight line OC. Also the ratio $\frac{OC}{OA}$ is constant, and since OA is described at a constant rate, so also is OC, and therefore the resultant of v_1 and v_2 is a constant velocity. Taking t to be *one* second, then $OA = v_1$, $AC = v_2$, and OC, the actual displacement of the body in a unit of time, is the resultant velocity in direction and magnitude. It is the diagonal, *drawn from the starting-point O,* of the parallelogram $OACB$.

Again, if OA and AC are taken to represent the increase (or *growth*) of component velocities, OC must represent the increase (or *growth*) of the resultant velocity.

Thus the parallelograms of velocity and acceleration are established and velocities and accelerations may be resolved and compounded in accordance with precisely the same rules that govern the triangle of forces.

Relative Velocity.—The velocity of one body *relatively* to another is the velocity with which the first body would *appear* to move if the observer were moving with the second body. The *relative* motion is of course unaffected if the same velocity is applied in the same direction to each body. Let a body at O move in the direction OA with a velocity of v_1 f/s., and let a body at D move in the direction DE with a velocity of v_2 f/s, Fig. 283. Apply to O and to D a velocity of v_2 f/s. in a direction opposite to the motion of D. Then D is brought to rest, while O has two simultaneous velocities, the one, v_1, in the direction of OA, and the other, v_2, in the direction of OB, parallel to DE.

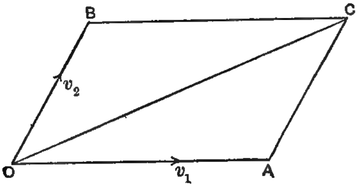


FIG. 282.

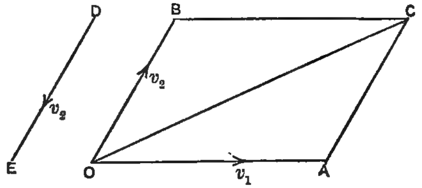


FIG. 283.

Taking $OA = v_1$ and $OB = v_2$, the diagonal OC is the resultant of these two velocities and represents in direction and magnitude the relative velocity, i.e., the velocity of O as seen from D.

7. Equation of Motion and Energy.—Let x_1, $\bar{y}_1$, $\bar{z}_1$ be the co-ordinates of the C. of G. of a moving body of mass M with respect to three rectangular axes at any given instant.

Let $\bar{x}_2$, $\bar{y}_2$, $\bar{z}_2$ be the co-ordinates of the same point after a unit of time.

Let x_1, y_1, z_1 be the co-ordinates of any particle of mass m at the given instant.

Let x_2, y_2, z_2 be the co-ordinates of the same particle after a unit of time. Then

$$M\bar{x}_1 = \Sigma(mx_1), \quad M\bar{y}_1 = \Sigma(my_1), \quad M\bar{z}_1 = \Sigma(mz_1);$$

$$M\bar{x}_2 = \Sigma(mx_2), \quad M\bar{y}_2 = \Sigma(my_2), \quad M\bar{z}_2 = \Sigma(mz_2);$$

therefore $M(\bar{x}_2 - \bar{x}_1) = \Sigma m(x_2 - x_1)$, $\quad M(\bar{y}_2 - \bar{y}_1) = \Sigma m(y_2 - y_1)$,

$$M(\bar{z}_2 - \bar{z}_1) = \Sigma m(z_2 - z_1),$$

or $$M\bar{u} = \Sigma mu, \quad M\bar{v} = \Sigma mv, \quad M\bar{w} = \Sigma mw,$$

$\bar{u}$, $\bar{v}$, $\bar{w}$ being the component velocities of the C. of G. at the given instant with respect to the three axes, and u, v, w the component velocities of the particle m at the same instant.

From these last equations,

$$M\bar{u}^2 = \Sigma mu\bar{u}, \quad M\bar{v}^2 = \Sigma mv\bar{v}, \quad M\bar{w}^2 = \Sigma mw\bar{w}.$$

Therefore $$M(\bar{u}^2 + \bar{v}^2 + \bar{w}^2) = \Sigma m(u\bar{u} + v\bar{v} + w\bar{w}),$$

which may be written in the form

$$M(\bar{u}^2 + \bar{v}^2 + \bar{w}^2) + \Sigma m\,\{(u - \bar{u})^2 + (v - \bar{v})^2 + (w - \bar{w})^2\} = \Sigma m(u^2 + v^2 + w^2),$$

or $$MU^2 + \Sigma mV^2 = \Sigma mv^2,$$

U being the resultant velocity of the C. of G., v that of the particle, and V that of the particle relatively to the C. of G.

The last equation may be written

$$\frac{MU^2}{2} + \frac{\Sigma mV^2}{2} = \frac{\Sigma mv^2}{2}.$$

Thus the energy of the total mass collected at the centre of gravity, together with the energy relatively to the centre of gravity, is equal to the total energy of motion.

If the body revolves around an axis through its C. of G. with an angular velocity ω, the second term of the last equation becomes

$$\frac{1}{2}\Sigma mr^2\omega^2 = \frac{\omega^2}{2}\Sigma mr^2 = \frac{\omega^2}{2}I,$$

r being the distance of the particle m from the axis and I the moment of inertia of the body with respect to the axis.

Again, let X, Y, Z be the forces parallel to the axes of x, y, z, respectively, acting upon a particle of mass m. Then

$$\Sigma m\frac{d^2x}{dt^2} = \Sigma X,$$

$$\Sigma m\frac{d^2y}{dt^2} = \Sigma Y,$$

and
$$\Sigma m\frac{d^2z}{dt^2}=\Sigma Z.$$

Therefore

$$\Sigma m\left(\frac{dx}{dt}\,\frac{d^2x}{dt^2}+\frac{dy}{dt}\,\frac{d^2y}{dt^2}+\frac{dz}{dt}\,\frac{d^2z}{dt^2}\right)=\Sigma\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)$$

Integrating with respect to t,

$$\frac{1}{2}\Sigma m\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)=\int\Sigma(Xdx+Ydy+Zdz)+H,$$

H being a constant of integration,

or
$$\tfrac{1}{2}\Sigma mv^2=\int\Sigma(Xdx+Ydy+Zdz)+H.$$

Hence, if v_0 is the initial velocity,

$$\tfrac{1}{2}(\Sigma mv^2-\Sigma mv_0^2)=\int\Sigma(Xdx+Ydy+Zdz)$$

= the work done on the system.

8. Angular Velocity. Centrifugal Force.—Angular velocity may be defined as the number of radians per second, a radian being the angle subtended at the centre of a circle by an arc equal in length to the radius. This angle is $\frac{180°}{\pi}=57.2958$ degrees, and if a wheel makes N revolutions per minute, its angular velocity is $\frac{2\pi N}{60}$ radians per second.

A body *constrained* to move in a plane curve exerts upon the body which constrains it a force called *centrifugal force*, which is equal and opposite to the *deviating* (or *centripetal*) force exerted by the constraining body upon the revolving body.

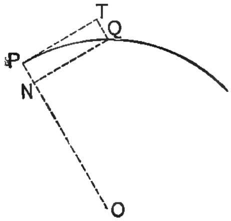


FIG. 284.

Let a particle of mass m move from a point P to a consecutive point Q (Fig. 284) of its path during an interval of time t under the action of a normal deviating force.

Let the normals at P and Q meet in O; PQ may be considered as the indefinitely small arc of a circle with its centre at O.

If there were no constraining force, the body would move along the tangent at P to a point T such that $PT=vt$, v being the linear velocity at P.

Under the deviating force the body is pulled towards O through a distance $PN=\frac{1}{2}ft^2$, f being the normal acceleration and QN being drawn perpendicular to OP.

Also, *in the limit,* $$PQ=PT=QN=vt.$$

But $$QN^2=PN\times 2OP.$$

Therefore $$v^2t^2=\tfrac{1}{2}ft^2 2R,$$

R being the radius OP, and hence

$$f=\frac{v^2}{R}=\omega^2 R,$$

ω being the angular velocity.

Hence the deviating force of the mass m

$$=mf=m\frac{v^2}{R}=m\omega^2 R,$$

and is equal and opposite to the centrifugal force.

Again, if a solid body of mass M revolve with an angular velocity ω about an axis passing through its C. of G., the total centrifugal force will be *nil,* provided the axis of rotation is an axis of symmetry, or is one of the principal axes of inertia at the C. of G.

If the axis of rotation is parallel to one of these axes, but at a distance $\overline{R}$ from the C. of G.,

$$\text{the centrifugal force}=\Sigma mr\omega^2=\omega^2\Sigma mr=\omega^2 M\overline{R}=\frac{W}{g}\omega^2\overline{R},$$

r being the distance of a particle of mass m from the axis and W the weight of the body. Thus the centrifugal force is the same as if the whole mass were concentrated at the C. of G.

If the axis of rotation is inclined at an angle θ to the principal axis, the body will be constantly subjected to the action of a couple of moment $2E \tan \theta$, E being the actual energy of the body.

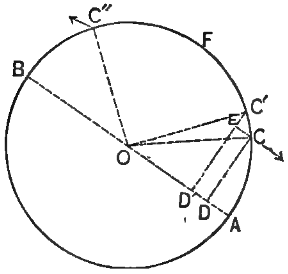


FIG. 285.

Consider, for example, the case of a ring of radius r rotating with angular velocity ω about its centre O. Let p be the weight of the ring per unit of length of periphery. Consider any half-ring AFB. The centrifugal force of any element

$$CC' = \frac{pCC'}{g} \omega^2 r.$$

The component of this force parallel to AB is balanced by an equal and opposite force at C'', the angle $C''OB$ being $=$ the angle COA. Thus the total centrifugal force parallel to AOB is *nil.*

The component of the force at C, perpendicular to AB,

$$= \frac{pCC'}{g} \omega^2 r \sin COD = \frac{pCC'}{g} \omega^2 r \cos C'CE$$

$$= \frac{pCC'}{g} \omega^2 r \frac{CE}{CC'} = p \frac{\omega^2 r}{g} DD'.$$

Hence, the total centrifugal force perpendicular to AB

$$= p \frac{\omega^2 r}{g} \Sigma (DD') = 2 \frac{p}{g} \omega^2 r^2.$$

If T is the force developed in the material at each of the points A and B,

$$2T = 2 \frac{p}{g} \omega^2 r^2,$$

since the direction of T is evidently perpendicular to AB,

and therefore $$T = \frac{p}{g} \omega^2 r^2 = \frac{p}{g} v^2,$$

v being the circumferential velocity.

Let f be the *intensity* of stress at A and B, and w the specific weight of the material.

Assuming that T is distributed uniformly over the sectional areas at A and B,

$$f=\frac{w}{g}v^2.$$

Thus, the stress is independent of the radius for a given value of v, and the result is applicable to every point of a flexible element, whatever may be the form of the surfaces over which it is stretched.

9. Ex. 1. *A train of W lbs. gross weight starts from a station at A and runs on the level to a station at D, l ft. away. If the average speed is not to exceed v f/s, find the time between the two stations.*

If the time is not limited, find the LEAST *time in which the run from A to D can be made and the maximum speed attained.*

Let P lbs. be the average pull exerted by the engine;
R " " " " road resistance;
B " " " " brake resistance.

Under the action of the force $P-R$ the speed of the train gradually increases from *nil* at A to v f/s at B, and the train then possesses a kinetic energy of $\frac{1}{2}\frac{W}{g}v^2$ ft.-lbs. The train runs at the uniform speed of v f/s from B to C, when steam is shut off, the brakes applied, and the train is gradually brought to rest at D, its kinetic energy having been absorbed by the force $B+R$ acting through the distance $CD=q$.

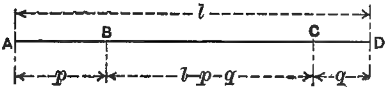


FIG. 286.

Hence, taking $AB=p$,

the time between A and D

$$=\text{time from } A \text{ to } B+\text{time from } B \text{ to } C+\text{time from } C \text{ to } D$$

$$=\frac{p}{\frac{1}{2}v}+\frac{l-p-q}{v}+\frac{q}{\frac{1}{2}v}=\frac{l}{v}+\frac{p+q}{v}. \quad \dots\dots\dots\dots (1)$$

Also,
$$(P-R)p=\frac{1}{2}\frac{W}{g}v^2=(B+R)q. \quad \dots\dots (2)$$

Substituting in (1) the values of p and q from (2), the time between A and D

$$=\frac{l}{v}+\frac{1}{2}\frac{Wv}{g}\frac{P+B}{(P-R)(B+R)}.$$

Secondly, if the speed is not limited, it gradually increases from *nil* at A to a maximum at E, where $AE=m$. Steam is then shut off, the brakes applied, and the train is gradually brought to rest in the distance $ED = l - m$.

Taking $v_{\text{max.}}$ f/s as the max. speed attained,

$$(P-R)m = \frac{1}{2}\frac{W}{g}(v_{\text{max.}})^2 = (B+R)(l-m),$$

and therefore

$$v_{\text{max.}} = \left\{\frac{2lg}{W}\,\frac{(P-R)(B+R)}{P+B}\right\}^{\frac{1}{2}} \text{f/s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also, the least time between A and D

$$= \text{time from } A \text{ to } E + \text{time from } E \text{ to } D$$

$$= \frac{2m}{v_{\text{max.}}} + \frac{2(l-m)}{v_{\text{max.}}} = \frac{2l}{v_{\text{max.}}}$$

$$= \left\{\frac{2lW}{g}\,\frac{P+B}{(P-R)(B+R)}\right\}^{\frac{1}{2}} \text{secs.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the trains runs up an incline of 1 *in* m, *the pull* P *of the engine has to do the additional work of lifting the weight* W *lbs. of the train through the vertical distance* $\frac{1}{m}$ *every second. The times and maximum speed can therefore be found from the expressions* (2), (3), *and* (4) *by substituting* $R + \frac{W}{m}$ *for* R.

Again, if E lbs. is the weight on the locomotive drivers, and if the adhesion is μ times the weight, then

$$B = \mu W,$$

and the max. pull $P = \mu E$.

The steepest incline up which an engine can climb is $\frac{W}{\mu E}$.

In practice μ usually varies from $\frac{1}{4}$ to $\frac{1}{6}$.

The H.P. required to run a train of W tons at a speed of M miles per hour up or down an incline having a slope of $\alpha°$, and against an average road resistance on the level at this speed of R lbs. per ton, is

$$(R \cos \alpha \pm 2000 \sin \alpha)\frac{WM}{375},$$

which for a light incline of 1 in m becomes

$$\left(R \pm \frac{2000}{m}\right)\frac{WM}{375}, \text{ approximately.}$$

Ex. 2. *The rim of a fly-wheel weighing* 5000 *lbs. has a velocity of* 40 *f/s; what is its kinetic energy? What is the loss of kinetic energy due to a reduction of* 4 *per cent in the rim velocity? How much is the rim velocity reduced* *by a diminution of 45,000 ft.-lbs. in the kinetic energy? If .08 is the coefficient of axle friction, and if the diameter of the fly-wheel is 12 times that of the axle, how many H.P. will be required to turn the wheel?*

The kinetic energy $=\frac{5000}{32}\frac{40^2}{2}=125{,}000$ ft.-lbs.

The rim velocity reduced 4 per cent becomes $(40-1.6)=38.4$ f/s., and the corresponding kinetic energy $=\frac{5000}{32}\frac{(38.4)^2}{2}$.

Therefore the loss of kinetic energy

$$=\frac{5000}{32}\left(\frac{40^2}{2}-\frac{\overline{38.4}^2}{2}\right)=9800 \text{ ft.-lbs.}$$

If v f/s. is the rim velocity when the kinetic energy is diminished by 45,000 ft.-lbs.,

$$\frac{5000}{32}\left(\frac{40^2}{2}-\frac{v^2}{2}\right)=45{,}000, \quad \text{and therefore} \quad v=32 \text{ f/s.},$$

so that the reduction of velocity is 8 f/s.

The circumferential velocity of the axle $=\frac{40}{12}=\frac{10}{3}$ f/s.

The weight on the axle $=5000$ lbs.

The frictional resistance at the axle surface $=.08\times 5000$
$=400$ lbs.

Therefore the H.P. required to turn the wheel

$$=\frac{400\times\frac{10}{3}}{550}=2\tfrac{14}{33}.$$

Ex. 3. *A weight of W_1 tons falls h ft. and by n successive blows drives an inelastic pile weighing W_2 tons a feet into the ground. Find the mean effective resistance of the ground. If the ground resistance is directly proportional to the depth of penetration, how far will the pile sink under the rth blow? If the head of the pile is crushed for a length of x ft., x being very small as compared with the depth $\frac{a}{n}$ of penetration, find* (1) *the mean thrust, during the blow, between the weight and the pile;* (2) *the time of penetration;* (3) *the time during which the blow acts.*

As the pile is inelastic, the weight does not rebound, but the pile and weight, immediately the blow is struck, move along together with a common velocity of u f/s., given by

$$\frac{W_1+W_2}{g}u=\frac{W_1}{g}v,$$

since the momentum does not change.

Thus the available energy of the pile and weight

$$=\frac{1}{2}\frac{W_1+W_2}{g}u^2=\frac{W_1^2}{W_1+W_2}\frac{v^2}{2g}=\frac{W_1^2h}{W_1+W_2}.$$

This energy is sufficient to carry the pile and weight through a distance of $\frac{a}{n}$ ft., against R_e, the mean effective ground resistance in tons, and therefore

$$R_e\frac{a}{n}=\frac{W_1^2h}{W_1+W_2},$$

or

$$R_e=\frac{W_1^2}{W_1+W_2}\frac{nh}{a},$$

so that the pile and weight will slowly sink under a superposed weight of R_e tons.

To allow for gravity, the value of R_e should be increased by W_1+W_2.

Let T be the mean thrust between the weight and pile for the very small interval of t seconds, during which the head of the pile is crushed through the distance of x ft. The thrust T is necessarily very great as compared with R_e, which may be disregarded without causing any sensible error. Hence

$$Tx=\text{the work expended in the crushing}$$

$$=W_1h-\frac{W_1^2h}{W_1+W_2}=\frac{W_1W_2h}{W_1+W_2}\text{ ft.-tons}$$

and

$$T=\frac{W_1W_2}{W_1+W_2}\frac{h}{x}\text{ tons.}$$

Also, T is necessarily very great as compared with R, which may be disregarded without causing any appreciable error.

Therefore

$$Tt=\text{impulse}$$

$$=\text{change of momentum}$$

$$=\frac{W_1}{g}(v-u)=\frac{W_1W_2}{W_1+W_2}\frac{v}{g}$$

$$=T\frac{x}{h}\times\frac{v}{g},$$

and

$$t=\frac{x}{h}\times\frac{v}{g}=\frac{x}{h}\times\text{time of fall of weight.}$$

After the blow the pile sinks $\frac{a}{n}$ ft. with an average velocity of $\frac{u}{2}$ f/s, and the *time of penetration* $=\frac{a}{n}\div\frac{u}{2}=\frac{2a}{n}\frac{W_1+W_2}{W_1 v}$ secs.

$$=\frac{W_1+W_2}{W_1}\frac{a}{4n\sqrt{h}}.$$

If the ground resistance is proportional to the depth of penetration, let

$x_r =$ penetration produced by r blows;

$x_{r-1} =$ " " " $r-1$ blows.

Then $x_r - x_{r-1} =$ " " " the rth blow against an average resistance of $\frac{x_r+x_{r-1}}{2}\frac{R_e}{a}$.

Hence $$\frac{x_x+x_{r-1}}{2}\frac{R_e}{a}\times(x_x-x_{r-1}) = \text{work done by each blow}$$

$$=\frac{R_e}{2}\frac{a^2}{n},$$

or $$x_r^2-x_{r-1}^2=\frac{a^2}{n}.$$

So, $$x_{r-1}^2-x_{r-2}^2=\frac{a^2}{n},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$x_2^2-x_1^2=\frac{a^2}{n},$$

$$x_1^2-0=\frac{a^2}{n},$$

and therefore $$x_r^2=r\frac{a^2}{n},\quad\text{or}\quad x_r=\frac{a}{\sqrt{n}}\sqrt{r},$$

so that the penetration under the rth blow

$$=x_r-x_{r-1}=\frac{a}{\sqrt{n}}(\sqrt{r}-\sqrt{r-1}).$$

If $R_{\text{max.}}$ is the maximum ground resistance,

$$R_{\text{max.}}=2R_e=\frac{W_1^2}{W_1+W_2}\frac{2nh}{a}.$$

The time of penetrating the depth $x = x \div \frac{1}{2}v = \frac{x}{4\sqrt{h}}$.

If x is not small as compared with the depth of penetration of a blow, let y and z be the distances, in feet, through which the pile moves *during the action of the blow* and *after* the blow. Then

$$Tx + R_ey = \text{work done in the crushing and in overcoming the ground resistance in } t \text{ seconds}$$

$$= W_1h - \frac{1}{2}\frac{W_1 + W_2}{g}u^2,$$

$$Tt = \text{change of momentum of } W_1$$

$$= \frac{W_1}{g}(v - u),$$

$$Tt - R_et = \text{change of momentum of pile}$$

$$= \frac{W_2}{g}u,$$

and

$$R_ez = \text{work done after the blow}$$

$$= \frac{1}{2}\frac{W_1 + W_2}{g}u^2 = W_1h - Tx - R_ey,$$

so that

$$Tx + R_e(y + z) = W_1h,$$

or

$$Tx + R_ea = W_1h.$$

Ex. 4. *Let a body of weight W_1 moving in a given direction with a velocity v_1 strike a body of weight W_2 moving in the same direction with a velocity v_2. After impact let the bodies continue to move in the same direction with a common velocity u. Find the energy lost in impact.*

$$\frac{W_1}{g}v_1 + \frac{W_2}{g}v_2 = \text{momentum before impact}$$

$$= \text{momentum after impact}$$

$$= \left(\frac{W_1}{g} + \frac{W_2}{g}\right)u,$$

or

$$W_1v_1 + W_2v_2 = (W_1 + W_2)u.$$

Energy before impact $= \frac{W_1}{g}\frac{v_1^2}{2} + \frac{W_2}{g}\frac{v_2^2}{2}$.

" after " $= \left(\frac{W_1 + W_2}{g}\right)\frac{u^2}{2}$.

Energy lost by impact
$$=\frac{1}{2g}(W_1v_1^2+W_2v_2^2)-\frac{u^2}{2g}(W_1+W_2)$$
$$=\frac{W_1W_2(v_1-v_2)^2}{2g(W_1+W_2)}.$$

If either of the bodies is subjected to any constraint, energy must be expended to overcome such constraint, and the loss of energy by impact will be less.

Ex. 5. *Let a hammer weighing W_1 lbs., moving with a velocity of v ft. per second, strike a nail weighing W_2 lbs. and drive it x ft. into a piece of timber, of weight W_2, against a mean resistance of R lbs.*

First, assume the timber to be fixed in position.

Let u_1 be the common velocity acquired by the hammer and nail.

$$(W_1+W_2)\frac{u_1^2}{2g}=\text{energy expanded in overcoming } R.$$
$$=Rx. \qquad (1)$$

But
$$\frac{W_1}{g}v=\text{change of momentum}=\frac{W_1+W_2}{g}u_1. \qquad (2)$$

Therefore
$$\frac{W_1^2}{W_1+W_2}\frac{v^2}{2g}=Rx, \qquad (3)$$

and the *time of the penetration* $=\frac{x}{\frac{1}{2}u_1}=\frac{W_1v}{Rg}$ sec. $\qquad (4)$

Second, let the timber be free to move, and let u_2 be the common velocity acquired by the hammer, nail, and timber.

$(W_1+W_2)\frac{u_1^2}{2g}=$ energy expended in overcoming R *plus* the energy expended in producing the velocity u_2

$$=Rx+(W_1+W_2+W_3)\frac{u_2^2}{2g}. \qquad (5)$$

But
$$\frac{W_1}{g}v=\frac{W_1+W_2}{g}u_1=\frac{W_1+W_2+W_3}{g}u_2. \qquad (6)$$

Hence, substituting these values of u_1 and u_2 in eq. (5),

$$\frac{W_1^2W_3}{(W_1+W_2)(W_1+W_2+W_3)}\frac{v^2}{2g}=Rx; \qquad (7)$$

also, the *time of the penetration*

$$=\frac{x}{\frac{1}{2}u_1}=\frac{W_1W_3}{W_1+W_2+W_3}\frac{v}{gR}\text{ sec.}, \qquad (8)$$

and the distance through which the timber moves

$$=\frac{u_2}{2}t=\frac{W_1{}^2W_3}{(W_1+W_2+W_3)^2}\frac{v^2}{2gR}\text{ ft.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Ex. 6. *An accumulator, loaded to a pressure of* 750 *lbs. per square inch, has a ram of* 21 *ins. diameter with a stroke of* 24 *ft. How much H.P. can be obtained for a period of* 50 *seconds?*

$$\text{H.P. required}=\frac{1}{4}\,\frac{22}{7}(21)^2\,\frac{750\times 24}{550\times 50}$$
$$=226.8.$$

Ex. 7. *Ten thousand* 50-*watt incandescent and two hundred and fifty* 450-*watt arc lamps are to be supplied with power from a waterfall,* 20 *miles away, having an effective head of* 40 *ft. The efficiency of the converting apparatus is* 92 *per cent, of the turbine* 85 *per cent, and the losses are* 5 *per cent between the lamps and converters at the receiving end of the transmission,* 10 *per cent on the line, and* 10 *per cent in the generators and transformers between the line and the turbine-shaft. Find the necessary flow of water per hour.*

The total watts required $=(50.10000+450.250)\frac{100}{92}\cdot\frac{100}{85}\cdot\frac{100}{95}\cdot\frac{100}{90}\cdot\frac{100}{90}$.

Hence, if Q is the water-supply in cubic feet per second,

$$62\tfrac{1}{2}\cdot Q\cdot\frac{40}{550}=\text{H.P.}=\frac{612{,}500}{746}\cdot\frac{(100)^5}{92\cdot 85\cdot 95\cdot 90^2}$$

and $$Q=300\cdot 175\text{ c.f/sec.,}$$

and the supply in cubic feet per hour

$$=60\cdot 60\cdot Q=1{,}080{,}630.$$

Ex. 8. *A* 1-*oz. bullet moving with a velocity of* 800 *ft. per second strikes a target and is stopped dead in the space of* $\frac{3}{20}$ *inch* ($g=32$). Then

$$\tfrac{1}{2}\cdot\tfrac{1}{16}\cdot\tfrac{1}{32}\cdot(800)^2=R'\cdot\tfrac{3}{20}\cdot\tfrac{1}{12};$$

and R', the mean resistance overcome by the bullet, $=5000$ lbs. The time in which the bullet is brought to rest

$$=\frac{\text{momentum}}{\text{force}}=\frac{\frac{1}{16}\cdot\frac{1}{32}\cdot 800}{5000}=\frac{1}{3200}\text{ sec.}$$

Ex. 9. *A volume of water of length l feet moves along a pipe with a velocity of v f/s and is quickly and uniformly shut off by the closing of a valve. Find the increase in the pressure per square inch near the valve.*

If p is the increased pressure per square foot, and t secs. the interval in which the closing is effected,

$$pt=\text{momentum of the fluid mass}$$

$$=\frac{62\frac{1}{2}}{g}lv,$$

and therefore

$$\text{the pressure per square inch} = \frac{p}{144} = \frac{62\frac{1}{2}}{144}\frac{lv}{gt}.$$

Ex. 10. *A weight of W pounds of water passes through a turbine wheel from a to f. It enters at a with a velocity of v_1 f/s in a direction inclined at an angle of $90° - \gamma$ to the radius $Oa = r_1$. It leaves at f with a velocity of v_2 f/s in a direction fh inclined at an angle δ to the radius $Of = r_2$. If ω is the angular velocity of the wheel, find the work done every second by the water on the wheel.*

Let p_1, p_2 be the perpendiculars from the axis O upon the directions of v_1, v_2, respectively. Then

$$\frac{W}{g}(v_1p_1 - v_2p_2) = \text{change of angular momentum per sec. between } a \text{ and } f$$

$$= M, \textit{ the moment of impulse.}$$

Therefore the work done in foot-pounds per second

$$= M\omega = \frac{W}{g}\omega(v_1p_1 - v_2p_2).$$

Let v'_w be the component of v_1 along the tangent at a;

Let v''_w be the component of v_2 along the tangent at f.

Then $\quad p_1v_1 = p_1v'_w \sec\gamma = r_1v'_w$

and $\qquad p_2v_2 = p_2v''_w \operatorname{cosec}\delta = r_2v''_w,$

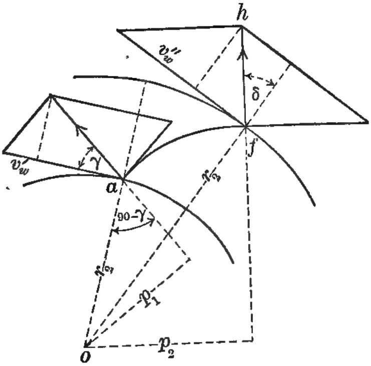


FIG. 287

so that the work in foot-pounds

$$= M\omega = \frac{W}{g}\omega(r_1v'_w - r_2v''_w)$$

$$= \frac{W}{g}(u_1v'_w - u_2v''_w),$$

where $u_1 = r_1\omega$ and $u_2 = r_2\omega$, are the peripheral lineal velocities at a and f respectively.

The components v'_w and v''_w are usually called the *whirling* velocities.

Ex. 11. *An ice-yacht travels in the direction of its keel with a velocity of v f/s under the action of a wind blowing with a velocity of w f/s in a direction making an angle β with the keel. The sail is set at an angle α with the keel,* *and it is assumed that there is no resistance to motion along the keel. Find the maximum speed of the yacht.*

In Fig. 288, take $OA=v$ and $BO=w$. Then AB is the *apparent* or relative velocity of the wind.

The yacht will be moving at full speed when AB is parallel to the sail OD,

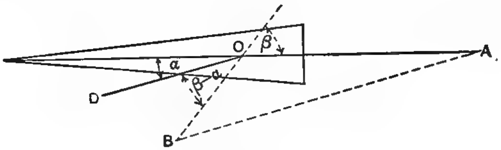


FIG. 288.

and the components of the velocities of the wind (OB) and the sail (AO) at right angles to the sail are then equal.

Therefore $$v\sin\alpha=w\sin(\beta-\alpha).$$

For given values of w and α, v is a maximum when $\beta=90°$, and then

$$v\sin\alpha=w\cos\alpha,$$

or $$v=w\cot\alpha.$$

The velocity of the boat to windward

$$=v\cos\beta=W\frac{\cos\beta\sin\overline{\beta-\alpha}}{\sin\alpha},$$

which, for a given set of sail, is greatest when

$$2\beta-\alpha=90°\quad\text{or}\quad\beta=45°+\frac{\alpha}{2}.$$

Thus the maximum speed to windward

$$=\frac{w}{2}(\operatorname{cosec}\alpha-1).$$

Ex. 12. *The charge of powder for a* 27-*ton breech-loader with a* 9-*ton carriage is* 300 *lbs.; the weight of the projectile is* 500 *lbs., its diameter is* 10 *in., and its radius of gyration* 3.535 *in.; the muzzle velocity is* 2020 *ft. per sec.; the velocity of recoil,* $16\frac{1}{4}$ *ft. per sec.; the gun is rifled so that the projectile makes one turn in* 40 *calibres.*

Total energy of explosion = energy of shot + energy of recoil;

Energy of shot = energy of translation + energy of rotation

$$=\frac{500}{32.2}\,\frac{(2020)^2}{2}+\frac{500}{32.2}\,\frac{1}{2}\left(\frac{\pi\frac{10}{12}}{\frac{5}{12}}\,\frac{2020}{40.\frac{10}{12}}\right)^2\left(\frac{3.535}{12}\right)^2$$

$$=31680124.2+97758.6$$

$$=31777882.8\text{ ft.-lbs.};$$

Energy of recoil $=\dfrac{36\times 2240}{32.2}\,\dfrac{(16\frac{1}{4})^2}{2}=330652.1$ ft.-lbs.

Hence, if C be the energy of 1 lb. of powder,

$$C\cdot 300 = 31777882.8 + 330652.1$$

$$= 32108534.9 \text{ ft.-lbs.},$$

and hence $\quad C = 107028.45$ ft.-lbs. $= 47.7$ ft.-tons.

Ex. 13. *Let W be the weight of a fly-wheel in pounds, and let its maximum and minimum angular velocities be ω_1, ω_2, respectively. The motion being one of rotation only, the energy stored up when the velocity rises from ω_2 to ω_1, or given out when it falls from ω_1 to ω_2, is*

$$\frac{I}{2}(\omega_1^2-\omega_2^2)=\frac{W}{2g}k^2(\omega_1^2-\omega_2^2)=\frac{W}{2g}(v_1^2-v_2^2),$$

v_1, v_2 being the linear velocities corresponding to ω_1, ω_2, and k being taken equal to the mean radius of the wheel.

It is usual to specify that the variation of velocity is not to exceed a certain fractional part of the mean velocity.

Let V be the mean velocity, and $\dfrac{1}{\mu}$ the fraction. Then

$$v_1-v_2=\frac{V}{\mu};\quad \text{also,}\quad v_1+v_2=2V.$$

Therefore $$\frac{v_1^2-v_2^2}{2}=\frac{V^2}{\mu}.$$

Hence the work *stored* or *given out* $=\dfrac{W}{g}\,\dfrac{V^2}{\mu}$.

Ex. 14. *An engine weighing* 64 *tons travels round a curve of* 1000 *ft. radius at the rate of* 45 *miles per hour. Find the horizontal thrust on the rails, and also find the direction and magnitude of the resultant thrust.*

The *horizontal thrust* in pounds $=\dfrac{64\times 2000}{32}\,\dfrac{1}{1000}\left(\dfrac{45\times 5280}{60\times 60}\right)^2=17{,}424.$

The *vertical* weight in pounds $=64\times 2000=128{,}000$.

Therefore the resultant thrust is inclined to the vertical at an angle whose tangent is $\dfrac{17424}{128000}=\dfrac{1.089}{8}=.136125.$

The resultant thrust in pounds

$$=\sqrt{(17{,}424)^2+(128000)^2}=129{,}181.$$

Ex. 15. *Find the total kinetic energy of a system of rigidly connected heavy particles revolving about a fixed axis with a uniform angular velocity ω.*

Let W_1, W_2, $W_3 \dots W_n$ be the weights of the particles in pounds, and let x, x_2, $x_3 \dots x_n$ be the distances, respectively, of the particles from the fixed axis.

The kinetic energies of the several weights are

$$\frac{1}{2}\frac{W_1}{g}(x_1\omega)^2,\ \frac{1}{2}\frac{W_2}{g}(x_2\omega)^2,\ \dots,\ \frac{1}{2}\frac{W_n}{g}(x_n\omega)^2.$$

Therefore the total kinetic energy

$$=\frac{\omega^2}{2}\left(\frac{W_1}{g}x_1^2+\frac{W_2}{g}x_2^2+\dots+\frac{W_n}{g}x_n^2\right)$$

$$=\frac{1}{2}\omega^2 I=\frac{1}{2}\frac{W}{g}(k\omega)^2,$$

I being the moment of inertia of the system, and k the radius of gyration.

10. Inertia—Balancing.—Newton's First Law of Motion, called also the *Law of Inertia,* states that "a body will continue in a state of rest or of uniform motion in a straight line unless it is made to change that state by external forces."

This property of resisting a change of state is termed *inertia,* and in dynamics is always employed to measure the *quantity of matter* contained in a body, i.e., its mass, to which the *inertia* must be necessarily proportional. Thus, to induce motion in a body, energy must be expended, and must again be absorbed before the body can be brought to rest. The inertia of the reciprocating parts of a machine may therefore heavily strain the framework, which should be bolted to a firm foundation, or must be sufficiently massive to counteract by its weight the otherwise unbalanced forces.

Ex. 16. Consider the case of a direct-acting horizontal steam-engine, Fig. 289. At any given instant let the crank OP and the connecting-rod CP make angles θ and ϕ, respectively, with the line of stroke AB.

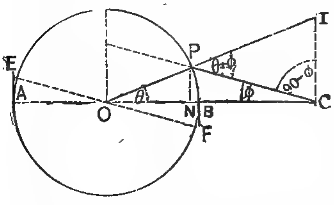


FIG. 289.

Let v be the velocity of the crank-pin centre P, and let u be the corresponding piston velocity, which must evidently be the same as that of the end C of the connecting-rod.

Let OP produced meet the vertical through C in I.

At the moment under consideration the points C and P are turning about I as an *instantaneous centre.* Therefore

$$\frac{u}{v}=\frac{IC}{IP}=\frac{\sin(\theta+\phi)}{\cos\phi}.$$

Let W be the weight of the reciprocating parts, i.e., the piston-head, piston-rod, cross-head (or motion-block), and a portion of the connecting-rod.

Assume (1) that the motion of the crank-pin centre is uniform;

(2) that the obliquity of the connecting-rod may be disregarded without sensible error, and hence $\phi = 0$.

Draw PN perpendicular to AB, and let $ON = x$; ON is equal to the distance of the piston from the centre of the stroke, corresponding to the position OP of the crank.

The kinetic energy of the reciprocating parts

$$= \frac{W}{g}\frac{u^2}{2} = \frac{W}{g}\frac{v^2 \sin^2\theta}{2} = \frac{W}{g}\frac{v^2}{2}\left(1 - \frac{x^2}{r^2}\right),$$

r being the radius OP.

Therefore the *change of kinetic energy*, or work done, corresponding to the values x_1, x_2 of x,

$$= \frac{W}{g}\frac{v^2}{2}\left(\frac{x_1^2 - x_2^2}{r^2}\right).$$

Let R be the *mean* pressure which, acting during the same interval, would do the same work. Then

$$\frac{W}{g}\frac{v^2}{2}\frac{x_1^2 - x_2^2}{r^2} = R(x_1 - x_2),$$

and therefore

$$R = \frac{W}{g}\frac{v^2}{2}\frac{x_1 + x_2}{r^2}.$$

Hence, in the limit, when the interval is indefinitely small, $x_1 = x_2 = x$, and the pressure corresponding to x becomes

$$R = \frac{W}{g}\frac{v^2}{r^2}x.$$

This is the *pressure due to inertia*, and may be written in the form

$$R = C\frac{x}{r},$$

$C\left(= \frac{W}{g}\frac{v^2}{r}\right)$ being the centrifugal force of W assumed concentrated at the crank-pin centre. R is a maximum and equal to C when $x = r$, i.e., at the points A, B, and its value at intermediate points may be represented by the vertical ordinates to AB from the straight line EOF drawn so that $AE = BF = C$. In low-speed engines C may be so small that the effect of inertia may be disregarded, but in quick-running engines C may become very large and the inertia of the reciprocating parts may give rise to excessive strains.

Another force acting upon the crank-shaft is the centrifugal force of the crank, crank-pin, and of that portion of the connecting-rod which may be supposed to rotate with the crank-pin.

Let w be the weight of the mass concentrated at the crank-pin centre which will produce the same centrifugal force as these rotating pieces (i.e., wr = sum of products of the weights of the several pieces into the distances of their centres of gravity from O).

The centrifugal force of $w = \frac{w}{g}\frac{v^2}{r}$.

Thus the total maximum pressure on the crank shaft

$$= C + \frac{w}{g}\frac{v^2}{r} = \frac{v^2}{gr}(W + w) = r(W + w)\frac{\omega^2}{g},$$

ω being the uniform *angular* velocity of the crank-pin.

This pressure may be counteracted by placing a suitable balance-weight (or weights) in such a position as to develope in the opposite direction a centrifugal force of equal magnitude.

Let W_1 be such a weight, and R its distance from O. Then

$$RW_1\frac{\omega^2}{g} = r(W + w)\frac{\omega^2}{g},$$

or

$$RW_1 = r(W + w),$$

from which, if R is given, W_1 may be obtained.

During the first half of the stroke an amount of energy represented by the triangle AEO is *absorbed* in accelerating the reciprocating parts, and the *same* amount, represented by the triangle BOF, is given out during the second half of the stroke when the reciprocating parts are being retarded.

During the up-stroke of a *vertical* engine the weights of the reciprocating parts act in a direction opposite to the motion of the piston, while during the down-stroke they act in the same direction.

In AE produced (Fig. 290) take EE' to represent the weight of the reciprocating parts on the same scale as AE represents the pressure due to inertia. Draw $E'O'F'$ parallel to EOF.

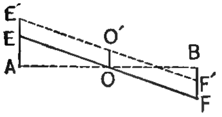


Fig. 290.

During the up-stroke the ordinates of $E'O'$ represent the pressures required to accelerate the reciprocating parts, the pressures while they are retarded being represented by the ordinates of $O'F'$.

The case is exactly reversed in the down-stroke.

N.B.—The formula $R = C\frac{x}{r}$ may be easily deduced as follows:

$$u = v \sin \theta;\ \text{the acceleration} = \frac{du}{dt} = v \cos \theta\frac{d\theta}{dt} = \frac{v^2}{r^2}x.$$

Therefore

$$\frac{W}{g}\frac{du}{dt} = \text{accelerating force} = \text{force due to inertia}$$

$$= \frac{W}{g}\frac{v^2}{r^2}x = C\frac{x}{r}.$$

Ex. 17. Consider a double-cylinder engine with two cranks at right angles and let d be the distance between the centre lines of the cylinders (Fig. 292).

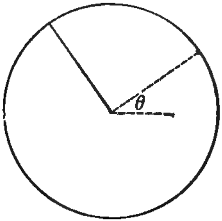


FIG. 291.

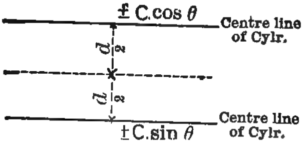


FIG. 292.

The pressures *due to inertia* transmitted to the crank-pins when one of the cranks makes an angle θ with the line of stroke are

$$P_1 = C\cos\theta \quad \text{and} \quad P_2 = C\sin\theta.$$

These are equivalent to a single alternating force

$$P = C\,(\cos\theta \pm \sin\theta)$$

acting half-way between the lines of stroke, together with a couple of moment

$$M = P\frac{d}{2} = C\frac{d}{2}(\cos\theta \pm \sin\theta).$$

The force and couple are twice reversed in each revolution, and their maximum values are

$$P_{\text{max.}} = C\sqrt{2} \quad \text{and} \quad M_{\text{max.}} = \frac{Cd}{2}\sqrt{2}.$$

In order to avoid the evils that might result from the action of the force and couple at high speeds, suitable weights are introduced in such positions that the centrifugal forces due to their rotation tend to *balance* both the force and the couple. For example, the weights may be placed upon the fly-wheel of a stationary engine, or, again, upon the driving-wheels of a locomotive.

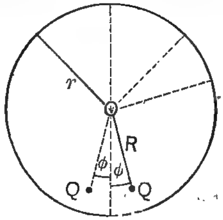


FIG. 293.

Let a balance-weight Q be placed nearly diametrically opposite to the centre of each crank-pin (Fig. 293), and let R be the distance from the axis to the centre of gravity of Q.

Let e be the horizontal distance between the balance-weights.

The centrifugal force F due to the rotation of Q

$$= \frac{Q}{g}\,\frac{(\text{velocity of } Q)^2}{R} = \frac{Q}{g}\,\frac{1}{R}\,\frac{R^2}{r^2}v^2 = \frac{Q}{g}\,\frac{R}{r^2}v^2,$$

and this force F is equivalent to a single force F acting half-way between the weights and to a couple of moment $F\frac{e}{2}$. Let ϕ be the angle between the radius to a balance-weight, and the common bisector of the angle between the two cranks (Fig. 294).

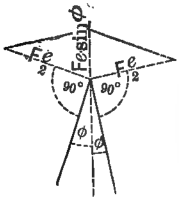


FIG. 294.

Since there are two weights Q, there will be two couples each of moment $F\frac{e}{2}$, and two forces each equal to F acting half-way between the weights, the angle between the axes of the couples being $180° - 2\phi$ and that between the forces being 2ϕ. The moment of the resultant couple is $Fe \sin \phi$, and its axis bisects the angle between the axes of the separate couples; the resultant force parallel to the line of stroke $=2F \cos \phi$.

Q and ϕ may now be chosen so that

$$2F \cos \phi = \text{maximum alternating force} = C\sqrt{2},$$

and $$Fe \sin \phi = \text{maximum alternating couple} = \frac{Cd}{2}\sqrt{2}.$$

Then $$\tan \phi = \frac{d}{e},$$

and $$F = \frac{C}{e}\sqrt{\frac{e^2+d^2}{2}},$$

or $$\frac{Q}{g}\frac{R}{r^2}v^2 = \frac{W}{g}\frac{v^2}{r}\frac{1}{e}\sqrt{\frac{e^2+d^2}{2}};$$

and therefore $$Q = \frac{W}{e}\frac{r}{R}\sqrt{\frac{e^2+d^2}{2}}.$$

Ex. 18. *Again, the pressure C at a dead-point may be balanced by a weight Q diametrically opposite.*

If R is the radius of the weight-circle, then

$$\frac{W}{g}\frac{v^2}{r} = C = \frac{Q}{g}\frac{R}{r^2}v^2,$$

and therefore $$Q = W\frac{r}{R}.$$

The weight Q may be replaced by a weight $Q\frac{e+d}{2e}$ on the near and a weight $Q\frac{e-d}{2e}$ on the far wheel. Thus, since the cranks are at right angles, there will be two weights 90° apart on each wheel, viz., $Q\frac{e+d}{2e}$ in line with the crank and

$Q\frac{e-d}{2e}$. These two weights, again, may be replaced by a single weight B whose centrifugal force is the resultant of the centrifugal forces of the two weights. Thus

$$\left(\frac{B}{g}\frac{v'^2}{R}\right)^2=\left(\frac{Q}{g}\frac{e+d}{2e}\frac{v'^2}{R}\right)^2+\left(\frac{Q}{g}\frac{e-d}{2e}\frac{v'^2}{R}\right)^2,$$

v' being the lineal velocity at the circumference of the weight-circle.

Therefore $$B^2=Q^2\frac{e^2+d^2}{2e^2},$$

or $$B=\frac{Q}{e}\sqrt{\frac{e^2+d^2}{2}}.$$

If α is the angle between the radius to the greater weight $Q\frac{e+d}{2e}$ and the crank radius,

$$\tan\alpha=\frac{\frac{Q}{g}\frac{e-d}{2e}\frac{v'^2}{R}}{\frac{Q}{g}\frac{e+d}{2e}\frac{v'^2}{R}}=\frac{e-d}{e+d}.$$

Note.—In outside-cylinder engines $e-d$ is approximately *nil*, and $B=Q=W\frac{r}{R}$.

11. Curves of Piston Velocity.—Consider the engine in Ex. 16.

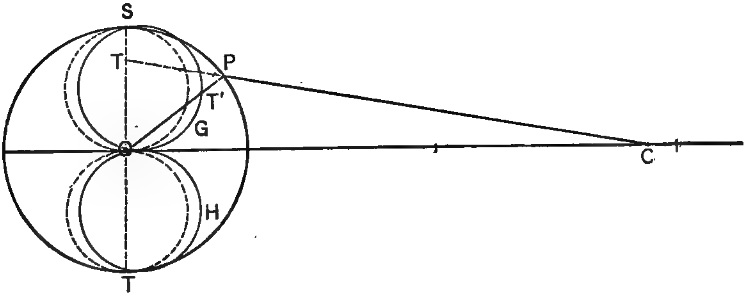


FIG. 295.

Let CP produced intersect the vertical through O in T, and in OP take $OT'=OT$.

The piston velocity u and the velocity v of the crank-pin centre are connected by the relation

$$\frac{u}{v}=\frac{\sin(\theta+\phi)}{\cos\phi}=\frac{OT}{OP}=\frac{OT'}{OP}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the velocity v is assumed constant, and if it is represented by OP, then on the same scale OT' will represent the piston velocity u. Drawing similar lines to represent the value of u for every position of the crank, the locus of T' will be found to consist of two closed curves OGS, OHT, called the *polar curves of piston velocity.* They pass through the point O and through the ends S and T of the vertical diameter. On the side towards the cylinder they lie outside the circles having OS and OT as diameters, while on the side away from the cylinder they lie inside the circles. If the connecting-rod is so long that its obliquity may be disregarded,

$$\phi=0, \quad u=v\sin\theta,$$

and the curves coincide with the circles.

A *rectangular* diagram of velocity may be drawn as follows.

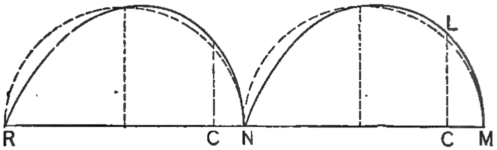


FIG. 296.

Upon the vertical through C, Fig. 296, take $CL=OT$; the locus of L is the curve required for one stroke. A similar curve may be drawn for the return-stroke either below MN or upon the prolongation $NR(=MN)$ of MN.

If the obliquity of the connecting-rod is neglected, the curves evidently coincide with the semicircles upon MN and NR, MN $(=NR)$ defining the extreme positions of C. The obliquity, however, causes the actual curve to fall above the semicircle during the first half of the stroke, and below during the second half.

Again, let the connecting-rod $(l)=n$ cranks (r). Then

$$\frac{\sin\theta}{\sin\phi}=\frac{l}{r}=n,$$

and by eq. (1),

$$u=v(\sin\theta+\cos\theta\tan\phi)=v\left(\sin\theta+\frac{\sin\theta\cos\theta}{\sqrt{n^2-\sin^2\theta}}\right). \quad . \quad . \quad (2)$$

If the obliquity is very small,

$$\tan\phi = \sin\phi = \frac{\sin\theta}{n}, \text{ approximately,}$$

and therefore $$u = v\left(\sin\theta + \frac{\sin\theta\cos\theta}{n}\right) = v\left(\sin\theta + \frac{\sin 2\theta}{2n}\right).$$

12. Curve of Crank-effort.—The *crank-effort* F for any position OP of the crank is the component along the tangent at P of the thrust along the connecting-rod.

$$\text{This thrust} = \frac{P}{\cos\phi};$$

therefore $$F = P\,\frac{\sin(\theta+\phi)}{\cos\phi}.$$

If the pressure P upon the piston is constant, and if it is represented by OP, then, *on the same scale*, OT', Fig. 297, will represent the crank-effort. Thus the curves of piston velocity already drawn may also be taken to represent curves of crank-effort. If the pressure P is variable, as is usually the case, let OP, the crank radius represent the *initial* value of P. After expansion has begun, take OP' in OP, for any position OP of the crank, to represent the corresponding pressure which may be directly obtained from the indicator-diagram. Draw $P'T'$ parallel to PT, and take $OT'' = OT'$. Then OT'' will represent the required crank-effort, and the linear and polar diagrams may be drawn as already described.

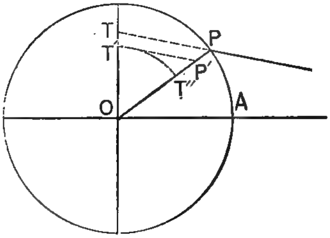


FIG. 297.

13. Curves of Energy—Fluctuation of Energy.—In the curve of crank-effort as usually drawn, the crank-effort for any position OP of the crank is the ordinate $S'H$, the abscissa DH being equal to the arc AP, i.e., to the distance traversed by the point of application of the crank-effort. Thus, DSE and EVG being the curves,

$$DE = EG = \text{semicircumference of crank-circle} = \pi r.$$

If the obliquity is neglected, the curves of crank-effort are the two curves of sines shown by the dotted lines.

The area $DS'H$ also evidently represents the *work done* as the crank moves from OA to OP, and the total work done is represented by the area DSE in the forward and by EVG in the return stroke.

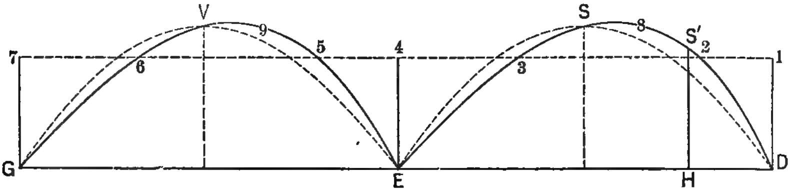


FIG. 298.

Let F_0 be the *mean* crank-effort. Then

$$F_0 \times 2\pi r = 2P \times 2r,$$

assuming P to be constant.

Therefore $$F_0 = \frac{2P}{\pi}.$$

Draw the horizontal line 1234567 at the distance $\frac{2P}{\pi}$ from DEG, and intersecting the verticals through D, E, and G in 1, 4, and 7, and the curves in 2, 3, 5, and 6. The engine may be supposed to work against a constant resistance R equal and opposite to the mean crank-effort F_0.

From D to 2, $R >$ crank-effort, and the speed must therefore continually diminish.

From 2 to 3, $R <$ crank-effort, and the speed must continually increase.

Thus 2 is a point of min. velocity, and therefore also of min. kinetic energy.

From 3 to E, $R >$ crank-effort, and the speed must continually diminish.

Thus 3 is a point of max. velocity, and therefore also of max. kinetic energy.

Similarly, in the return-stroke, 5 and 6 are points of min. and max. velocity respectively.

The change or *fluctuation* of kinetic energy from 2 to 3 = area 283, bounded by the curve and by 23.

The fluctuation from 3 to 5 = area $3E5$, bounded by 35 and by the curve.

Again, since $\frac{F}{P}=\frac{Fr}{Pr}$, the ordinates of the curves may be taken to represent the *moments* of crank-effort, and the abscissæ are then the corresponding values of θ.

The work done between A and any other position P of the crank-pin

$$=\int_0^\theta Fr d\theta = Pr\int_0^\theta\left(\sin\theta+\frac{\sin\theta\cos\theta}{\sqrt{n^2-\sin^2\theta}}\right)d\theta$$

$$=Pr(1-\cos\theta+n-\sqrt{n^2-\sin^2\theta}).$$

If there are two or more cranks, the ordinates of the crank-effort curve will be equal to the algebraic sums of the several crank-efforts. For example, if the two cranks are at right angles, and if F_1, F_2 are the crank-efforts when one of the cranks (F_1) makes an angle θ with the line of stroke,

$$F_1=P\left(\sin\theta+\frac{\sin 2\theta}{2n}\right)$$

and

$$F_2=P\left(\cos\theta-\frac{\sin 2\theta}{2n}\right).$$

Therefore $\quad F_1+F_2=P(\sin\theta+\cos\theta)=$ combined crank-effort,

P being supposed constant.

Note.—In the case of the polar curves of crank-effort, if a circle is described with O as centre and a radius = mean crank-effort $=\frac{2P}{\pi}$, it will intersect the curves in four points, which are necessarily points of max. and min. velocity.

EXAMPLES.

1. A stone weighing 8 oz. falls for 5 seconds. What is its momentum, and what force will stop it in 3 seconds? *Ans.* 80; $1\frac{1}{3}$ lbs.

2. What force, acting for 6 seconds on a mass of 12 lbs., will change its velocity from 200 to 320 ft. per second? *Ans.* $7\frac{1}{2}$ lbs.

3. The velocity of a body is observed to increase by four miles per hour in every minute of its motion. Compare the force acting on it with the force of gravity. *Ans.* 11 to 3600.

4. A railway train whose mass is 100 tons, moving at the rate of a mile a minute, is brought to rest in 10 seconds by the action of a uniform force. Find how far the train runs during the time for which the force is applied. Also determine the force, stating the units employed.

Ans. 440 ft.; $27\frac{1}{2}$ tons.

5. If the unit of mass is the mass of 12 lbs., and the units of length and time are 14 ft. and 12 seconds respectively, find the measures of the mass, velocity, and momentum of a body which weighs 1 cwt. and is moving with a velocity of 35 ft. per second.

Ans. mass $=9\frac{1}{3}$; velocity $=30$; momentum $=280$.

6. A goods truck of 6 tons, travelling at 3 miles an hour, collides with another truck at rest, and both move on together at 2 miles an hour. Find the mass of the second truck. *Ans.* 3 tons.

7. Find the average force which will bring to rest, in 2 ft., an ounce bullet moving at the rate of 1500 ft. per second. How long will it take to bring it to rest?

Ans. $351\frac{9}{16}$ poundals; $\frac{\sqrt{2}}{.75}$ secs.

8. A man of 12 stone weight climbs up a mine-shaft 800 ft. deep by a ladder. What work does he do? If he exerts $\frac{1}{5}$ horse-power, how long will he be?

Ans. 134,400 ft.-lbs.; $20\frac{4}{11}$ minutes.

9. A shot is fired from a gun, which is fixed, with a certain charge of powder. If the quantity of powder be quadrupled, in what proportion will the velocity of the shot be increased? *Ans.* Doubled.

10. A body falling from a mast took .25 second to fall from the hatchway to the bottom of the hold, a distance of 20 ft. From what height did it fall and with what velocity did it strike? *Ans.* $110\frac{1}{4}$ ft.; 84 f/s.

11. A train going at the rate of 45 miles an hour takes half a minute in passing another train, 230 yards long, going in the same direction at the rate of 15 miles an hour. What is the length of the first train? *Ans.* 210 yards.

12. A train 215 yards long, going at the rate of 55 miles an hour, takes 10 seconds in passing another train going in the opposite direction, at the rate of 35 miles an hour. What is the length of the second train?

Ans. 225 yards.

13. Just as a tramcar reaches a man standing by the tramway it has a velocity of $8\frac{1}{2}$ ft. per second; the man takes hold of and mounts the car. What change of velocity takes place, the weights of the car and man being 1 ton and 10 stone respectively? *Ans.* 8 f/s.

14. One hundred and fifty pounds is drawn up the shaft of a coal-pit, and, starting from rest, acquires a velocity of 3 miles an hour in the first minute. Assuming that the acceleration is uniform, find how heavy the mass appears to one drawing it up. *Ans.* $150\frac{11}{32}$ lbs.

15. A train of 100 tons, running on a level line, is kept going by the locomotive at a uniform pace of 50 miles per hour; the steam is suddenly shut off, and the train comes to rest after it has travelled 2 miles farther. What

was the force applied by the locomotive to the train, supposing the resistance of the rail and air to be constant? *Ans.* A weight of $\frac{1875}{1728}$ ton.

16. If a 700-lb. shot be fired from a 75-ton gun with a speed of 1200 feet per second, find the speed of recoil of the gun. *Ans.* 5 f/s.

17. What force must be applied for one tenth of a second to a mass of 10 tons in order to produce in it a velocity of 3840 ft. per minute? What would be the energy of the mass so moving?

Ans. A weight of 200 tons; 640 ft.-tons/sec.

18. A bullet moving at the rate of 1100 ft. per second passes through a thin plank, and comes out with a velocity of 1000 ft. per second. If it then passes through another plank exactly like the former, with what velocity will it come out of this second plank? *Ans.* $100\sqrt{79}$ f/s.

19. A shot of 600 lbs. is fired from a 10-ton gun with a velocity of 1000 ft. per second. If the mass of the powder be neglected, find the velocity of recoil. *Ans.* 26.8 f/s.

20. A bullet whose mass is 1 oz. is fired with a velocity of 1210 ft. per second into a mass of 1 cwt. of wood at rest. What is the velocity with which the wood begins to move? *Ans.* 8.098 in./s.

21. An 1800-lb. shot moving with a velocity of 2000 ft. per second impinges on a plate of 10 tons, passes through it, and goes on with a velocity of 400 ft. per second. If the plate be free to move, find its velocity.

Ans. 12.857 f/s.

22. To find the velocity of an 8-lb. shot that will just penetrate an armor-plate 10 ins. thick, the resistance being 84 tons. If the velocity of the shot be doubled, what must be the thickness of the plate in order that the shot may only just penetrate it? *Ans.* 1120 f/s; 40 ins.

23. The velocity of flow of water in a service-pipe 48 ft. long is 64 ft. per second. If the stop-valve is closed in $\frac{1}{9}$ of a second, find the increase of pressure near the valve. *Ans.* 375 lbs. per sq. in.

24. A body approaches an observer with a velocity due east. If the observer moves due north with an equal speed, in what direction will the body appear to move? *Ans.* South-east.

25. A horseman at full gallop fires at a stationary animal. Show that he must aim behind the animal.

26. If the animal is also running in a parallel direction, show that the horseman must aim in front or behind according as his speed is less or greater than that of the animal.

27. Rain is falling vertically, and it is observed that the splashes made by the drops on the window of a moving railway carriage are inclined to the vertical. Explain this, and point out in which direction the splashes are inclined.

If the train is also travelling at 60 miles per hour, and the inclination of the splashes to the vertical is 30°, what is the velocity of the falling drops?

28. A steamboat is going north at 15 miles per hour while an east wind

is blowing at 5 miles per hour. Find the angle the direction of the smoke makes with the ship's keel. *Ans.* W. by S. at $\cot^{-1} 3$.

29. Knowing the direction of the true wind and the velocity and direction of the apparent wind on a ship, as shown by the direction of the vane on the mast, determine the velocity of the ship, supposing there is no leeway.

30. To determine (*a*) the direction taken by the smoke of a steamer, (*b*) the direction and velocity with which the wind appears to blow to a passenger on board. (N.B.—The smoke is carried along with the wind.)

31. A train travels at the rate of 45 miles an hour. Rain is falling vertically, but owing to the motion of the train, the drops appear to fall past the window at an angle $\tan^{-1} 1.5$ with the vertical. Find the velocity of the rain drops.

32. The rim of the wheel of a centrifugal pump moves at 30 ft. per second; water flows radially at 5 ft. per second; the vanes are inclined backward at an angle of 35° to the rim. What is the absolute velocity of the water? What is the component of this parallel to the rim? *Ans.* 23.4 f/s; 22.8 f/s.

33. Find the time in which it is possible to cross a road, of breadth 100 ft., in a straight line with the least velocity, between a stream of vehicles of breadth 40 ft., following at intervals of 20 ft., with velocity 5 f/s.

Ans. 50 secs.

34. A man stands on a platform which is ascending with a uniform acceleration of 6 ft. per second per second, and at the end of four seconds after the platform has begun to move he drops a stone. Find the velocity of the stone after three more seconds. *Ans.* 72 f/s.

35. A man in a lift throws up a ball vertically with a velocity of v f/s and catches it again after t seconds; prove that the vertical acceleration of the lift is $g-2\frac{v}{t}$.

36. A railway train is going at 30 ft. per second; how must a man throw a stone from the window so that it shall leave the train laterally at 1 ft. per second, but have no velocity in the direction of the train's motion?

Ans. 30.03 f/s at 178° 6′ with direction of train's motion.

37. If a bucket of water, weighing 20 lbs., is pulled up from a well with an acceleration of 8 ft. per second per second, find, in pounds weight, the force which must be applied to the rope. *Ans.* 25 lbs.

38. Find the maximum velocity of an ice-yacht sailing at right angles to a wind of 10 miles an hour if the angle between the sail and the keel is $\sin^{-1} .6$ (neglecting all resistances to motion along the keel).

Ans. $13\frac{1}{3}$ f/s.

39. Find the angle between the sail and the keel of the ice-yacht if, when the wind is perpendicular to the keel, the maximum velocity of the yacht is four times the velocity of the wind. *Ans.* $\tan^{-1} .25$.

40. An ice-yacht is sailing at right angles to the wind at the rate of 20 miles

an hour. Its sail makes with the keel an angle of 30°. Find the least possible velocity of the wind. *Ans.* $6\frac{2}{3}\sqrt{3}$ f/s.

41. Show that it is possible with an ice-boat to beat to windward faster than the wind is blowing in an opposite direction.

Show also that it is quicker to beat to leeward instead of sailing dead before the wind.

42. A steamer takes 3 minutes on the measured sea-mile with the tide and 4 minutes against the tide; find the speed of the tide and the speed of the vessel through the water. *Ans.* $17\frac{1}{2}$ knots; $2\frac{1}{2}$ knots.

43. A ship sailing north-east by compass through a tide running 4 knots finds after 2 hours she has made good 4 miles south-east; determine the direction of the current and the speed of the ship.

44. A ship is sailing at $6\frac{3}{4}$ knots directly towards a battery where artillery practice is going on. The man at the helm observes the flash of a gun and hears the report 15 seconds afterwards; five minutes after the first flash he observes a second flash, and hears the report 12 seconds afterwards. Assuming that a sea-mile is 6080 ft., find the velocity of sound.

45. Two particles are started simultaneously from the points A and B, 5 ft. apart, one from A towards B with a velocity which would cause it to reach B in 3 seconds, and the other at right angles to the former and with three fourths of its velocity. Find their relative velocity in magnitude and direction, the shortest distance between them, and the time at which they are nearest to one another.

46. From the edge of a cliff two stones are thrown at the same time, one vertically downwards with a velocity of 30 ft. per second, the other vertically upwards with the same velocity. The first stone reaches the ground in $7\frac{1}{2}$ seconds. How much longer will the other be in the air? *Ans.* $1\frac{7}{8}$ secs.

47. A steam-engine moves a train of mass 60 tons on a level road from rest, and acquires a speed of 5 miles an hour in 5 minutes. If the same engine move another train and give it a speed of 7 miles an hour in 10 minutes, find the mass of the second train. (The mass of the engine is included in that of the train, and the forces exerted by it are the same in both cases.)

Ans. $85\frac{5}{7}$ tons.

48. A force which can just support 200 lbs. acts for one minute on 6000 lbs. What velocity does v produce in it? *Ans.* 64 f/s.

49. In a ship sailing at 16 miles per hour it is observed that the direction of the wind is apparently 30° to the line of keel and from the bows; its velocity is apparently 4 miles per hour. What is its true direction and magnitude?

Ans. $\text{Cosec}^{-1}\,6.34$; 12.69 m/h.

50. The velocity of a ship in a straight course on an even keel is $8\frac{1}{8}$ miles an hour; a ball is bowled across the deck, perpendicular to the ship's length, with a uniform velocity of 3 yards in a second. Describe the true path of the ball in space, and show that it will pass over 45 ft. in 3 seconds nearly.

51. A train passes two men walking beside a railway in $3\frac{1}{2}$ seconds and $3\frac{3}{5}$ seconds respectively; a second train passes the men in $4\frac{2}{5}$ seconds and $4\frac{1}{2}$ seconds respectively. · Show that this train will overtake the first train and, if on a different line, could pass it completely in 36 seconds.

52. Two ships, A and B, of which B bears from A 35° E. of N., lie eastwards of a coast which presents seawards a long vertical precipice running N. and S. A cannon is fired from B, and 5 seconds after the flash is seen from A the report is heard, and 7 seconds later the echo from the precipice. Find the distance of A from the coast, assuming 1100 f/s as the velocity of sound.

53. A goods train is running uniformly at 15 m/h; a passenger train runs on a parallel line with an average speed of 25 m/h, but stops at a station every mile for 1 minute. The goods train has a start of 1 minute. When and where will the trains pass? Solve this problem graphically.

54. A man rows a boat through the water at the rate of 3 miles an hour in a direction 60° east of north, in a current flowing southwards at the rate of $1\frac{1}{2}$ miles an hour. Show that the boat will travel due eastwards and find rate of progress. *Ans.* $\frac{3}{2}\sqrt{3}$ m/h.

55. AB is a given straight line, and P a given fixed point without it; a particle Q moves along AB with a given constant velocity. When Q is in any assigned position, find its angular velocity with respect to P.

Ans. pv/PQ^2, v being the constant velocity and p the perpendicular from P to AB.

56. A particle under the action of a number of forces moves with a uniform velocity in a straight line. What condition must the forces fulfil?

57. A lift is descending and coming to rest with a uniform retardation of 4 F.P.S. units. A man in the lift weighs out a pound of tea with an ordinary balance, and a pound of sugar with a spring balance. How many pounds of each does he really obtain? *Ans.* 1 lb. of tea; $\frac{8}{9}$ lb. of sugar.

58. A 3-ton cage, descending a shaft with a speed of 9 yards a second, is brought to a stop by a uniform force in the space of 18 ft. What is the tension in the rope while the stoppage is occurring? *Ans.* $4\frac{125}{128}$ tons.

59. A 1-oz. bullet is fired with a velocity of 1000 ft. per second. Find the velocity with which a 2-oz. bullet could be fired from the same rifle with treble the charge of powder. *Ans.* 1225 f/s.

60. If a 14-lb. shot leave the muzzle of a 2-ton gun with a *relative* speed of 540 ft. per second, find the speed of recoil. *Ans.* 1.682 f/s.

61. A gun weighing 5 tons is charged with a shot weighing 28 lbs. If the gun be free to move, with what velocity will it recoil when the ball leaves it with a velocity of 100 ft. per second? *Ans.* $\frac{1}{4}$ f/s.

62. A railway train travels $\frac{1}{4}$ mile on a smooth level line, while its speed increases uniformly from 15 to 20 miles an hour. What proportion does the pull of the engine bear to the weight of the train? *Ans.* $\frac{77}{17280}$.

63. A 13-ton gun recoils on being fired with a velocity of 10 ft. per second, and is brought to rest by a uniform friction equal to the weight of $\frac{1}{16}$ ton. How far does it recoil? *Ans.* 5 ft.

64. Determine the charge of powder required to send a 32-lb. shot to a range of 2500 yards with an elevation of 15°, supposing the initial velocity is 1600 ft. a second when the charge is half the weight of the shot, and that the initial energy of the shot is proportional to the charge of powder.

65. A 5-oz. ball moving at the rate of 1000 f/s pierces a shield and moves on with a velocity of 400 f/s; what energy is lost in piercing the shield? *Ans.* 4076 ft.-lbs.

66. A half-ton shot is discharged from an 81-ton gun with a velocity of 1620 ft. per second. What will be the velocity with which the gun will begin to recoil if the mass of the powder be neglected? Will the gun or the shot be able to do more work before coming to rest, and in what proportion? *Ans.* 10 f/s; shot does 162 times that of gun.

67. A ball weighing 12 lbs. leaves the mouth of a cannon horizontally with a velocity of 1000 ft. per second; the gun and carriage, together weighing 12 cwt., slide upon a smooth plane whose inclination to the horizon is 30°. Find the space through which the gun and carriage will be driven up the plane by the recoil. *Ans.* 4.982 ft.

68. The pressure of water in a hydraulic company's main is 750 lbs. per square inch, and the average flow is 25 cubic feet per minute. What horse-power does this represent? If the charge for the water is twopence per 100 gals., what is the cost per horse-power hour?

69. An accumulator loaded to a pressure of 750 lbs. per square inch has a ram of 21 ins. diameter, with a stroke of 24 ft. How much horse-power can be obtained for a period of 50 seconds? *Ans.* 226.8.

70. Find the weight which will give an average fluid pressure of 750 lbs. per square inch in an accumulator with a 14-in. ram and a stroke of 16 ft. How much energy can be stored up? *Ans.* 115,500 lbs.; 1,848,000 ft.-lbs.

71. A steam-pump raises 11 tons of water 15 ft. high every minute. What is its horse-power? *Ans.* 10.

72. Find the horse-power required to raise a weight of 10 tons up a grade of 1 in 12 at a speed of 6 miles per hour against a resistance of 9 lbs. per ton. *Ans.* 31.3.

73. Find the horse-power of an engine that would empty a cylindrical shaft full of water in 32 hours if the diameter of the shaft be 8 ft. and its depth 600 ft., the weight of a cubic foot of water being 62.5 lbs. *Ans.* 30.

74. Determine the horse-power transmitted by a belt moving with a velocity of 600 ft. per minute, passing round two pulleys, supposing the difference of tension of the two parts is the weight of 1650 lbs. *Ans.* 30.

75. A cylindrical shaft of 10 ft. diameter has to be sunk to a depth of 100 fathoms through chalk, the weight of the chalk being $143\frac{3}{4}$ lbs. per cubic foot.

What horse-power is required to lift out the material in 12 working days of 8 hours each?

76. An engine is required to raise in 3 minutes a weight of 13 cwt. from a pit whose depth is 840 ft. Find the horse-power of the engine. *Ans.* 12.35.

77. An accumulator-ram is 8.8 ins. in diameter and has a stroke of 21 ft. Find the store of energy in foot-pounds when the ram is at the top of the stroke and is loaded until the pressure is 750 lbs. per square inch.
Ans. 958,320 ft.-lbs.

78. In a differential accumulator the diameters of the spindle are 7 ins. and 5 ins.; the stroke is 10 ft. Find the store of energy when full and loaded to 2000 lbs. per square inch. *Ans.* 377,000 ft.-lbs.

79. In a differential press the diameters of the upper and lower portions of the ram are 6 ins. and 8 ins. respectively. The pressure is 1000 lbs. per square inch., and the stroke is 10 ft. Find the load on the accumulator, the maximum store of energy, and the store of water.
Ans. 22,000 lbs.; 220,000 ft.-lbs.; $1\frac{19}{36}$ cu. ft.

80. Find the horse-power of a fall where 18,000,000 cu. ft. passes per minute, falling 162 ft. *Ans.* 5,522,727.

81. A horizontal axle 10 ins. diameter has a vertical load upon it of 20 tons, and a horizontal pull of 4 tons. The coefficient of friction is 0.02. Find the heat generated per minute, and the horse-power wasted in friction, when making 50 revolutions per minute. *Ans.* 155 thermal units; 3.63 H.P.

82. A 3-H.P. steam-crane is found to raise a weight of 10 tons to a height of 50 ft. in 20 minutes; what part of the work is done against friction? If the crane is kept at similar work for 8 hours, how many foot-pounds of the work are wasted on friction? *Ans.* $\frac{48}{99}$; 20,640,000 ft.-lbs.

83. Find the energy per second of a waterfall 30 yds. high and $\frac{1}{4}$ mile broad where the mass of water is 20 ft. deep and has a velocity of $7\frac{1}{2}$ miles an hour when it arrives at the fall. The weight of water is 1024 oz. per cubic foot.
Ans. 496,849.2 ft.-tons/sec.

84. A shaft 560 ft. deep and 5 ft. in diameter is full of water; how many foot-pounds of work are required to empty it, and how long would it take an engine of $3\frac{1}{2}$ horse-power to do the work? (N.B. Of course it is to be assumed that there is no flow of water into the shaft.)
Ans. 192,500,000 ft.-lbs.; $1666\frac{2}{3}$ minutes.

85. Find the work expended in raising the materials (112 lbs. per cubic foot) for a brick tower 125 ft. in height and of 24 ft. external and 16 ft. internal diameter. In what time could the materials be raised by a $2\frac{1}{2}$-H.P. engine?
Ans. 220,000,000 lb.-ft.; $44\frac{4}{9}$ hours.

86. A uniform beam weighs 1000 lbs. and is 20 ft. long. It hangs by one end, round which it can turn freely. How many foot-pounds of work must be done to raise it from its lowest to its highest position? *Ans.* 20,000 ft.-lbs.

87. Electric lamps give 1 candle-power for 4 watts; how many 10- or how many 16-candle lamps may be worked per electric horse-power? The combined efficiency of engine, dynamo, and gearing being 70 per cent, what is the candle-power available for every indicated horse-power?
Ans. 18; 11; 130.55.

88. A cast-iron fly-wheel of 36 sq. ins. section and 120 ins. mean diameter makes 60 revolutions per minute. Find, approximately, the mean energy of rotation. Also find the number of revolutions per minute after losing 800 ft.-lbs. of energy. *Ans.* 54,424 ft.-lbs.; 59.

89. If the earth be assumed to be spherical, how much heat would be developed if its axial rotation were suddenly stopped, a unit of heat corresponding to 778 ft.-lbs.? Weight of mass of earth $=10^{21}\times 6.029$ tons; diameter of earth $=8000$ miles.

90. A fly-wheel supported on a horizontal axle 2 ins. in diameter is pulled round by a cord wound round the axle carrying a weight. It is found that a weight of 4 lbs. is just sufficient to overcome the friction. A further weight of 16 lbs., making 20 in all, is applied, and after two seconds starting from rest it is found that the weight has gone down 12 ft. Find the moment of inertia of the wheel. *Ans.* .014.

91. Find the horse-power of an engine which pumps up water from a depth of 50 ft. and delivers it at the rate of 1000 gals. per minute through a pipe whose cross-section is 1 sq. ft. *Ans.* $15\frac{5}{27}$.

92. A fly-wheel with a rim of uniform axial thickness weighs 1000 lbs., has a 60-in. external and a 48-in. internal diameter, and makes 60 revolutions per minute. The greatest fluctuation of energy is 1000 ft.-lbs. Find the variation in speed.

Ans. 9.36 revolutions per minute.

93. A beam will safely carry 1 ton with a deflection of 1 in. From what height may a weight of 100 lbs. drop without injuring it, neglecting the effect of inertia? *Ans.* 10.2 in.

94. A cut of .06 in. depth is being made on a 4-in. wrought-iron shaft revolving at 10 revolutions per minute; the traverse feed is .3 in. per revolution; the pressure on the tool is found to be 400 lbs. What is the horse-power expended on the tool? How much metal is removed per hour per horse-power?

Ans. .1381; 98.28 cu. ins.

95. The travel of the table of a planing-machine which cuts both ways is 9 ft.; taking the resistance to be overcome at 400 lbs. and the number of double strokes per hour at 80, find the horse-power absorbed in cutting.

Ans. 29.

96. The fly-wheel of a 3-H.P. riveting-machine fluctuates between 80 and 120 revolutions per minute; every two seconds an operation occurs which requires seven eighths of all the energy supply for two seconds. Find the wheel's moment of inertia. *Ans.* 65.6.

97. A fly-wheel weighing 5 tons has a mean radius of gyration of 10 ft. The wheel is carried on a shaft of 12 ins. diameter, and is running at 65 revolutions per minute. How many revolutions will the wheel make before stopping, if the coefficient of friction of the shaft in its bearing is 0.065? (Other resistances may be neglected.) *Ans.* 352.

98. In a gas-engine, using the Otto cycle, the I.H.P. is 8, and the speed is 264 revolutions per minute. Treating each fourth single stroke as effective and the resistance as uniform, find how many foot-lbs. of energy must be

stored in a fly-wheel in order that the speed shall not vary by more than one-fortieth above or below its mean value. *Ans.* 30,000 ft.-lbs.

99. A sphere, mass 16 lbs., at rest is struck by another, mass 8 lbs., moving with a velocity of 20 miles per hour in a direction making an angle of 45° with the line of centres at the moment of impact; the coefficient of rebound is $\frac{1}{3}$. Determine the subsequent motion.

100. A steamer of 8000 tons displacement sailing due east at 16 knots an hour collides with a steamer of 5000 tons displacement sailing at 10 knots an hour. Find the energy of collision if the latter at the moment of collision is going (1) due west; (2) northwest; (3) northeast.
Ans. 92,823; 79,951; 17,820 ton-ft.

101. A fly-wheel weighs 10,000 lbs., and is of such a size that the matter composing it may be treated as if concentrated on the circumference of a circle 12 ft. in radius; what is its kinetic energy when moving at the rate of 15 revolutions a minute?

How many turns would it make before coming to rest, if the steam were cut off and it moved against a friction of 400 lbs. exerted on the circumference of an axle 1 ft. in diameter? *Ans.* 55,561.2 ft.-lbs.; 44.2 turns.

102. Two inelastic bodies, the one of 100 lbs. moving due W. at 20 f/s, the other of 50 lbs. moving due E. at 10 f/s, collide. Find the energy of collision. What will be the energy of collision if they move in the same direction, i.e., due E. or due W.? *Ans.* $468\frac{3}{4}$ ft.-lbs.; $52\frac{1}{12}$ ft.-lbs.

103. Prove that for a rope round pulleys, running at 3000 ft. a minute, to transmit 40 horse-power, the tension on the driving side must be 880 lbs., supposing it double the tension on the slack side.

104. A fly-wheel weighs 20 tons and its radius of gyration is 5 ft. How much work is given out while the speed falls from 60 to 50 revolutions per minute? *Ans.* 94.3 ft.-tons.

105. A tower is to be built of brickwork, and the base is a rectangle 22 ft. by 9, and the height is 66 ft., the walls being 2 ft. thick. Find the number of units of work expended on raising the bricks from the ground, and the number of hours in which an engine of 3 horse-power would raise them, a cubic foot of brickwork weighing 1 cwt. *Ans.* 26,345,088 ft.-lbs.

106. A rifle-bullet .45 in. in diameter weighs 1 oz.; the charge of powder weighs 85 grains; the muzzle-velocity is 1350 ft. per second; the weight of the rifle is 9 lbs. *Neglecting* the *twist*, determine the energy of 1 lb. of powder. If the bullet loses $\frac{1}{3}$ of its velocity in its passage through the air, find the average force of the blow on the target into which the bullet sinks $\frac{1}{8}$ in.

If there is a twist of 1 in in 20 in., find the charge to give the same muzzle velocity, the length of the barrel being 33 in.

107. The table of a small planing-machine, which weighs 1 cwt., make six double strokes of $4\frac{1}{2}$ ft. each per minute. The coefficient of friction between the sliding surfaces is 0.07. What is the work performed in foot-pounds per minute in moving the table? *Ans.* 423.3

108. The fly-wheel of a 40-H.P. engine, making 50 revolutions per minute, is 20 ft. in diameter and weighs 12,000 lbs. What is its kinetic energy?

If the wheel gives out work equivalent to that done in raising 5000 lbs. through a height of 4 ft., how much velocity does it lose?

The axle of the fly-wheel is 12 in. in diameter. What proportion of the horse-power is required to turn the wheel, the coefficient of friction being .08?

If the fly-wheel is disconnected from the engine when it is making 50 revolutions per minute, how many revolutions will it make before it comes to rest?

Ans. 511,260.4 ft.-lbs.; 1.04 ft. per sec.; $\frac{4}{35}$; 169.4.

109. A uniform circular disc, whose radius is 4 ins. and weight 2 lbs., can turn about a horizontal axis through its centre at right angles to its plane. When started at the rate of 200 revolutions per minute, and left to itself, it is observed to come to rest in one minute.

Prove that the retarding couple (supposed constant) has the same moment as a weight of $25\frac{1}{2}$ grains attached to the rim at the extremity of a horizontal radius. Also find how many revolutions it makes.

110. A particle is placed on a rough horizontal plate at a distance of 9 ins. from a vertical axis about which the plate can turn. Find the greatest number of revolutions per minute the plate can make without causing the particle to slip upon it, the coefficient of friction being $\frac{2}{3}$. *Ans.* $50\frac{10}{11}$.

111. A wrought-iron fly-wheel 10 ft. in diameter makes 63 revolutions per minute. Find the intensity of stress on a transverse section of the rim, disregarding the influence of the arms. If the wheel, which weighs W lbs. gives out work equivalent to that done in raising W through a height of $5\frac{1}{4}$ ft. in 1 sec., what velocity will it lose? If the axle of the wheel is 10 in. in diameter and if .08 is the coefficient of friction, show that it will take $\frac{W}{2500}$ H.P. to turn the wheel. ($g=32.2$.)

Ans. 16,335 lbs.; 5.6 ft. per sec.

112. An engine of 400 horse-power can draw a train of 200 tons gross up an incline of 1 in 280 at 30 miles an hour. Determine the resistance of the road in pounds per ton. *Ans.* 17 lbs.

113. Prove that a train going 45 miles an hour will be brought to rest in about 378 yards by the brakes, supposing them to press with two-thirds of the weight on the wheels of the engine and brake-vans, which are half the weight of the train, taking a coefficient of friction 0.18.

Prove that an engine capable of exerting a uniform pull of 3 tons can take this train, weighing 120 tons, on the level from one station to stop at the next, 2 miles off, in about 3 mins. $38\frac{1}{2}$ secs., the speed being kept uniform when it has reached 45 miles an hour.

114. Determine the constant effort exerted by a horse which does 1,650,000 ft.-lbs. of work in one hour when walking at the rate of $2\frac{1}{2}$ miles per hour.

Ans. 125 lbs.

115. A train is drawn by a locomotive of 160 H.P. at the rate of 60 miles an hour against a resistance of 20 lbs. per ton. What is the gross weight of the train? *Ans.* 50 tons.

116. A train of $292\frac{3}{4}$ tons is drawn up an incline of 1 in 75, $5\frac{1}{2}$ miles long, against a resistance of 10 lbs. per ton, in ten minutes. Find the H.P. of the engine. The speed on the level, the engine exerting 769.42 H.P., is 43.4 miles per hour. What is the resistance in pounds per ton?

Ans. 1027 H.P.; 22.7 lbs. per ton.

117. An engine with its tender weighs 80 tons. It is moving uniformly at the rate of 20 miles an hour, against a resistance of 7 lbs. a ton. At what horse-power is it working?

If it drew after it a train of 12 carriages, each weighing 10 tons, at the rate of 40 miles an hour, against a resistance of 8 lbs. a ton, at what horse-power would it now be working? *Ans.* $29\frac{13}{15}$; $170\frac{2}{3}$.

118. Given a locomotive with two 18″×26″ cylinders, the connecting-rod = 6 ft., the boiler-pressure = 140 lbs., and driving-wheels of 7′ 0″ diameter, calculate the adhesion-friction, i.e., the ratio $\frac{\text{force at periphery}}{\text{weight on drivers}}$.

119. A railway wagon weighing 20 tons, with two pairs of wheels 8′ 0″ centre to centre, and with its centre of inertia 7′ 0″ above top of rails, has its wheels skidded while running. Take $\mu = 0.15$. Required the total retarding force and pressure of each wheel.

Ans. 7.375; 12.625, and 3 tons on rail.

120. Prove that if a motor car going at 100 km./h. can be stopped in 200 m., the brakes can hold the car on an incline of 1 in 5, and determine the time required to stop.

121. What horse-power is given up in lowering by 2 ft. the level of the surface of a lake 2 square miles in area in 300 hours, the water being lifted to an average height of 5 ft.? *Ans.* 29.23.

122. A cistern at a height of 250 ft. above the river is to be filled with 20,000 gals. of water every 24 hours by pumps worked by a turbine with a fall of 4 ft. Determine how much water the turbine will require, and its horse-power with an efficiency of 70 per cent.

Ans. 875,000 gals. per 24 hours; 1.05.

123. Taking the average power of a man as one tenth of a horse-power, and the efficiency of the pump used as 0.4, in what time will ten men empty a tank of 50′×30′×6′ filled with water, the lift being an average height of 30 ft.? *Ans.* 1274 minutes.

124. A fire-engine pump is provided with a nozzle the sectional area of which is 1 square inch, and the water is projected through the nozzle with an average normal velocity of 130 ft. per second; find (1) the number of cubic feet discharged per second, (2) the weight of water discharged per minute, (3) the kinetic energy of each pound of water as it leaves the nozzle, (4) the horse-power of the engine required to drive the pump, assuming the efficiency to be 70 per cent. *Ans.* .9 cu. ft.; 1.51 tons; 262.3 ft.-lbs.; 38.3

125. Find the shortest time from rest to rest in which a chain capable of

bearing a safe load of 25 tons can raise a weight of 10 tons out of a hold 15 ft. deep; also find the greatest load which can be raised or lowered in $2\frac{1}{2}$ seconds.

Ans. $1\frac{1}{4}$ seconds; $21\frac{1}{4}$ tons.

126. A heavy vertical chain is drawn upwards by a given force of P lbs-weight, which exceeds its weight W. Find its acceleration and its tension at any assigned point. Show that the tension at its middle point is $\frac{1}{2}P$.

Ans. $\frac{P-W}{W}g$; nP, n being the fraction of the chain below the point considered.

127. A rope 500 ft. long, and weighing 2 lbs. a foot, is wound on a roller. What is the difference of its potential energy in this position and in its position when 200 ft. of the rope have rolled out, neglecting friction and the weight of the roller and supposing that no part of the rope touches the ground?

Ans. 1,280,000 ft.-poundals.

128. A chain hanging vertically, 520 ft. long, weighing 20 lbs. per foot, is wound up; what work is done? *Ans.* 1352 ft.-tons.

129. What is the kinetic energy of a tramcar moving at 6 miles per hour, laden with thirty-six passengers, each of the average weight of 11 stones, weight of car $2\frac{1}{2}$ tons? What is its momentum? If stopped in two seconds, what is the average force? If the force is constant, this must also be the space average force; find the distance of stopping if the force is constant.

Ans. 13,400 ft.-lbs.; 3045.565; 1522.8 lbs.; 88 ft.

130. Find the horse-power of a locomotive which moves a train of mass 50 tons at the rate of 30 miles an hour along a level railroad, the resistance from friction and the air being 16 lbs. weight per ton. *Ans.* 64.

131. A cylinder and a ball, each of radius R, start from rest and roll down an inclined plane without slipping. If V is the velocity of translation after descending through a vertical distance h, show that

$$V^2=\tfrac{2}{3}(2gh) \text{ in the case of the cylinder}$$

and

$$V^2=\tfrac{5}{7}(2gh) \text{ in the case of the ball.}$$

132. A wheel having an initial velocity of 10 ft. per second ascends an incline of 1 in 100. How far will the wheel run along the incline, neglecting friction? $g=32.2$. *Ans.* 232.9 ft.

133. The fly-wheel of an engine of 4 H.P. running at 75 revolutions per minute is equivalent to a heavy rim of 45 ins. mean diameter, weighing 500 lbs. Determine the ratio of the kinetic energy in the fly-wheel to the energy developed in a revolution, and also find the maximum and minimum speeds of rotation when the fluctuation of energy is one fourth of the energy of a revolution. *Ans.* 1; 1.94; 75.2, 74.8.

134. It was found that when a length of 12 ins. was cut off the muzzle of a 6-in. gun, the velocity fell from 1490 to 1330 f/s. If the weight of the shot was 100 lbs., calculate the pressure on its base as it left the muzzle.

Ans. 11.13.

135. Three goods trucks, weighing respectively 5 tons, 7 tons, and 8 tons.

are placed on the same line of rails. The first is made to impinge on the second with a velocity of 60 ft. per second without rebounding. The first and second together impinge in the same way on the third. Find the final velocity. *Ans.* 15 f/s.

136. A steel punch $\frac{3}{4}$ in. in diameter is employed to punch a hole in a plate $\frac{5}{8}$ in. in thickness; what will be the least pressure necessary to drive a punch through the plate when the shearing strength of the material is 35 tons per square inch? *Ans.* 51.56 tons.

137. Find the weight of rim required for the fly-wheels of a punching-machine, intended to punch holes $1\frac{1}{2}$ in. diameter through $1\frac{1}{4}$-in. plates; speed of rim 30 ft. per second.

138. In a fly-press for punching holes in iron plates, the two balls weigh 30 lbs. each, and are placed at a radius of 30 ins. from the axis of the screw, the screw itself having a pitch of 1 in. What diameter of hole could be punched by such a press in a wrought-iron plate of $\frac{1}{4}$ in. thickness, the shearing strength of which is 25.2 tons per square inch? Assume that the balls are revolving at the rate of 60 revolutions per minute when the punch comes into contact with the plate, and that the resistance is overcome in the first sixteenth of an inch of the thickness of the plate. *Ans.* 1.136 in.

139. Find the stress due to centrifugal force in the rim of a cast-iron wheel 8 ft. diameter, running at 160 revolutions per minute.

140. Find in pounds the horizontal thrust on the rails, of an engine weighing 20 tons, going round a curve of 600 yds. radius at 30 miles an hour.
Ans. Weight of $13\frac{4}{9}$ cwts.

141. A leather belt runs at 2400 ft. per minute. Find how much its tension is increased by centrifugal action, the weight of leather being taken at 60 lbs. per cubic foot. *Ans.* $20\frac{5}{6}$ lbs. per square inch.

142. Find the centrifugal force arising from a cylindrical crank-pin 6 in. long and $3\frac{1}{2}$ in. in diameter, the axis of the pin being 12 in. from the axis of the engine-shaft, which makes 100 revolutions per minute. How would you balance such a pin? *Ans.* 55.02 lbs.

143. A railway carriage of mass 1000 lbs. is travelling with velocity 50 f/s round a curve. If the radius of the curve be 1000 ft., find the magnitude and direction of the resultant thrust on the rails. If the rails are 4 ft. $8\frac{1}{2}$ ins. apart, find how much the outer rail must be raised so that the carriage may press perpendicularly on the rails.
Ans. 1003.04 lbs.; $\tan^{-1}\frac{5}{64}$; $4\frac{1}{2}$ ins.

144. In a fly-wheel weighing 12,000 lbs. and making 50 revolutions per minute, the centre of gravity is *one seventeenth* of an inch out of the centre. Find the centrifugal force. *Ans.* 50.4 lbs.

145. In the preceding question, if the axis of rotation is inclined to the plane of the wheel at an angle $\cot^{-1}.001$, find the centrifugal couple, the radius of gyration being 10 ft. *Ans.* 1028.9 ft.-lbs.

146. A rubber tire weighing 2 lbs. per foot is stretched over the circumference of a wheel 3 ft. in diameter, the tangential pull in the rubber being

10 lbs. Find the radial pressure exerted by the tire on the circumference of the wheel per inch of length when the wheel is at rest, and the speed at which the wheel must revolve to make the tire cease to exert any radial pressure.

147. An engine of mass 1 ton is travelling round a curve at the rate of 30 miles an hour. If the curve is an arc of a circle whose radius is 1210 ft., determine the horizontal thrust between the engine and the rails.

Ans. .05 ton.

148. Prove that the india-rubber band of a bicycle will become slack when running at more than $(\pi gdT/W)^{\frac{1}{2}}$ f/s, where W denotes the weight of the band in pounds, T the tension in pounds, and d the diameter of the wheel in feet.

149. What is the angular velocity of the 4-ft. wheel of a car which is travelling at the rate of 30 miles per hour? What is the relative velocity of the centre and the highest point of the wheel? *Ans.* 22; 30 m/h.

150. A locomotive engine weighing 9 tons passes round a curve 600 ft. in radius with a velocity of 30 miles an hour. What force tending towards the centre of the curve must be exerted by the rails so that the engine may move on this curve? *Ans.* Weight of 18.15 cwts.

151. If trains are to run at 30 miles an hour, find how much the outer rail should be raised on a curve of half a mile radius, the gauge being 4 ft., so that there shall be no side thrust on the flange. *Ans.* 1.1 in.

152. A railway train is running smoothly along a curve at the rate of 60 miles an hour, and in one of the cars a pendulum which would ordinarily oscillate seconds is observed to oscillate 121 times in two minutes. Show that the radius of the curve is =1650 ft. nearly.

153. If a railway carriage without flanges to its wheels moves on a circular curve, show how the effect of the centrifugal force may be counteracted by a rise of the outer rail, and find what the rise of the outer above the inner rail should be if the radius of the circle be 1320 ft., with a velocity of the train 30 miles an hour, and the breadth of the track 5 ft. *Ans.* $2\frac{3}{4}$ ins.

154. A train weighing 4000 lbs. per lineal ft. has a speed of 30 miles per hour around a 10-degree curve on a steel viaduct. Find (*a*) the transverse pressure per lineal ft. of structure due to the centrifugal load; (*b*) the superelevation of the outer rail. *Ans.* (*a*) 420 lbs.; (*b*) 5.95 inches.

155. The fly-wheel of an engine makes 80 revolutions per minute, and the reciprocal of the coefficient of fluctuation of the velocity is not to exceed 40. Determine the least moment of energy required, the fluctuation of energy per second being 8000 lbs. If the weight of the wheel is 4000 lbs., find the radius of gyration. *Ans.* 4500; 6 ft.

156. A circular disk of cast iron (sp. g. =7.1) 10 ins. in diameter and 1 in thick acts as a pulley for a cord carrying 10 lbs. on one end and 5 lbs. on the other. Find the angular velocity of rotation of the pulley, and the linear velocities of the weights 50 seconds after starting from rest, disregarding the fluctuation of the shaft and its inertia. *Ans.* 770; 320 ft. per second.

157. A fly-wheel of a shearing-machine has 150,000 ft.-lbs. of kinetic energy stored in it when its speed is 250 revolutions per minute; what energy does it part with during a reduction of speed to 200 revolutions per minute? If 82 per cent of this energy given out is imparted to the shears during a stroke of 2 ins., what is the average force due to this on the blade of the shears?

158. A 4 in.×3 in. diameter crank-pin is to be balanced by two weights on the same side of the crank; the length of the crank is 12 ins.; the engine makes 100 revolutions per minute; the distance of the C. of G. of each weight from the axis of the shaft is 6 in. Find the weights.

159. A heavy ball attached by a string to a fixed point O revolves in a horizontal circle with a given uniform angular velocity ω. Find the vertical depth of the centre of the ball below the point of attachment.

If a uniform rod be substituted for the ball and string, find its position.

Also find the position when the ball is attached to the fixed point by a uniform rod; r being the ratio of the weight of the rod to the weight of the ball.

Ans. $h=\frac{g}{\omega^2}$; $h=\frac{3}{2}\frac{g}{\omega^2}$; $h=\frac{3}{2}\frac{g}{\omega^2}\frac{n+2}{n+3}$.

160. The deflection of a truss of l ft. span is $l\times.001$ under a stationary load W. What will be the increased pressure due to centrifugal force when W crosses the bridge at the rate of 60 miles an hour?

Ans. $\frac{242}{125}\frac{W}{l}$.

161. A fly-wheel 20 ft. in diameter revolves at 30 revolutions per minute. Assuming weight of iron 450 lbs. per cubic foot, find the intensity of the stress on the transverse section of the rim, assuming it unaffected by the arms. *Ans.* 96 lbs. per square inch.

162. Diameter of a pipe is 18 in.; at one point it is curved to an arc of 6 ft. radius. Water flows round the curve with a velocity of 6 ft. per second. Determine the centrifugal force per foot of length of bend measured along the axis. *Ans.* 20.717 lbs.

163. A disk of weight W and area A square feet makes n revolutions per second about an axis through its centre, inclined at an angle θ to the normal to the plane of the disk. Find the centrifugal couple.

Ans. $\frac{WAn^2}{5.12}\tan\theta$ ft.-lbs.

164. Assuming 15,000 lbs. per square inch as the tensile strength of cast iron, and taking 5 as a factor of safety, find the maximum working speed and the bursting speed for a cast-iron fly-wheel of 20 ft. mean diameter and weighing 24,000 lbs., the section of the rim being 160 sq. in.

165. A 60-in. driving-wheel weighs $3\frac{1}{2}$ tons, and its C. of G. is 1 in. out of centre. Find the greatest and the least pressure on the rails.

166. A wheel of weight W, radius of gyration k, and making n revolutions

per second on an axle of radius R, comes to rest after having made N revolutions. Find the coefficient of friction.

Ans. $\sin\phi = \frac{\pi n^2 k^2}{Ng}$, and coeff. of fric. $= \tan\phi$.

167. The maximum variation in the energy of a fly-wheel whose mean speed is 101 revolutions per minute is 16,500 ft.-lbs. Find the weight and approximate section of the rim of the wheel, the mean diameter being 10 ft., and the greatest variation of speed being $2\frac{1}{2}$ per cent from its mean value.

168. A hammer weighing 2 lbs. strikes a nail with a velocity of 15 ft. per second, driving it in $\frac{1}{8}$ in. What is the mean pressure overcome by the nail?

Ans. 675 lbs.

169. A mass of 50 lbs. falls from a height of 50 ft. and penetrates 2 ft. into loose sand. To find the resistance of the sand in pounds weight.

Ans. 1300 lbs.

170. A pile-driver of 300 lbs. falls 20 ft., and is stopped in $\frac{1}{10}$ second. What is the average force exerted on the pile? *Ans.* 3344 lbs.

171. Determine in tons the mean thrust on the terminus buffers, which stop in 6 ft. a train of 200 tons going at 6 m/h; also find the time it brakes in seconds. *Ans.* $40\frac{1}{3}$ tons; $1\frac{1}{11}$ seconds.

172. Show that a brake resistance of 385 lbs./ton will destroy a velocity of 60 m./h. in 704 ft. and in 16 seconds.

173. With full brake power it is possible to pull up a train in 225 ft. when running at 30 m/h, and in 900 ft. at 60 m/h. How many seconds does it take to pull up, and what will be the distance overshot by the delay of a second in the action of the brakes?

174. The brakes of a train reduce its speed $3\frac{1}{2}$ m/h every second, and take one second to be applied. In what distance can a train be stopped when going at 30 m/h and at 60 m/h?

175. Determine the length and time in which a barge of 50 tons moving at 2 m/h can be brought up by a rope round a post, supposing the breaking pull of the rope is 1 ton.

176. An express train, timed to run at the full speed of 60 m/h, is checked by a signal to 20 m/h over a mile of road under repair. The train takes one mile from rest to get up full speed, and half a mile to pull up. Show that the train will be 2 minutes 40 seconds late.

177. A weight falls 16 ft. and does 2560 ft.-lbs. of work upon a pile which it drives 4 ins. against a uniform resistance. Find the weight of the ram, and the resistance. *Ans.* 160 lbs.; 7680 lbs.

178. When a nail is driven into wood, why do the blows seem to have little if any effect unless the wood is backed up by a piece of metal or stone?

179. A hammer weighing 2 lbs. strikes a steel plate with a velocity of 10

ft. per second, and is brought to rest in .0001 second. What is the average force on the steel? *Ans.* 6250 lbs.

180. A hammer weighing 10 lbs. strikes a blow of 10 ft.-lbs. and drives a nail .5 in. into a piece of timber. Find the velocity of the hammer at the moment of contact, and the mean resistance to entry. Also find the steady pressure that will produce the same effect as the hammer.

Ans. 8 f/s; 240 lbs.; 480 lbs.

181. In Ex. 179, taking the weight of the nail to be 4 oz. and the weight of the piece of timber to be 100 lbs., find the depth and time of the penetration (*a*) when the timber is fixed; (*b*) when the timber is free to move.

Also in case (*b*) find the distance through which the timber moves.

Ans. (*a*) $\frac{20}{41}$ in.; $\frac{1}{96}$ second; (*b*) .44245 in.; .0009448 second; .04113 in.

182. An inelastic pile weighing half a ton is driven 12 ft. into the ground by 30 blows of a hammer weighing 2 tons falling 30 ft.

Prove that it would require 120 tons in addition to the hammer to be superposed on the pile to drive it down slowly, supposing the resistance of the ground uniform; but 240 tons if the resistance increases as the penetration.

Prove that, with uniform resistance, each movement of the pile takes 0.0228 second.

183. If the resistance of the ground to the penetration of an inelastic pile is 60 tons, prove that 15 blows of a hammer weighing 1 ton falling 20 ft. will drive the pile 4 ft. into the ground, the pile weighing $\frac{1}{4}$ ton.

Prove also that the time of each movement of the pile is 0.01863 second.

184. An *inelastic* pile weighing 788 lbs. is driven $3\frac{1}{2}$ ft. into the ground by 120 blows from a weight of 112 lbs. falling 30 ft Find the steady load upon the pile which will produce the same effect, assuming the ground-resistance to be (*a*) uniform; (*b*) proportional to the depth of penetration. If the resistance is uniform, how long (*c*) does each movement of the pile last? How many blows (*d*) are required to drive the pile the first half of the depth, viz., $1\frac{3}{4}$ ft., the ground-resistance being 7168 lbs.? How far (*e*) does the pile sink under the last blow?

Ans. (*a*) 14,336 lbs.; (*b*) 28,672 lbs.; (*c*) .0107 second; (*d*) 30; (*e*) .0146 ft.

185. A pile is driven *a* ft. vertically into the ground by *n* blows of a steam-hammer fastened to the head of the pile. Prove that, if *p* is the mean pressure of the steam in pounds per square inch, *d* the diameter of piston in inches, *l* the length of stroke in feet, *W* the weight in pounds of the moving parts of the hammer, *w* lbs. the weight of the pile and the fixed parts of the hammer attached to it, then the mean resistance of the ground in pounds is

$$\frac{nW}{W+w}\left(W+\frac{1}{4}\pi d^2p\right)\frac{l}{a}.$$

186. Prove that if a hammer-head weighing 2 lbs., striking with a velocity of 50 f/s a nail $\frac{1}{16}$ in. in diameter and weighing 1 oz., drives the nail 1 in. into a plank of wood, then a bullet 0.5 in. in diameter and weighing 1 oz., striking with a velocity 1500 f/s, will penetrate 1.16 in. of the wood; supposing the resistance uniform and proportional to the sectional area of the hole.

Determine also the penetration of the bullet, supposing the resistance proportional to the penetration.

Determine also the time of movement in each case.

187. Prove that the mean resistance of the wood is 204 lbs. to a nail weighing 1 oz., supposing a hammer weighing 1 lb., striking it with a velocity of 34 f/s drives the nail 1 in. into a fixed block of wood.

If the block is free to move and weighs 68 lbs., prove that the hammer will drive the nail only $\frac{64}{65}$ in.

Prove that the nail is 0.0052 and 0.005128 second in penetrating the wood in the two cases, during which the block if free will move 0.015 in.

188. Show that the total work done in raising a number of weights through to a given level is the product of the sum of the weights and the vertical displacement of their centre of gravity.

189. An engine has to raise 4000 lbs. 1000 ft. in 5 minutes. What is its horse-power? How long will the engine take to raise 10,000 lbs. 100 ft.?
Ans. $24\frac{8}{33}$ H.P.; $1\frac{1}{4}$ minutes.

190. How many men will do the same work as the engine in the preceding question, assuming that a man can do 900,000 ft.-lbs. of work in a day of 9 hours? *Ans.* 480 men.

191. Determine the horse-power which will be required to drag a heavy rock weighing 10 tons at the rate of 10 miles an hour on a level road, the coefficient of friction being 0.8. What will be the speed up a gradient of 1 in 50, the same power being exerted? *Ans.* $477\frac{13}{15}$; $9\frac{31}{41}$ miles per hour.

192. Two horses draw a load of 4000 lbs. up an incline of 1 in 25 and 1000 ft. long. Determine the work done. *Ans.* 160,000 ft.-lbs.

193. At what speed do the horses walk if each horse does 16,000 ft.-lbs. of work per minute? *Ans.* $2\frac{8}{11}$ miles per hour.

194. It is said that a horse can do about 13,200,000 ft.-lbs. of work in a day of 8 hours, walking at the rate of $2\frac{1}{2}$ miles per hour. What pull (in pounds) could such a horse exert continuously during the working day? How many such horses would be required to do as much work as an engine of 10 H.P., working day and night? *Ans.* 125 lbs.; 36.

195. Find the shortest distance in which a train going at 30 m/h can be brought up by continuous brakes pressing on the wheels with *three fourths* of the weight of the train. Take .16 as the coefficient of friction.
Ans. About 252 ft.

196. A train of 60 tons travels 20 miles in 1 hour with *nine* intermediate stoppages, each of 2 minutes, at intervals of 2 miles. The resistance of the road is 10 lbs. per ton, and the brake-power of the engine and brake-van, half the weight of the train, is *one sixth* of its weight. Find the pull of the engine and its horse-power when running at full speed.

197. An express train reduced speed from 60 to 20 miles an hour in 800 yds., the distance between the distant and home signals. How much farther

out should the distant signal be placed, or how much should the brake-power be increased?

198. A train of 100 tons fitted with continuous brakes is to be run on a level line between stations one third of a mile apart at an average speed of 12 m/h, including two thirds of a minute stop at each station. Prove that the weight on the driving-wheels must exceed $22\frac{1}{2}$ tons, with an adhesion of one sixth.

Prove that this line can be worked principally by gravity if the road is curved down between the stations to a radius of 11,740 ft., implying a dip of 33 ft. between the stations, a gradient at a station of 1 in 13, and a maximum running velocity of 31 m/h.

199. Show that the time lost by a train with continuous brakes, overshooting a station a ft. and backing in, is $(\sqrt{n+1}+1)\sqrt{\dfrac{2a}{\mu g}}$ seconds *at least*, the coefficient of adhesion being μ and the weight on the driving-wheels $\dfrac{1}{n}$ of the weight of the train.

200. (*a*) A train weighing 160 tons (of 2240 lbs.) travels at 30 miles an hour against a resistance of 10 lbs. per ton. What horse-power is exerted?

(*b*) With the same horse-power what will be the speed up a gradient of 1 in 100?

(*c*) If the steam is shut off, how far will the train run before stopping (1) on the incline; (2) on the level?

(*d*) If the draw-bar suddenly breaks, in what distance would the carriages (100 tons in weight) be stopped if the brakes are applied immediately the fracture occurs, the weight of the brake-van being 20 tons and the coefficient of friction .2?

(*e*) If the engine (weight = 60 tons) continued to exert the same power after the fracture, what would be its ultimate speed?

(*f*) What resistance would be required to stop the whole train, after steam is shut off, in 1000 yds. on the level?

Ans. (*a*) 128; (*b*) $9\frac{7}{27}$ miles per hour; (*c*) (1) 199.2 ft., (2) 6776 ft.; (*d*) 680.3 ft. on the level, 52.9 ft. on the incline; (*e*) 80 miles an hour on the level, 24.6 miles on the incline; (*f*) 22.58 lbs. per ton.

201. Prove that a train going 60 miles an hour can be brought to rest in about 313 yds. by the brakes, supposing them to press on the wheels with two thirds of the weight of the train, taking a coefficient of friction 0.18 in addition to a passive resistance of 20 lbs. a ton.

Prove that the mean uniform pull to be exerted by an engine to take this train, weighing 100 tons, on the run from one station to a stop at the next, 2 miles off, in 4 minutes, is about 2.6 tons.

202. A train of 120 tons is to be taken from one station to the next, a

mile off, up an incline of 1 in 80, in 4 minutes without using the brakes. Prove that with no road resistance the engine must exert a pull, until steam is turned off, of about 6203 lbs., and the weight on the drivers must be 37,218 lbs. =16.6 tons, with an adhesion of one sixth.

203. With a coefficient of adhesion μ a motor car actuated and braked on the hind axle can get up a speed v f/s in x ft., or be brought to rest again in y ft. given by $g\mu ax=v^2(a-\mu h)$ and $g\mu ay=v^2(a+\mu h)$, a being the distance between the axles and h the weight from the ground of the C. G. midway between the wheels.

204. Determine in tons the greatest train an engine capable of exerting a uniform pull of 3 tons can take on the level, from one station to the next, a mile off, in 4 minutes, supposing the resistance of the road estimated at 20 lbs. a ton, and the brake power at 400 lbs. a ton in addition.

205. Supposing that 1 in m is the steepest incline a train can crawl up with uniform velocity, and 1 in n is the steepest incline on which the brakes can hold the train, prove that the quickest run up an incline of 1 in p, from one station to stop at the next, a distance of a ft., which can be made is $\sqrt{\frac{2a(m+n)p^2}{g(p-m)(p+n)}}$ seconds. Also determine the proportion of the weight which must be carried by the driving-wheels of the engine with a coefficient of adhesion μ, assuming the resistance on the level to be equivalent to an incline of 1 in q.

206. If the end of a railway wagon exposes a surface of 6×4 ft. to the wind, what is the greatest gradient up which a 20 lb. to the square foot gale will drive it? Take the weight at 10 tons, the friction 10 lbs. per ton.

Ans. 1 in 59.

207. A locomotive and tender weigh 70 tons, of which 26 tons are carried, by the driving-wheels. Taking the adhesion at $\frac{1}{6}$, friction 10 lbs. per ton-what maximum gradient can the engine ascend? *Ans.* 1 in 16.

208. Determine the pull of an engine and the weight on the drivers, on which $d=20$, $1=24$, $D=60$, for a mean effective pressure $p=100$ lbs. on the square inch, taking an adhesion of one sixth.

209. The weight upon the driving-wheels (D in. in diameter) of a locomotive is W tons; the adhesion = *one fifth*; the cylinders have a diameter of d in. and a stroke of l in. Find the steam-pressure required to skid the wheels

Ans. $400\frac{WD}{d^2l}$ lbs. per square inch.

210. In an express engine the driving-wheels are 8 ft. in diameter, and the load on them is 15 tons; the cylinders are 18″×28″. Find the pressure of steam which will skid the wheels with an adhesion of one sixth. Determine the ratio of the velocity of the engine to the velocity of the piston at any point of the stroke.

211. A locomotive capable of exerting a uniform pull of 2 tons, with a 24-in. stroke, 20-in. cylinder, and 60-in. driving-wheels, hauls a train between two stations 3 miles apart. The gross weight of the train and locomotive = 200 tons; the road resistance = 12 lbs. per ton (of 2000 lbs.); the brakes, when applied, press with two thirds of the weight on the wheels of the engine and brake-van, viz., 90 tons, the coefficient of friction being .18. Find (*a*) the least time between the stations; (*b*) the distance in which the train is brought to rest; (*c*) the maximum speed attained; (*d*) the pressure of the steam; (*e*) the weight upon the driving-wheels.

Ans. (*a*) 513.8 seconds; (*b*) 990 ft.; (*c*) 42 miles per hour; (*d*) 25 lbs. per square inch.; (*e*) $11\frac{1}{3}$ tons.

212. If the speed in the last question is limited to 30 miles an hour, find (*a*) the time between the stations; (*b*) the distance in which the train is brought to rest; (*c*) the distance traversed at 30 miles an hour.

Ans. (*a*) $543\frac{1}{3}$ seconds; (*b*) $504\frac{1}{6}$ ft.; (*c*) $7773\frac{1}{3}$ ft.

213. If the steam-pressure in the above locomotive is increased to 50 lbs. per square inch, find (*a*) the weight of the heaviest train which can be hauled between the stations in 10 minutes, the road-resistance being 20 lbs. per ton (of 2000 lbs.), and the braking-power being sufficient to bring the train to rest in a distance of 720 ft.

Also find (*b*) the braking-power; (*c*) the weight thrown upon the drivers, the coefficient of friction being $\frac{1}{6}$; (*d*) the maximum speed attained.

Ans. (*a*) $310\frac{1}{2}$ tons; (*b*) 15.6 tons; (*c*) 24 tons; (*d*) 36 miles per hour.

214. Two trains, each with a brake-power of 190 lbs. per ton (of 2000 lbs.), run between Montreal and Toronto, a distance of 333 miles, against an average resistance of 10 lbs. per ton. One train runs through, and the other stops at N intermediate stations. Show that the saving of fuel in the former is $\frac{9N}{25}$ per cent; the speed is not to exceed 30 miles per hour.

215. Prove that the horse-power of an engine drawing a train of 120 tons up an incline of 1 in 224 at 30 miles an hour is 336, taking the resistance of the road on the level at this speed at 25 lbs. a ton.

Determine also the horse-power of an engine drawing a train of 200 tons up an incline of 1 in 140 at 30 miles an hour, resistance of road 18 lbs. a ton; or of the engine drawing the train down an incline of 1 in 140 at 50 miles an hour, resistance at this speed 50 lbs. a ton.

Determine the pull of the engine, and supposing it constant, find how far the train would go up an incline of 1 in 100, before the velocity dropped from 50 to 30 miles an hour, taking an average uniform resistance of the road of 20 lbs. a ton.

216. A locomotive exerting a uniform pull of 4 tons hauls a train of 200 tons up an incline of 1 in 200, between two stations 2 miles apart, the greatest allowable speed being 30 miles an hour. If the road-resistance is 10 lbs. per ton (of 2000 lbs.), and if the brakes are capable of exerting a pressure of 100 tons, the adhesion being *one fifth*, find (*a*) the time between the stations; (*b*)

the distance in which the train is brought to rest; (*c*) the distance traversed at 30 miles.

Also, if the speed is not limited to 30 miles, find (*d*) the least time in which the distance can be accomplished; (*e*) the maximum speed attained; (*f*) the distance in which the train is brought to rest.

Ans. (*a*) $5\frac{1}{4}$ minutes; (*b*) 275 ft.; (*c*) 7260 ft.; (*d*) 4.47 minutes; (*e*) 53.8 miles per hour; (*f*) 880 ft.

217. With the same brake-power, adhesion, and road-resistance, find the weight of the heaviest train which the locomotive in the preceding question, exerting the uniform pull of 4 tons, can haul between the two stations in 6 minutes. *Ans.* 360 tons.

218. If the locomotive has 60-in. drivers and $24'' \times 20''$ diameter cylinders find the weight required upon the drivers when the steam-pressure is 50 lbs per square inch. *Ans.* 20 tons.

219. Prove that the loss of time in going from A to C, two points on a railway at the same level 8 miles apart, due to an incline of 1 in 100 from A up to B, and an incline of 1 in 300 from B down to C, instead of going on a level line from A to C at a uniform velocity of 45 miles an hour, is 2 minutes 20 seconds, equivalent in time to a detour on the level of 1 mile 60 chains.

It is supposed that, with full steam on, the velocity drops from 45 to 15 miles an hour at the summit B, and that in descending the incline full steam is kept on till the velocity is again 45, after which the velocity is kept uniform by partly shutting off steam; and prove that this happens at a point Q, distant from B about 1 mile 892 yards.

Prove that the train would be "stalled on the grade" if the incline from A to B was a quarter of a mile longer.

220. Prove that if the weight of the train is 200 tons, and the resistance of the road is 14 lbs. a ton, the pull of the engine from A to Q is $2\frac{5}{48}$ tons, and from Q to B is $\frac{7}{12}$ ton; and determine whether there is any extra expenditure of work due to the inclines on this and the return journey.

Determine the requisite weight on the driving-wheels, taking an adhesion of one sixth.

221. The distance between two stations is 105 miles, and there are 27 intermediate stations. The average resistance of a parliamentary train stopping at all stations is taken as 8 lbs. a ton, while the resistance to an express train which runs through without stopping is taken at 10 lbs. a ton, the brake-power in each case being taken at 80 lbs. a ton additional.

Supposing the speed to be kept constant by reducing steam when it has reached 30 miles an hour, find which train is most expensive in fuel, and by how much per cent.

Work out the same problem with their resistances and full-speed doubled.

222. Determine the loss of time and the extra expenditure of work—if any—due to crossing a pass by a railway having an incline of 1 in m up, and 1 in n down, instead of going in a level tunnel of length l ft. through the pass;

supposing that V f/s is the maximum speed allowed on the line, and that the velocity of the train drops from V to v f/s at the summit of the pass; determine the pull of the engine P in tons for a train of W tons weight, taking the resistance of the road at r lb. per ton.

Determine the length of detour of equal time.

223. The crank of a horizontal engine is 3 ft. 6 ins. and the connecting-rod 9 ft. long. At half-stroke the pressure in the connecting-rod is 500 lbs. What is the corresponding twisting moment on the crank-shaft?

Ans. 1716½ ft.-lbs.

224. A piston and rod and cross-head weigh 330 lbs. At a certain instant when the resultant total forces due to steam-pressure is 3 tons, the piston has an acceleration of 370 ft. per second in the same direction. What is the actual force acting at the cross-head?

225. If the connecting-rod is 5 ft. long and the crank is 1 ft. in an engine making 200 revolutions per minute, what are the accelerations of the piston when it is farthest from and nearest to the crank? The piston and rod and cross-head weigh 330 lbs.; the area of piston = 120 sq. ins.; at the beginning of either the in- or the out-stroke the pressure is 80 lbs. per square inch on one side in excess of what it is on the other. Find the total forces on the cross-head in these two cases.

226. A revolving weight W is at a distance r from the axis of rotation and is to be balanced by two weights each r_1 from the axis, the one being a on the right and the other b on the left of the weight to be balanced. Find, the weights.

Ans. $W\frac{r}{r_1}\frac{b}{a+b}$; $W\frac{r}{r_1}\frac{a}{a+b}$.

227. There is a balance-weight of 180 lbs. at a distance of 3.4 ft. from the centre, and another weight of 150 lbs. at a distance of 2.56 ft. from the centre, in a direction at right angles to the first, both on the same driving-wheel of a locomotive. Find the amount and position of any single weight which would have the same balancing effect as these two.

228. Each piston of a locomotive weighs 300 lbs. What balance-weights will completely balance one piston so that there may be no couple and no horizontal force? Stroke = 24 in.; distance C. to C. of cylinders = 42 ins.; radial distance of balance weights = 39 ins.; distance between centres of gravity of balance-weights = 57 ins.

Ans. 12.15 lbs.; 80.15 lbs.

229. The piston and rod and cross-head weigh 330 lbs.; the connecting-rod 300 lbs.; the 12-in. crank is equivalent to 270 lbs. at crank-pin; the C. of G. of the connecting-rod is 0.38 of its length from the crank-pin. If the centre lines of the cylinders are 27 ins. apart, and the middles of the wheels are 59 ins. apart, neglecting valve-motions, etc., calculate the balance-weights and their positions. Place their centres of mass 3 ft. from the centres of the wheels. Is the locomotive in perfect balance? What is the nature of the balance?

230. Find the acceleration-pressure at each end of the stroke of a vertical

inverted high-speed steam-engine when running at 500 revolutions per minute stroke 9 ins., weight of reciprocating parts 110 lbs., diameter of cylinder 8 ins., length of connecting-rod 1.5 ft.

Ans. 54.8 lbs. / sq. m. at bottom; 85.5 lbs. / sq. m. at top.

231. In a horizontal marine engine with two cranks at right angles distant 8 ft. from one another, weight of reciprocating parts attached to each crank is 10 tons, revolutions 75 per minute, stroke 4 ft., find the alternating force and couple due to inertia. *Ans.* 54.2 tons; 218.26 ft.-tons.

232. An inside-cylinder locomotive is running at 50 miles an hour; the driving-wheels are 6 ft. in diameter; the distance between the centre lines of the cylinders is 30 in., the stroke 24 in., the weight of one piston and rod 300 lbs., and the horizontal distance between the balance-weights $4\frac{3}{4}$ ft.; the diameter of the weight-circle is $4\frac{1}{2}$ ft. Find the alternating force and couple, and also the magnitude and position of suitable balance-weights.

Ans. 7871 lbs.; 9839 ft.-lbs.; 106.5 lbs.; $27\frac{3}{4}°$.

233. The pressure equivalent to the weight of the reciprocating parts of an engine is 3 lbs. per square inch; the stroke is 36 in.; the number of revolutions per minute is 45; the back-pressure is 2 lbs. per square inch; the absolute, initial steam-pressure is 60 lbs. per square inch; the rate of expansion is 3. Find the pressure necessary to start the piston, and also the effective pressure at each $\frac{1}{3}$ of the stroke.

234. An engine with a 24-in. cylinder and a connecting-rod = six cranks = 6 ft., makes 60 revolutions per minute. Show that the pressure required to start and stop the engine at the dead-points $= \frac{11}{24}$ of the weight of reciprocating parts.

235. Find the ratio of thrust at cross-head to tangential effort on crank-pin when the crank is 45° from the line of stroke, the connecting-rod being = four cranks. *Ans.* 6 to 5.

236. Draw the linear diagram of crank-effort in the case of single crank, the connecting-rod being = four cranks. Assume the resistance uniform and a constant pressure of 9000 lbs. on the piston, the stroke being 4 ft. and the number of revolutions per minute 55. Also find the fluctuation of energy in foot-pounds for one revolution.

237. An engine with a connecting-rod = six cranks = 6 ft. receives steam at 70 lbs. pressure per square inch, and cuts off at *one-quarter* stroke. Find the crank-effort when the piston has travelled *one third* of its forward stroke. Diameter of piston = 2 ft. Also find the position of the piston where its velocity is a maximum.

238. Data: Stroke = 3 ft.; number of revolutions per minute = 60; cut-off at one-half stroke; initial pressure = 56 lbs. per square inch absolute; diameter of piston = 10 in.; weight of reciprocating parts = 550 lbs.; back-pressure = $1\frac{1}{2}$ lbs. per square inch absolute. Find the effective pressure at each fourth of the stroke, taking account of the inertia of the piston. Also find the pressure equivalent to *inertia* at commencement of stroke.

239. A pair of 250-H.P. engines, with cranks at 90°, and working against a uniform resistance and under a uniform steam-pressure, are running at 60 revolutions per minute. Assuming an indefinitely long connecting-rod, find the maximum and minimum moments of crank-effort, the fluctuation of energy, and the coefficient of energy.

240. An inside-cylinder locomotive runs 25 miles per hour; its drivers are 60 in. in diameter; the stroke is 24 ins.; the distance between the centre lines of the cylinders = 30 in.; weight of reciprocating parts = 500 lbs.; horizontal distance between balance-weights = 59 in.; diameter of weight-circle = 42 in. Find the alternating force, alternating couple, and the magnitude and position of suitable balance-weights.

Ans. 226.8 lbs.; 4113.8 ft.-lbs.; $\phi = 26°$.

241. Draw a diagram of crank-effort for a single crank, the connecting-rod being equal to *four* cranks, the stroke 4 ft., and the number of revolutions per minute 55. Assume a uniform resistance and a constant pressure of 9000 lbs. on the piston.

CHAPTER IV.

STRESS; STRAIN; ELASTICITY; OSCILLATION; THIN CYLINDER.

1. **Stress and Strain; Resilience.**—The science relating to the strength of materials is partly theoretical, partly practical. Its primary object is to investigate the forces developed within a body, and to determine the most economical dimensions and form, consistent with stability, of that body. Certain hypotheses have to be made, but they are of such a nature as always to be in accord with the results of direct observation.

The materials in ordinary use for structural purposes may be termed, generally, *solid bodies*, i.e., bodies which offer an appreciable resistance to a change of form.

A body acted upon by external forces is said to be *strained* or *deformed*, and the straining or deformation induces *stress* amongst the particles of the body.

The state of strain is *simple* when the stress acts in *one* direction only, and the strain itself is measured by the ratio of the deformation to the original length.

The state of strain is *compound* when *two* (or *more*) stresses act simultaneously in different directions.

A strained body tends to assume its natural state when the straining forces are removed; this tendency is called its *elasticity*. A thorough knowledge of the laws of elasticity, i.e., of the laws which connect the external forces with the internal stresses, is absolutely necessary for the proper comprehension of the strength of materials. This property of elasticity is not possessed to the same degree by all bodies. It may be almost perfect or almost zero, but in the majority of cases it has a mean value. Hence it naturally follows that solid bodies may be classified between two extreme, though ideal, states, viz., a *perfectly elastic* state and a *perfectly soft* state.

Perfectly elastic bodies which have been strained resume their original forms exactly when the straining forces are removed. Perfectly soft bodies are wholly devoid of elasticity and offer no resistance to a change of form.

Bodies capable of undergoing an indefinitely large deformation under stress are said to be *plastic.*

Every body may be subjected to five distinct kinds of stresses, viz.:

(*a*) A longitudinal pull, or tension.

(*b*) A longitudinal thrust, or compression.

(*c*) A shear, or tangential stress, which may be defined as a stress tending to make one surface slide over another with which it is in contact.

(*d*) A transverse stress.

(*e*) A twist or torsion.

Under any one of these stresses a body may suffer either an elastic deformation of a temporary character or a plastic deformation of a permanent character.

Let a wire or a bar, of length L and uniform sectional area A, be fixed at the upper end and hang vertically. A load P uniformly distributed over the lower end will stretch the bar by an amount l which can be readily measured by an extensometer or by the aid of a telescope and a vertical scale. If this bar is now replaced by one of the same length but of twice the sectional area, it is found that it requires twice the load to stretch it by the same amount. Also, the load upon a column or strut producing a certain shortening of the length is twice as great as that required to produce the same amount of shortening in a strut of the same length but one half the sectional area.

The ratio $\frac{l}{L}$, i.e., *the change of length* (or deformation) *per unit of length,* is called the *strain.*

Again, the load acts at every cross-section in a similar manner, and it is well to remember that, according to St. Venant, the actual distribution of load on a small area is not of much importance. Whatever the distribution may be, the strain at a point not very near is the same. To eliminate any difference which might exist at a point where the bar has a greater or less sectional area, it is advisable to use the *load per unit of area*, viz., $\frac{P}{A}$, instead of the total load P.

The *load per unit of area* is called the *stress.*

Experiment then shows that, *within certain limits, the stress is practically proportional to the strain.*

This relation between stress and strain was enunciated by Hooke and is known as Hooke's Law.

It may be expressed by the equation

$$\mathbf{f = Ee},$$

where $f = \frac{P}{A}$ = the load per unit of area, or stress, and

$e = \frac{l}{L}$ = the strain per unit of length.

The multiplier E is a number whose value depends upon the character of the material. It is called Young's *coefficient* or *modulus of elasticity,* and tables at the end of the chapter give the value of E for different materials. It must not be forgotten that the more homogeneous a material is the more accurate is the relation $f = Ee$.

The longitudinal strain is accompanied by an alteration in the transverse dimensions, the lateral unit strain being $\mp \frac{e}{\sigma}$, where $\frac{1}{\sigma}$ is a coefficient which usually varies from $\frac{1}{3}$ to $\frac{1}{4}$ for solid bodies and is approximately $\frac{1}{4}$ for the metals of construction. In the case of India-rubber, if the deformation is small, $\frac{1}{\sigma}$ is about $\frac{1}{2}$.

Generally the deformation may be calculated per unit of *original* length without sensible error, but for India-rubber it is more accurate to make the calculation per unit of *stretched length* $\left(= \frac{e}{1+e} \right)$.

The ratio of the lateral to the longitudinal strain is called Poisson's ratio.

Ex. 1. A cable 1000 ft. long stretches 6 ins. under a given load; what is the strain?

$$\text{The strain} = \frac{6}{1000 \times 12} = .0005.$$

Ex. 2. *A short cast-iron strut* 24 *ins. long and of* 8 *sq. ins. sectional area bears a load of* 32 *tons. If* $E = 8000$ *tons/sq. in., how much is the strut shortened? What is the stress? What the strain?*

$$\text{The stress} = \tfrac{32}{8} = 4 \text{ tons/sq. in.}$$

$$4 = 8000 \times \text{the strain},$$

and therefore $\quad$ the strain $= \frac{4}{8000} = .0005.$

The diminution of length $= 24 \times$ the strain $= .012$ in.

Ex. 3. *A mild-steel bar* 100 *ft. long and of* 2 *sq. ins. sectional area carries a load which developes in the material a stress of* 16,000 *lbs./sq. in. If* $E =$ 30,000,000 *lbs./sq. in., by how much is the bar lengthened? What is the strain? What is the total load?*

$$\text{The total load} = 16000 \times 2 = 32000 \text{ lbs.}$$
$$16000 = 30000000 \times \text{the strain.}$$

Therefore $\quad$ the strain $= \dfrac{16000}{30000000} = \dfrac{1}{1875}$,

and $\quad$ the stretch $= 100 \times \dfrac{1}{1875} = \dfrac{4}{75}$ ft. $= .64$ in.

Again, the change of length l is produced by a load which gradually increases from 0 to P, and therefore

$$\text{the work done in changing the length} = \frac{P}{2}l = \frac{fA}{2}\,\frac{fL}{E} = \frac{f^2}{2E}AL,$$

and this is the energy stored up in the strained bar. It is called the *resilience* of the bar if the load is only a little less than that which produces a *permanent* change of length, i.e., a permanent deformation or *set*. Such a load is the *proof load*, and the stress f becomes the *proof stress*. Also, since AL is the volume, the resilience per unit of volume is $f^2/2E$.

Thus the *resilience* is the greatest amount of energy which can be stored by a material without becoming permanently deformed. If the stress is variable, the resilience is necessarily less, while a blow or a shock may develope in the material a larger amount of strain energy than it can bear, so that a local deformation (or *set*) and a plastic yielding may occur.

Exs. 4 and 5. *Find the work done in producing the changes of length in Exs.* 2 *and* 3.

$$\text{In Ex. 2, the work in in.-tons} = \frac{32}{2} \times .012 = .192.$$

In Ex. 3, the work in in.-lbs. $=\frac{32000}{2}\times .64=10240,$

or again, the work in in.-lbs. $=\frac{(16000)^2\times 2\times 100\times 12}{2\times 30000000}=10240.$

Ex. 6. *A bar of weight W_1, length L, and uniform sectional area A is fixed at its upper end, hangs vertically, and carries a weight W_2 at its lower end. Determine the stretch of the bar and the work of stretching.*

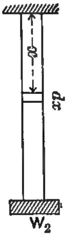


FIG. 299.

Consider a slice of thickness dx at the distance x from the fixed end.

The weight on the slice $=\frac{W_1}{L}(L-x)+W_2$, and if dl is the extension of dx under this weight,

$$\frac{\frac{W_1}{L}(L-x)+W_2}{A}=E\frac{dl}{dx}.$$

Therefore, $l=\frac{1}{EA}\int_0^L\left\{\frac{W_1}{L}(L-x)+W_2\right\}dx=\frac{L}{EA}\left(\frac{W_1}{2}+W_2\right).$

The work done in stretching dx by the amount dl, under a weight which gradually increases from 0 to $\frac{W_1}{L}(L-x)+W_2$,

$$=\frac{1}{2}\left\{\frac{W_1}{L}(L-x)+W_2\right\}dl=\frac{1}{2EA}\left\{\frac{W_1}{L}(L-x)+W_2\right\}^2dx,$$

and therefore the work done in stretching the whole bar

$$=\frac{1}{2EA}\int_0^L\left\{\frac{W_1}{L}(L-x)+W_2\right\}^2dx=\frac{1}{2}\frac{L}{EA}\left(\frac{W_1^2}{3}+W_1W_2+W_2^2\right).$$

Ex. 7. *A vertical steel bar of 200 ft. length is fixed at its upper end, hangs vertically, and carries a load of 100 tons at the lower end. The stress in the material is not to exceed 7 tons/sq. in. and is to be the same at every point of the bar. Determine the form of the bar.*

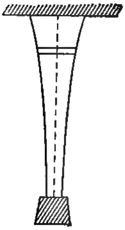


FIG. 300.

Consider a slice of thickness dx and sectional area A at a distance x from the fixed end.

At $x+dx$ the area is $A-dA$, the amount dA being the additional area required at x to support the weight of the slice dx. Therefore

$$-f dA=wA dx,$$

w being the specific weight, and f the stress developed in the material.

This equation may be written

$$-\frac{w}{f}dx=\frac{dA}{A}.$$

Integrating, $$\frac{w}{f}(L-x)=\log_e\frac{A}{A_1},$$

A_1 being the sectional area at the lower end and L being the length of the bar.

Therefore the form of the bar is given by

$$\frac{A}{A_1}=e^{\frac{w}{f}(L-x)}.$$

If A_2 is the area at the upper end, i.e., when $x=0$,

$$\frac{A_2}{A_1}=e^{\frac{w}{f}L}.$$

Such a bar is called a bar of uniform strength.

In the present case $A_1=\frac{100}{7}=14\frac{2}{7}$ sq. ins.

and $$A_2=\frac{100}{7}e^{\frac{490\times 200}{7\times 2000\times 144}}=\frac{100}{7}e^{\frac{7}{144}}=\frac{100}{7}\times 1.05=15 \text{ sq. ins.}$$

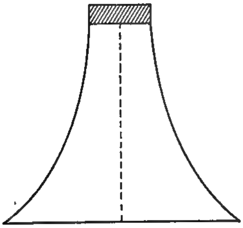


FIG. 301.

Ex. 8. *A pier of height L, with its axis vertical and carrying a weight on the top, is to have the same stress f per unit of area at every point, i.e., is to be of uniform strength. Determine the form of the pier, w being its specific weight.*

If A is the area of the pier at x from the base, then $-dA$ is the diminution of area at $x+dx$, corresponding to the weight of the slice dx by which the load upon A is reduced. Therefore

$$wAdx=-fdA,$$

or $$-\frac{w}{f}x=\frac{dA}{A}.$$

Integrating, $$-\frac{w}{f}x=\log_e\frac{A}{A_1},$$

A_1 being the area of the base of the pier.

Thus the form of the pier is given by

$$\frac{A}{A_1}=e^{-\frac{w}{f}x}.$$

The volume of the pier $= \int_0^L A\,dx = A_1 \int_0^L e^{-\frac{w}{f}x}\,dx$

$$= A_1 \frac{f}{w}\left(1 - e^{-\frac{w}{f}L}\right).$$

Also, if A is the area of the top of the pier,

$$\frac{A_2}{A_1} = e^{-\frac{w}{f}L}$$

and the surcharge $= fA_2 = fA_1 e^{-\frac{w}{f}L}$.

If $L = \infty$, i.e., if the pier is infinitely high,

$$e^{-\frac{w}{f}L} = 0.$$

Therefore A_2 and the surcharge are each *nil*, and the volume of the pier $= A_1 \frac{f}{w}$.

Ex. 9. *A steel bar of 2 sq. ins. sectional area has its ends fixed between two immovable blocks when the temperature is at 40° F. What pressure will be exerted upon the blocks if the temperature of the bar is raised to 100° F., the coefficient (α) of linear dilatation per degree being .00108÷180? $E = 30{,}000{,}000$ lb./sq. in.*

If the length of the bar is L, and if it were allowed to lengthen freely under an increasing temperature, then, when the temperature is t° F.,

the *additional* length $= L\alpha t$.

Hence the stress developed when the extension is prevented

$$= E\frac{L\alpha t}{L} = E\alpha t,$$

and, in the present case, the pressure on the blocks

$$= 2 \times 30000000 \times \frac{.00108}{180} \times 60$$
$$= 21600 \text{ lbs.}$$

2. Theory of Vibrations.—Let a particle of weight W be acted upon by a force which is proportional and acts directly opposite to its displacement from a fixed point O, Fig. 302. Then

$$\frac{W}{g}\frac{d^2x}{dt^2} = -\mu x,$$

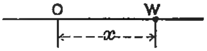


FIG. 302.

μ being some constant.

Multiplying each side by $2\frac{dx}{dt}$ and integrating,

$$\frac{W}{g}\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2) = \frac{W}{g}v^2,$$

where $\frac{dx}{dt} = 0$, when $x = a$ and the velocity of the particle at any distance x from O is given by

$$v^2 = \frac{g\mu}{W}(a^2 - x^2).$$

The equation may be written

$$\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\frac{g\mu}{W}} \cdot dt.$$

Integrating again,

$$\sin^{-1}\frac{x}{a} = \sqrt{\frac{g\mu}{W}}t + c,$$

c being a constant of integration.

But $x = a$, when $t = 0$, and therefore $c = \frac{\pi}{2}$.

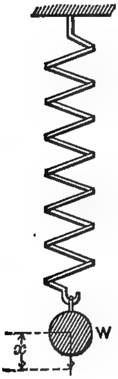


FIG. 303.

Hence $$\sin^{-1}\frac{x}{a} = \sqrt{\frac{g\mu}{W}}t + \frac{\pi}{2},$$

or $$x = a \cos \sqrt{\frac{g\mu}{W}}t,$$

and therefore the time of a complete oscillation is

$$2\pi\sqrt{\frac{W}{g\mu}}.$$

Ex. 10. *A spiral spring, Fig. 303, whose coefficient of stiffness is e is loaded with a weight W and then depressed a distance x below its position of equilibrium.*

Disregarding the weight of the spring,

$$\frac{W}{g}\frac{d^2x}{dt^2} = \text{the "restoring" force of the spring}$$
$$= -ex,$$

and therefore the spring will oscillate, the time of complete oscillation being

$$2\pi\sqrt{\frac{W}{ge}}.$$

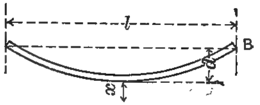


FIG. 304.

Ex. 11. *A straight spring AB is loaded in the middle with a weight W. It is then deflected still further by an amount x.*

Let the "restoring" force F due to this additional deflection be such that

$$x = n\frac{Fl^3}{EI},$$

l being the length of the beam. (See Chap. VII.)

Then $$\frac{W}{g}\frac{d^2x}{dt^2} = -F = -\frac{EI}{nl^3}x.$$

Thus, the weight will oscillate about its position of equilibrium, and the time of a complete oscillation $=2\pi\sqrt{\frac{Wnl^3}{gEI}}$.

Ex. 12. *A revolving shaft is supported at the point B of a cast-iron standard ABD*

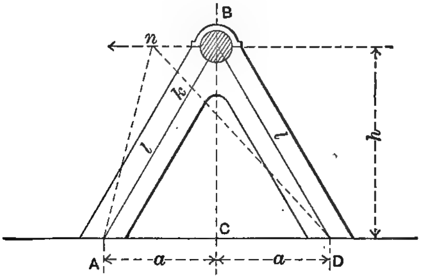


FIG. 305.

Let W be the weight carried at B. Each rod supports in the direction of its length a force (either a tension or a compression) $=\frac{W}{2}\frac{l}{h}$.

Let B be displaced to n and draw nk perpendicular to AB. Then AB is shortened by the length Bk. Also,

$$\frac{Bk}{Bn} = \frac{AC}{AB} = \frac{a}{l}.$$

Therefore $$Bk = x = \frac{a}{l}Bn$$

and the restoring force is $Ex = -\frac{1}{g}\frac{W}{2}\frac{l}{h}\frac{d^2x}{dt^2}$,

E being Young's modulus of elasticity for the material of the bars. Hence the period of a complete vibration of the rod

$$=2\pi\sqrt{\frac{Wl}{2ghE}}.$$

General Equation of Motion.—The preceding examples seem to indicate that if vibrations are once started they will never die out. They are destroyed, however, sooner or later by other forces, some of the nature of *friction* which are constant, and others of the nature of viscosity which are directly proportional to the velocity $\left(\frac{dx}{dt}\right)$.

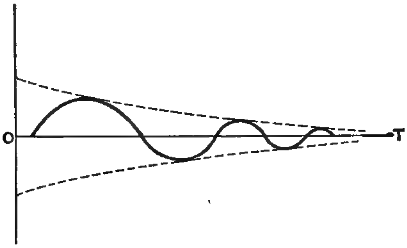


FIG. 306.

The complete equation of motion may be written in the form

$$\frac{d^2x}{dt^2}+2p\frac{dx}{dt}+qx+r=0.$$

Put $$x=-\frac{r}{q}+ye^{-pt}.$$

Then the equation becomes

$$\frac{d^2y}{dt^2}+(q-p^2)y=0,$$

from which $$y=D\cos\sqrt{q-p^2}t,$$

D being a coefficient whose value is to be found.

Hence $$x=-\frac{r}{q}+De^{-pt}\cos\sqrt{q-p^2}t,$$

and the extent of the vibrations diminishes as t increases, i.e., they die out.

Forced Oscillation.—The term *free oscillation* is applied to a condition of vibration in which the forces acting upon the body depend only on the displacements of the several particles from the position of equilibrium. If other forces which are due to external causes and are functions of the time also act, the oscillation is called a forced oscillation. For example, let the spiral spring be loaded with a weight W and then depressed through a distance x below its neutral position, the restoring force being ex. Let a force $q \sin nt$, which is a function of the time, also act upon the weight. Then its equation of motion becomes

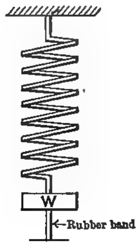


FIG. 307.

$$\frac{d^2x}{dt^2} = -\frac{g}{W}ex + q \sin nt.$$

Integrating,
$$x = A \cos \sqrt{\frac{ge}{W}}t - \frac{q}{n^2} \sin nt.$$

A being a constant whose value is to be determined.

Thus the oscillation is compounded of a free and a forced oscillation, the periods being $2\pi\sqrt{\frac{W}{ge}}$ and $\frac{2\pi}{n}$, respectively.

3. On the Oscillatory Motion of a Weight at the End of a Vertical Elastic Rod.—An elastic rod of natural length $L(OA)$ and sectional area A is suspended from O, and carries a weight P at its lower end, which elongates the rod until its length is $OB = L + l$.

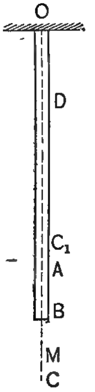


FIG. 308.

Assume that the mass of the rod as compared with P is sufficiently small to be disregarded, then

$$P = EA\frac{l}{L}.$$

If the weight is made to descend to a point C, and is then left free to return to its state of equilibrium, it must necessarily describe a series of vertical oscillations about B as centre.

Take B as the origin, and at any time t let the weight be at M distant x from B; also let $BC = c$.

Two cases may be considered.

First, suppose the end of the rod to be *gradually* forced down to C and then suddenly released.

According to the principle of the conservation of energy,

$$\frac{P}{g}\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \text{the } \textit{work done} \text{ between } C \text{ and } M$$

$$= \frac{EA}{L}\left(\frac{c^2}{2}-\frac{x^2}{2}\right),$$

or

$$\frac{P}{g}\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{P}{l}\frac{1}{2}(c^2-x^2),$$

and hence

$$v, \text{ the velocity of the weight at } M, = \sqrt{\frac{g}{l}}(c^2-x^2)^{\frac{1}{2}}.$$

Now v is zero when $x=\pm c$, so that the weight will rise above B to a point C_1 where $BC_1=c=BC$.

Again, from the last equation,

$$dt\sqrt{\frac{g}{l}} = \frac{dx}{(c^2-x^2)^{\frac{1}{2}}},$$

and integrating between the limits 0 and x,

$$t\sqrt{\frac{g}{l}} = \sin^{-1}\frac{x}{c},$$

and the oscillations are therefore isochronous.

When $x=c$, $$t=\frac{\pi}{2}\sqrt{\frac{l}{g}},$$

and the time of a complete oscillation is

$$\pi\sqrt{\frac{l}{g}}.$$

Next, suppose the oscillatory motion to be caused by a weight P falling without friction from a point D, and being suddenly checked and held by a catch at the lower end of the rod.

Take the same origin and data as before, and let $AD=h$.

The elastic resistance of the rod at the time t is

$$EA\frac{l+x}{L},$$

and the equation of motion of the weight is

$$\frac{P}{g}\frac{d^2x}{dt^2}=P-EA\frac{l+x}{L}=P-\frac{P}{l}(l+x)=-P\frac{x}{l},$$

or

$$\frac{d^2x}{dt^2}=-\frac{g}{l}x.$$

Integrating,

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{l}x^2+c_1,$$ c_1 being a constant of integration.

But $\frac{dx}{dt}$ is zero when $x=c$, and therefore $c_1=\frac{g}{l}c^2$.

Hence

$$\left(\frac{dx}{dt}\right)^2=\frac{g}{l}(c^2-x^2)=v^2.$$

This is precisely the same equation as was obtained in the first case, and between the limits 0 and x

$$t\sqrt{\frac{g}{l}}=\sin^{-1}\frac{x}{c},$$

so that the motion is isochronous, and the time of a complete oscillation is

$$\pi\sqrt{\frac{l}{g}}.$$

When $x=-l$,

$$\left(\frac{dx}{dt}\right)^2=2gh,$$

and hence

$$\frac{g}{l}(c^2-l^2)=2gh,$$

or

$$c^2=l^2+2lh.$$

If $h=0$, i.e., if the weight is merely placed upon the rod at the end A, $c=\pm l$, and the amplitude of the oscillation is *twice* the statical elongation due to P.

The rod may be safely stretched until its length is $L+l$, while a further elongation c might prove most injurious to its elasticity, which shows the detrimental effect of vibratory motion. If a small downward force Q is applied to P when it has reached the end of its vibration, it will produce a corresponding descent, and the weight P will then ascend an equal distance above its neutral position. At the end of the interval corresponding to P's natural period of vibration, apply the force again, and P will descend still further. This process may be continued indefinitely, until at last rupture takes place, however small P and Q may be. If Q is applied at irregular intervals, the amplitude of the oscillations will still be increased, but the increase will be followed by a decrease, and so on continually. In practice the problem becomes much more complex on account of local conditions, but experience shows that a *fluctuation* of stress is always more injurious to a structure than the stress due to the maximum load, and that the injury is aggravated as the periods of fluctuation and of vibration of the structure become more nearly synchronous.

An example of a fluctuating load is a procession marching in time across a suspension bridge, which may strain it far more severely than a much greater dead load, and may set up a synchronous vibration which may prove absolutely dangerous. In fact, a bridge has been known to fail from this cause.

The coefficient of elasticity of the rod may be approximately found by means of the formula

$$T=\pi\sqrt{\frac{l}{g}},$$

T being the time of a complete oscillation. For suppose that the rod emits a musical note of n vibrations per second, then

$$\pi\sqrt{\frac{l}{g}}=T=\frac{1}{2n}$$

is the time of travel from C to C_1;

therefore $\qquad l=\dfrac{g}{4\pi^2n^2},\quad$ and hence $\quad E=\dfrac{PL}{A}\dfrac{4\pi^2n^2}{g}.$

Suppose that the weight is perfectly free to slide along the rod. When it returns to A, it will leave the end of the rod and rise with a certain initial velocity. This velocity is evidently $\sqrt{2gh}$, and the weight accordingly ascends to D, then falls again, repeats the former operation, and so on. The equations of motion are in this case only true for values of x between $x=+c$ and $x=-l$.

4. On the Oscillatory Motion of a Weight at the End of a Vertical Elastic Rod of Appreciable Mass.—Suppose the mass of the rod to be taken into account, and assume:

(*a*) That all the particles of the rod move in directions parallel to the axis of the rod.

(*b*) That all the particles which at any instant are in a plane perpendicular to the axis remain in that plane at all times.

As before, the rod OA of natural length L and sectional area A is fixed at O and carries a weight P_1 at A.

Take O as the origin, and let OX be the axis of the rod.

Let ξ, $\xi+d\xi$, and x, $x+dx$, be respectively the *actual* and *natural* distances from O of the two consecutive sections MM, $M'M'$.

Let ρ_0 be the natural density of the rod, and ρ the density of the section MM, distant ξ from O.

The forces which act upon the rod are:

(*a*) The upward and constant force P_0 at O.

(*b*) The weight P_1 at A.

(*c*) The weight of the rod.

(*d*) A force X per unit of mass through the slice bounded by the planes MM, $M'M'$, distant ξ and $\xi+d\xi$, respectively, from O.

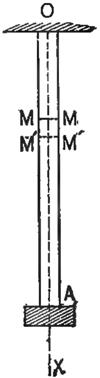


FIG. 309.

Suppose the rod, after equilibrium has been established, to be cut at the plane $M'M'$. In order to maintain the equilibrium of the portion $OM'M'$ it will be necessary to apply to the surface of this plane a certain force P, and the equation of equilibrium becomes

$$-P_0+\int_0^\xi \rho A d\xi X+P+\rho_0 gAx=0.$$

But if the thickness $d\xi$ of the slice MM' is indefinitely diminished, P is evidently the elastic reaction, and its value is

$$EA\frac{d\xi-dx}{dx}=EA\left(\frac{d\xi}{dx}-1\right).$$

Hence

$$-P_0+\int_0^\xi \rho AXd\xi+EA\left(\frac{d\xi}{dx}-1\right)+\rho_0 gAx=0.$$

Differentiating with respect to x,

$$\rho AX\frac{d\xi}{dx}+EA\frac{d^2\xi}{dx^2}+\rho_0 gA=0.$$

But $\rho d\xi=\rho_0 dx$,

therefore
$$\rho_0 AX+EA\frac{d^2\xi}{dx^2}+\rho_0 gA=0,$$

or
$$X+\frac{E}{\rho_0}\frac{d^2\xi}{dx^2}+g=0.$$

Also, $\rho_0 AXdx$ is the resistance to acceleration arising from the inertia of the slice, and is therefore equal to

$$-\rho_0 Adx\frac{d^2\xi}{dt^2},$$

so that

$$X=-\frac{d^2\xi}{dt^2}.$$

Hence

$$\frac{d^2\xi}{dt^2}=\frac{E}{\rho_0}\frac{d^2\xi}{dx^2}+g. \qquad \text{(1)}$$

To solve this equation.—In the state of equilibrium,

$$EA\left(\frac{d\xi}{dx}-1\right)$$

is the tension in the section of which the distance from O is x, and counterbalances the weight P_1 and the weight $\rho_0 A(l-x)g$ of the portion AMN of the rod.

Therefore
$$EA\left(\frac{d\xi}{dx}-1\right)=P_1+\rho_0 Ag(l-x),$$

or
$$\frac{d\xi}{dx}=1+\frac{P_1}{EA}+\frac{\rho_0 g}{E}(l-x).$$

Integrating, $$\xi = x + \frac{P_1}{EA}x + \frac{\rho_0 g}{E}\left(lx - \frac{x^2}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

There is no constant of integration, as x and ξ vanish together.

This value of ξ is a *particular* solution of (1), and is independent of t.

Put $$\xi = x + \frac{P}{EA}x + \frac{\rho_0 g}{E}\left(lx - \frac{x^2}{2}\right) + z,$$

z being a new function of x and t. Then

$$\frac{d^2\xi}{dx^2} = -\frac{\rho_0}{E}g + \frac{d^2z}{dx^2}, \quad \text{and} \quad \frac{d^2\xi}{dt^2} = \frac{d^2z}{dt^2}.$$

Hence, from eq. (1),

$$\frac{d^2z}{dt^2} = \frac{E}{\rho_0}\frac{d^2z}{dx^2} = v_1{}^2\frac{d^2z}{dx^2}, \quad \text{where} \quad v_1{}^2 = \frac{E}{\rho_0}.$$

The integral of this equation is of the form

$$z = F(x + v_1 t) + f(x - v_1 t),$$

$v_1\left(=\sqrt{\frac{E}{\rho_0}}\right)$ being the velocity of propagation of the vibrations.

The full solution of (1) is therefore of the form

$$\xi = x + \frac{P_1}{EA}x + \frac{\rho_0 g}{E}\left(lx - \frac{x^2}{2}\right) + F(x + v_1 t) + f(x - v_1 t).$$

5. Specific Weight; Coefficient of Elasticity; Limit of Elasticity; Breaking Stress.—Before the strength of a body can be fully known certain physical constants whose values depend upon the material must be determined.

(*a*) *Specific Weight.*—The specific weight is the weight of a unit of volume. The specific weights of most of the materials of construction have been carefully found and tabulated. If the specific weight of any new material is required, a convenient approximate method is to prepare from it a number of regular solids of determinate volume and weigh them in an ordinary pair of scales. The ratio of the total weight of these solids to their total volume is the specific weight. It must be remembered that the weight may vary considerably with time, etc.; thus a sample of greenheart weighed 69.75 lbs. per cubic foot when first cut out of the log, and only 57 lbs. per cubic foot at the end of six months. When the strength of a timber is being determined, it is important to note the amount of water present in the test-piece, since this appears to have a great influence upon the results.

The straining of a structure is generally largely due to its own weight.

The *total load* upon a structure includes *all* the external forces applied to it, and in practice is designated *deaa* (*permanent*) or *live* (*moving*), according as the forces are gradually applied and steady, or suddenly applied and accompanied with vibrations. For example, the weight of a bridge is a dead load, while a train passing over it is a live load; the weight of a roof, together with the weight of any snow which may have accumulated upon it, is a dead load; *wind* causes at times excessive vibrations in the members of a structure, and although often treated as a dead load, should in reality be considered a live load.

The dead loads of many structures (as masonry walls, etc.) are so great that extra or accidental loads may be safely disregarded. In cold climates, great masses of snow and the penetrating effect of the frost necessitate very deep foundations, which proportionately increase the dead weight.

(*b*) *Coefficient of Elasticity.*—Generally speaking, a knowledge of the external forces acting upon a structure discloses the manner of their distribution amongst its various members, but the deformation of these members can only be estimated by means of the coefficient of elasticity, which expresses the relation between a stress and the corresponding strain.

In practice it is usually sufficient to assume that a material is elastic, homogeneous, and isotropic, and its deformation under stress may be found if the coefficients of elasticity, of form, and of volume are known.

In a homogeneous solid there may be twenty-one distinct coefficients of elasticity, which are usually classified under the following heads:

(1) *Direct,* expressing the relation between longitudinal strains and normal stresses in the same direction.

(2) *Transverse,* expressing the relation between tangential stresses and strains in the same direction.

(3) *Lateral,* expressing the relation between longitudinal strains and normal stresses at right angles to the strains, i.e., a lateral resistance to deformation.

(4) *Oblique,* expressing other relations of stress and strain.

If a body is isotropic, i.e., equally elastic in all directions, the *twenty-one* .coefficients reduce to *two,* viz., the coefficients of direct elasticity and of lateral elasticity. Such bodies, however, are almost wholly ideal. In a perfectly elastic body E would be the same both for tension and compression. In the ordinary materials of construction it is slightly less for compression than for tension; but if the stresses do not exceed a certain limit, the difference is so slight that it may be disregarded.

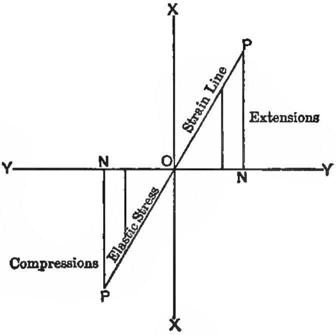


FIG. 310.

The equation $f = Ee$ may be illustrated graphically by the straight line POP, the ordinate at any point being the stress required to produce the strain represented by the corresponding abscissa.

The work done per unit of volume in producing the extension (or compression) ON

$$= \text{area of triangle } PON = \frac{1}{2} ON \cdot PN = \frac{fe}{2} = \frac{f^2}{2E}.$$

Also, $\tan PON = \frac{f}{e} = E.$

Coefficients of elasticity must be determined experimentally.

The coefficients of direct elasticity for the different metals and timbers are sometimes obtained by subjecting bars of the material to forces of extension or compression, or by observing the deflections of beams loaded transversely. The coefficients for blocks of stone and masonry might also be found by transverse loading; they are of little, if any, practical use, as, on account of the inherent stiffness of masonry structures, their deformations, or *settlings,* are due rather to defective workmanship than to the natural play of elastic forces.

The *torsional* coefficient of elasticity, i.e., the coefficient of elastic resistance to torsion, has been shown by experiment to vary from two fifths to three eighths of the coefficient of direct elasticity.

When, e.g., it is required to find the E for a material a test-piece is prepared and is held in the grips (or holders) of a testing-machine. An intermediate portion of a definite length L (usually 8 or 10 ins.) and far enough from the ends of the grips to be unaffected by their action is then marked off. Before the loading of the specimen commences the telescope is set at zero. Under a load P the telescope gives a reading R, indicating that the length L has been increased by R.

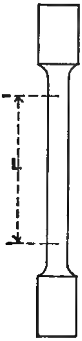


FIG. 311.

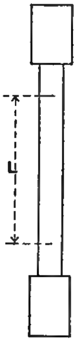


FIG. 312.

FIG. 313.

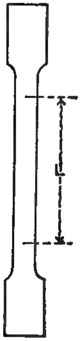


FIG. 314.

Therefore
$$P = EA\frac{R}{L},$$

A being the sectional area of the specimen.

If the load is now increased by an amount ΔP, there will be a corresponding increase of ΔR in the extension.

Therefore
$$P + \Delta P = EA\frac{R + \Delta R}{L}.$$

Hence
$$E = \frac{\Delta P}{\Delta R}\,\frac{L}{A},$$

from which the value of E can be calculated.

Ex. 13. *A mild-steel specimen of* 1 *in. diameter is placed in the testing-machine and a distance of* 10 *ins.* (L) *between the measuring-points is marked off; determine the average E from the following tabulated observations:*

Load in Tons of 2240 Lbs., P.	Increment of Load, ΔP.	Extension in Inches, R.	Increment of Extension, ΔR.
0	0	0	0
4	4	.0042	.0042
8	4	.0081	.0039
12	4	.0121	.0040

Thus, for an increment in the load of 4 tons, the corresponding *average* increment in the extension

$$=\frac{.0042+.0039+.0040}{3}=\frac{.0121}{3},$$

and therefore

$$E=\frac{4}{\frac{.0121}{3}}\,\frac{10}{\frac{22}{7}\,\frac{1}{4}(1)^2}=12{,}629 \text{ tons.}$$

The ends of the test-piece are enlarged and are connected with the main body of the specimen by giving the shoulder a suitable curve (Fig. 311). Abrupt changes of section (Figs. 312 and 313) must be avoided, as at such points great stresses, of which the distribution is unknown, are induced. The fracture of a *hard* specimen almost invariably takes place at an abrupt change of section, while with a ductile material there will be a *flow* (p. 254) toward the narrower portion, which will prevent its full contraction before fracture takes place. The *apparent* strength of the material is therefore increased. When test-pieces are *sheared* out of boiler-plate, about a quarter of an inch of the metal should be removed on each side between the measuring-points (Fig. 314), in order to eliminate the upsetting effect of the shearing action. All specimens should be long enough to allow the portion between the measuring-points to contract freely, as otherwise they seem to be under constraint and fracture takes place before the contraction is complete. Thus the *apparent* strength is greater than the actual strength, which is the breaking load per square inch of the *original* sectional area.

In ordinary practice, however, the engineer does not care to know this actual ultimate stress, but demands the stress which the material would bear if the sectional area of the test-piece remained unchanged up to the point of fracture. In other words, he requires the breaking load per square inch of the *original sectional area,* and he can then decide what fraction of this load may be safely applied to any structural member.

Again, the distance between the "measuring-points" of a test-piece is subdivided into a number (8 or 10) of 1-inch lengths, and

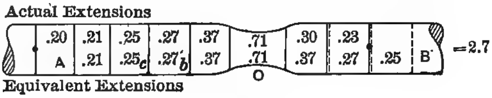


FIG. 315.

after fracture has taken place the extension of each division is carefully determined. If the fracture is near the centre, the percentage of elongation is $\frac{100l}{L}$. If the fracture, however, is at a distance from the centre, as in Fig. 315, the equivalent elongation is obtained in the following manner:

The elongation of each division is shown in the figure. On the right of the division containing the fracture and in which the local elongation is necessarily the greatest, *imaginary* divisions, shown by dotted lines, are added, and these divisions have the same extensions as the corresponding divisions b and c on the left of O. The total *equivalent elongation* is then $AB-8''$, the length AB including 8 divisions. In the specimen in question $AB=10''.7$.

Therefore the equivalent elongation $=2''.7$ and the equivalent percentage $=\frac{100}{8}\times 2.7=33\frac{3}{4}$, very nearly.

Also, if a is the *fractured* area, the percentage of reduction in area

$$=100\left(1-\frac{a}{A}\right).$$

These two percentages are measurements of the *ductility* of the material.

In the above specimen the initial diameter $=1.''059$, and the final diameter $=0''.681$.

Therefore $$\frac{a}{A}=\left(\frac{.681}{1.059}\right)^2=.4135,$$

and the reduction of area $=100(1-.4135)=58.65$ per cent.

(*c*) *Limit of Elasticity.*—When the forces which strain a body fall below a certain limit, the body, on the removal of the forces, will resume its original form and dimensions without sensible change (disregarding any effects due to the development of heat) and may be treated as perfectly elastic. But if the forces exceed this limit, the body will receive a permanent deformation, or, as it is termed, a *set*.

Such a limit is called a *limit of elasticity*, and is the greatest stress that can be applied to a body without producing in it an appreciable and permanent deformation.

This is an unsatisfactory definition, as a body passes from the elastic to the non-elastic state by such imperceptible degrees that it is impossible to fix any exact line of demarcation between the two states.

Bauschinger's experiments also indicate that the application to a body of any stress, however small, produces a plastic or permanent deformation. This, perhaps, is sometimes due to a want of uniformity in the material, or to the bar being not quite straight initially. In any case, the deformations under loads which are less than a load known as the elastic limit are so slight as to be of no practical account and may be safely disregarded, and for such loads Hooke's Law may be regarded as substantially correct.

So, too, the *hysteresis* effect, i.e., the lagging of the relation of stress to strain, is so slight within the elastic limit as to be safely disregarded. Beyond the elastic limit the creeping becomes very marked for the first few minutes and may continue much longer, but at a diminished rate. At last a point is reached at which the material draws out and breaks without any further increase of load.

Fairbairn defines the limit of elasticity more correctly as the stress below which the deformation is approximately proportional to the load which produces it, and beyond which the deformation increases much more rapidly than the load. In fact, both the elastic and ultimate strengths of a material depend upon the *nature* of the stresses to which they are subjected and upon the *frequency* of their application. For example, in experimenting upon bars of iron having an ultimate tenacity of 46,794 lbs. per square inch and a ductility of 20 per cent, Wöhler found that with repeated stresses of equal intensity, but alternately tensile and compressive, a bar failed after 56,430 repetitions when the intensity was 33,000 lbs. per square inch; a second bar failed only after 19,187,000 repetitions when the intensity was 18,700 lbs. per square in; while a third bar remained intact after more than 132,000,000 repetitions when the intensity was 16,690 lbs. per square inch. These experiments therefore indicated that the *limit of elasticity* for the iron in question, under repeated stresses of equal intensity, but alternately tensile and compressve, lay between 16,000 and 17,000 lbs. per square inch, which is much less than the limit under a steadily applied stress. Similar results have been shown to follow when the stresses fluctuate from a maximum stress to a minimum stress of the *same kind.*

Generally speaking, then, the limit of elasticity of a material subjected to repeated stresses is a certain maximum stress below which the condition of the body remains unimpaired.

Bauschinger defines the elastic limit as the point at which the stress ceases to be sensibly proportional to the strain, the latter being measured by a mirror apparatus reading to about .00001 inch. If the yield point is exceeded it will rise so long as the bar remains loaded. If the load exceeds the elastic limit this limit will rise until it approaches the yield point, when it rapidly falls. If the load is then increased beyond the yield point the limit will rise again until it reaches a much higher point than its former value.

The main object, then, of the *theory* of the strength of materials is to determine whether the stresses developed in any particular member of a structure exceed the limit of elasticity. As soon as they do so, that member is permanently deformed, its strength is impaired, it becomes predisposed to rupture, and the safety of the whole structure is threatened. Still, it must be borne in mind that it is not absolutely true that a material is always weakened by being subjected to forces superior to this limit. In the manufacture of iron bars, for instance, each of the processes through which the metal passes changes its elasticity and increases its strength. Such a material is to be treated as being in a new state and as possessing new properties.

Again, when a bar has been overstrained so that its molecular condition is changed, the application of a small load produces an immediate extension followed by a *creeping* which only *slowly* disappears when the bar is relieved of load. If the bar is allowed to rest it gradually recovers its elastic properties and the recovery becomes more and more complete as the time of rest increases. Muir's experiments indicate that the bar may regain its elasticity by being immersed for a few minutes in boiling water. It is also found that this process gives the metal a higher elastic limit. The overstraining developes a resistance to plastic deformation or produces a hardening effect which may be eliminated by heating the metal to redness and then allowing it to cool slowly, i.e., by *annealing*.

The strength of a material is governed by its tenacity and rigidity, and the essential requirement of practice is a *tough* material with a high elastic limit.

This is especially necessary for bridges and all structures liable to constantly repeated loads, for it is found that these repetitions lower the elastic limit and diminish the strength.

In the majority of cases experience has fixed a practical limit for the stresses much below the limit of elasticity. This insures greater safety and provides against unforeseen and accidental loads which may exceed the *practical* limit, but which do no harm unless they pass beyond the *elastic* limit.

Certain operations have the effect of raising the limit of elasticity; a wrought-iron bar steadily strained almost to the point of its ultimate strength and then released from strain and allowed to rest, experiences an elevation both of tenacity and of the elastic limit.

If the bar is stretched until it breaks, the tensile strength of the broken pieces is greater than that of the bar. A similar result follows in the various processes employed in the manufacture of iron and steel bars and wires: the wire has a greater ultimate strength than the bar from which it was drawn.

Again, iron and steel bars subjected to *long-continued* compression or extension have their resistance increased mainly because time is allowed for the molecules of the metal to assume such positions as will enable them to offer the maximum resistance; the increase is not attended by any appreciable change of density.

Under an increasing stress a *brittle* material will be fractured without any great deformation, while a *tough* material will become plastic and undergo a large deformation.

(*d*) *Breaking Stress.*—When the load upon a material increases indefinitely the material may merely suffer an increasing deformation, but generally a limit is reached at which fracture suddenly takes place.

Cast iron is perhaps the most doubtful of all materials, and the greatest care should be observed in its employment. It possesses little tenacity or elasticity, is very hard and brittle, and may fail suddenly under a shock or an extreme variation of temperature. Unequal cooling may predispose the metal to rupture, and its strength may be still further diminished by the presence of air-holes.

Cast iron and similar materials receive a sensible set even under a small load, and the set increases with the load. Thus at no point will the stress-strain curve be absolutely straight, and the point of fracture will be reached without any *great* change in the slope of the curve and without the development of much plasticity.

Wrought iron and *steel* are far more uniform in their behaviour, and obey with tolerable regularity certain theoretical laws. They are tenacious, ductile, have great compressive strength, and are most reliable for structural purposes. Their strength and elasticity may be considerably reduced by high temperatures or severe cold.

When a bar of such material is tested, the *stress-strain* curve ($f = \pm Ee$), as has already been pointed out, is almost absolutely straight within the elastic limit, e.g., from O to A in tension and from O to B in compression (Fig. 316). As the load increases beyond the elastic limit, the increasing deformation becomes plastic and permanent, and the stress-strain diagram takes an appreciable curvature between the limits A and D and the points B and E corresponding to the maximum loads. In tension, as soon as the point D is reached, the bar rapidly elongates and is no longer able to sustain the maximum load, its sectional area rapidly diminishes, and fracture ultimately takes place under a load much less than the maximum load. The point of fracture is represented in the figure by the point F, the ordinate of F being the *actual ultimate intensity*

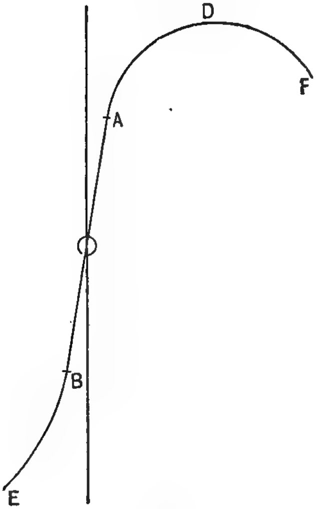


FIG. 316.

$$\textit{of stress} = \frac{\textit{final load on the bar}}{\textit{area of fractured section}}.$$

The exact form of the stress-strain curve between D and F is unknown, as no definite relation has been found to exist between the stress and strain during the elongation from D to F.

It is also important to note that, as the deformation gradually increases under the increasing load, the molecules of the material require greater or less time to adjust themselves to the new condition.

During the tensile test of a ductile material there is, at some point beyond the elastic limit, an abrupt break GH in the continuity of the stress-strain curve, the curve again becoming continuous from H to D, Fig. 317. This phenomenon is always very marked in mild steel and other ductile materials, and the deformation after passing GH is almost wholly plastic or permanent.

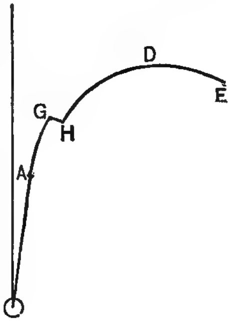


FIG. 317.

The point G is called the *yield-point* and seems to be always higher than A, the true elastic limit.

In compression there is no local stretch as in tension, and there is consequently no considerable change in the curvature of the compression stress-strain curve up to the point of fracture.

Timber is usually tested by being subjected to the action of tensile, compressive, or transverse loads. Other characteristics, however, must be known before a full conception of the strength of the wood can be obtained. Thus the specific weight must be found; the amount of water present, the loss in drying, and the corresponding shinkage should be determined; the *structural* differences of the several specimens, the rate of growth, etc., should be observed.

The chief object of experiments upon *masonry* and *brickwork* is to discover their resistance to compression, i.e., their crushing strength. In fact, their stiffness is so great that they may be compressed up to the point of fracture without sensible change of form, and it is therefore very difficult, if not impossible, to observe the limit of elasticity.

The *cement* or *mortar* uniting the stones and bricks is most irregular in quality. In every important work it should be an invariable rule to prepare specimens for testing. The crushing strength of cement and of mortar is much greater than the tensile strength, the latter being often exceedingly small. Hence it is advisable to avoid tensile stresses within a mass of masonry, as they tend to open the joints and separate the stones from one another. Attempts are frequently made to strengthen masonry and brickwork walls by inserting in the joints tarred and sanded strips of hoop-iron. Their utility is doubtful, for, unless well protected from the atmosphere, they oxidize, to the detriment of the surrounding material, and besides this they prevent an equable distribution of pressure. They are, however, far preferable to bond-timbers.

The working stress is the greatest stress which a material is to bear in ordinary practice, and the ratio

$$\frac{\text{ultimate strength}}{\text{working stress}}$$

is called the *factor of safety.*

The value of this factor is governed by many important considerations, as, for example, the nature of the material, the character of the stresses, the effect of bad workmanship, etc. The material slowly deteriorates with time, and its life is also influenced in an important degree by variations of temperature. A marked change in the magnetic properties of iron and steel when exposed for a considerable length of time to a temperature as low as 100° to 150° F. shows that these metals undergo a gradual molecular change, but they may be restored to their original condition by re-annealing. Again, the behavior of a material largely depends upon the existence of initial internal stresses which may be due to *set* caused either by temperature variations or by the action of previously applied forces. This is well illustrated in a cylindrical casting in which an outer shell cools first and contracts, and is then compressed tangentially by the cooling and contraction of an adjoining inner shell, while the metal of this inner shell is pulled out radially. It has been pointed out by J. Thomson that the defect of elasticity under small loads observed by Hodgkinson in cast iron is probably due to the existence of internal stresses. There must necessarily be an equilibrium between these initial stresses at any section of the material, as there is no external load, and if the material is wholly free from such stresses it is said to be in a *state of ease* (Pearson). Annealing will give a condition of almost complete ease to a plastic metal, but even then test-pieces, when first loaded, show elastic defects which may arise in part from initial internal stress and which are reduced or may disappear under repeated applications of the loads.

A factor of safety is rarely less than 3, and the following factors are in accordance with the best practice:

For timber.	3	for dead loads
“ “	6	“ live “
“ metals	3	“ dead “
“ “	6	“ live “
“ masonry	6	“ dead “
“ “	10	“ live “

Experiments also indicate that, under a steady (or static) load, timber may be strained almost to the point of fracture without apparent injury to the material.

The factor of safety is largely influenced by the variability of the stresses to which a material may be subjected. The extension of a bar under a *gradually* applied load is much less than when the load is placed on the bar and *suddenly* released, while it is still greater if the body is in *motion* and commences its action on the bar by a *shock* or a *blow*. This is followed by a vibration (Art. 2) of the weight about a neutral position and the vibration continues, but with a diminishing amplitude, until it finally disappears under the action of the molecular frictional resistance of the material. The effect, then, of a suddenly applied load or a shock, even if it strains the material only slightly beyond the limit of elasticity, is detrimental, while the *same* load may be gradually applied without doing any harm. The kinetic energy due to a shock or blow is expended in doing the work of straining, and if the resulting strain is so great that the limit of elasticity is exceeded, a local hardening is produced which renders the material less able to take up the work as an elastic strain, and its capacity for doing so may be rapidly exhausted by repetitions of such shocks.

6. Wöhler's Experiments.—Fatigue.—It is known that variable forces, constantly repeated loads, and continued vibrations diminish the strength of a material, whether they produce stresses approximating to the elastic limit, or exceedingly small stresses occurring with great rapidity. Indeed many structures are designed so that the several members are always subjected to the same kind of stress, thus avoiding the detrimental effect of alternating tensile and compressive stresses. Although the fact of a variable ultimate strength had long been tacitly acknowledged and often allowed for, Wöhler was the first to give formal expression to it, and, as a result of observation and experiment extending over a period of twelve years, enunciated the following law:

"That if a stress *t*, due to a static load, cause the fracture of a bar, the bar may also be fractured by a series of often-repeated stresses, each of which is less than *t*; and that, as the differences of stress increase, the cohesion of the materials is affected in such a manner that the minimum stress required to produce fracture is diminished."

This law is manifestly incomplete. In Wöhler's experiments the applications of the load followed each other with great rapidity, yet a certain length of time was required for the resulting stresses to attain their full intensity; the influence due to the rapidity of application, to the rate of increase of the stress, and to the duration of individual strains still remains a subject for investigation.

The experiments, however, show that the rate of increase of repetitions of stress required to produce fracture is much more rapid than the rate of decrease of the stresses themselves, and depends both upon the *maximum* stress and upon the difference or *fluctuation of stress.*

The effect of repeated stresses of equal intensity, but alternately tensile and compressive, has been already pointed out in Art. 4.

Bars of the same material repeatedly bent in one direction bore 31,132 lbs. per square inch when the load was wholly removed between each bending, and 45,734 lbs. per square inch when the stress fluctuated between 45,733 lbs. and 24,941 lbs.

The table on page 245 gives the results of similar experiments on steel.

The axle steel was found to bear 22,830 lbs. per square inch when subjected to repeated shears of equal intensity but *opposite* in kind, and 29,440 lbs. per square inch when the shears were of the *same* kind. It would therefore appear that the shearing strengths of the metal in the two cases are about $\frac{4}{5}$ of the strengths of the same metal under alternate bending and under bending in one direction respectively.

Character of Fluctuation.	Maximum Resistance to Repeated Stresses in Lbs. per Square Inch.	
	Axle Steel.	Spring Steel (unhardened).
Alternating stresses of equal intensity	29,000, −29,000	
Complete relief from stress between each bending	49,890, 0	52,000, 0
Partial relief from stress between each bending	83,110, 36,380	93,500, 62,240

From torsion experiments with various qualities of steel the important result was deduced that the maximum resistance of the steel to alternate twisting was $\frac{4}{5}$ of the maximum resistance of the same steel to alternate bending.

For shearing stresses in opposite directions Wöhler found, in the case of Krupp cast steel (untempered), that $u=39{,}500$ lbs./sq. in. and $s=23{,}000$ lbs./sq. in., or about $\frac{4}{5}$ of the corresponding values for stresses which are alternately tensile and compressive, and it may be generally assumed that the value of the working stress for shearing stresses is $\frac{4}{5}$ of its value for stresses which are alternately tensile and compressive, and for which the ratio of the maximum tensile to the maximum compressive stress is the same.

Wöhler proposed 2 as a factor of safety, and considered that the maximum permissible working stresses should be in the ratios of 1:2:3, according as members are subjected to alternate tensions and compressions (alternate bending), to tensions alternating with entire relief, or to a steady load.

Fatigue.—It is a fact of practical and scientific importance that iron and steel, and probably all materials, are weakened by repeated variations of stress, the weakening effect being called *fatigue.* A certain variation may be permissible provided that the stresses are well within the elastic limits, but it must not be forgotten that when these limits are exceeded even the toughest bar may be fractured by a very few bendings. It is strange that the material in old rails, tires, etc., which have been long in use and have been almost exhausted by fatigue should have become a seemingly totally different material, but yet, under the ordinary tests for plasticity and strength, should give results which do not differ in any marked degree from those obtained for the new material. A period of rest tends to restore the elasticity and possibly the strength of a fatigued piece, and the effect of fatigue may be entirely eliminated by heating the piece to redness and then allowing it to cool slowly, i.e., by *annealing.*

The phenomena of fatigue have been fully explained by Prof. J. Thomson's deductions, verified by Bauschinger's experiments. There are in reality *two limits of elasticity.* Bauschinger first loaded a number of wrought-iron bars until a point was reached at which the tensile stress was no longer sensibly proportional to the strain. He then reversed the process and loaded the bars in compression until the compressive stress was no longer sensibly proportional to the strain. He found that these operations had the effect of lowering the elastic limit in tension and raising that in compression until finally the two limits were equidistant from the line of *no load.* These he called the *natural* elastic limits of the wrought iron, the corresponding stress being 8½ tons per square inch, which is practically the same as that obtained by Wöhler in bars of the same material tested alternately in tension and compression. Bauschinger's experiments also showed that although either *one* of the limits might be changed, even to the ordinary breaking stress, the *range of elasticity* remains about the same. This accords with Thomson's theoretical deductions in which he defines the limits as a *superior* and an *inferior* limit, for the reason that they are not necessarily equal and equidistant from the line of no load, and, if the overstraining is sufficient, may both lie on the same side of this line. The tensile elastic limit of bars fresh from the rolls is raised by processes of manufacture and treatment to a higher point than Bauschinger's *natural* limit, and is at once lowered when subjected to stresses which are alternately tensile and compressive. At the same time the limit in compression is raised and this change continues until both limits are the same. It is *not* the straining beyond *one* elastic limit which is injurious to a material, but the repeated straining beyond the *two* elastic limits.

On the basis of Wöhler's experiments, empirical formulæ have been deduced which, it is claimed, are more in accordance with the results of experiment, give smaller errors, and insure greater safety than the incorrect assumption of a constant ultimate strength.

The formulæ necessarily depend upon certain experimental results, but in applying them to any particular case, it must be remembered that only such results should be employed as have been obtained for a material of the same kind and under the same conditions as the material under consideration. The effects due to faulty material, rust, etc., are altogether indeterminate, so that no formula can be perfectly universal in its application. Hence the necessity for factors of safety, with values depending upon the character of the stresses as well as upon the nature of the structure still exists.

In the formulæ the following assumptions are used:

t is the ultimate strength of the material under a static or under a very gradually applied load.

u is the strength when the material is subjected to a number of repeated stresses, the stress in each repetition remaining unchanged in kind; that is, being wholly tensile or wholly compressive or wholly shearing.

s is the ultimate strength of the material when subjected to repeated stresses which are of equal intensity and are alternately tensile and compressive.

f is the working stress per unit of sectional area.

A is the effective sectional area.

fA is the numerically absolute maximum load which the material has to carry.

Launhardt's Formula.—Let a_1 be the ultimate strength of a bar of a sectional area A when the bar is subjected to stresses which vary between a_1 and a minimum stress a_2 of *the same kind.*

Let $a_1 - a_2$, *the fluctuation of stress,* $= d$.

As the result of experiment,

$$a_1 \propto d = Fd,$$

F being a numerical coefficient whose value must be determined by experiment.

If $a_2 = 0$, i.e., if the fluctuation is between the maximum load and complete rest, then

$$a_1 = d = u \quad \text{and} \quad F = 1.$$

If $d=0$, i.e., if there is no fluctuation of stress so that the bar is under a static load, then

$$a_1=a_2=t \quad \text{and} \quad F=\infty.$$

Launhardt's assumption that

$$F=\frac{t-u}{t-a_1}$$

satisfies these extreme conditions and also gives intermediate values of a_1 which approximately agree with the results of the most reliable experiments. Hence

$$a_1=Fd=\frac{t-u}{t-a_1}(a_1-a_2)$$

or

$$t-a_1=(t-u)\left(1-\frac{a_2}{a_1}\right),$$

and therefore

$$a_1=u\left(1+\frac{t-u}{u}\frac{a_2}{a_1}\right)=u\left(1+\frac{t-u}{u}\phi\right),$$

where

$$\phi=\frac{a_2}{a_1}.$$

Taking 3 as a factor of safety,

$$\text{the } \textit{working load}=fA=A\frac{a_1}{3}=\frac{Au}{3}\left(1+\frac{t-u}{u}\phi\right). \quad . \quad . \quad \text{(L)}$$

Weyrauch's Formula.—The bar is now subjected to stresses which vary from a *numerically* maximum stress a_1 to a minimum stress a_2 of an *opposite kind.* Thus

$$\text{the } \textit{actual fluctuation of stress}=a_1+a_2=d,$$

and again as the result of experiment

$$a_1=Fd.$$

If $a_2=0$, i.e., if the fluctuation is between the maximum load and complete rest,

$$a_1=d=u \quad \text{and} \quad F=1.$$

If $a_1=a_2=s$, i.e., if the fluctuation is between maximum stresses of equal intensity but of opposite kinds,

$$a_1=a_2=\frac{d}{2}=s \quad \text{and} \quad F=\frac{1}{2}.$$

Weyrauch's assumption that

$$F=\frac{u-s}{2u-s-a_1}$$

satisfies these extreme conditions and gives intermediate values of a_1 which approximately agree with the most reliable results of the few experiments yet recorded. It is also in accordance with Wöhler's deductions that a_1 increases as d diminishes and vice versa.

Hence
$$a_1=Fd=\frac{u-s}{2u-s-a_1}(a_1+a_2)$$

or
$$2u-s-a_1=(u-s)\left(1+\frac{a_2}{a_1}\right)$$

$$=(u-s)(1+\phi),$$

where
$$\phi=\frac{a_2}{a_1}.$$

Therefore
$$a_1=u\left(1+\frac{s-u}{u}\phi\right),$$

and taking 3 as a factor of safety,

$$\text{the } \textit{working load}=fA=A\frac{a_1}{3}$$

$$=\frac{Au}{3}\left(1+\frac{s-u}{u}\phi\right). \quad . \quad . \quad . \quad . \quad (W)$$

(L) and (W) may be written in the form

$$\text{the working stress}=\frac{\text{the working load}}{A}$$

$$=\frac{u}{3}\left(1+\frac{f-u}{u}\phi\right),$$

and in ordinary practice it may be assumed that

u is 30,000 lbs. for wrought iron and 48,000 lbs. for steel.

Also, if the varying stresses are of the *same* kind,

$$\frac{f-u}{u} \text{ is } \frac{1}{2} \text{ for wrought iron and } \frac{9}{11} \text{ for mild steel,}$$

while if they are *opposite* in kind,

$$\frac{f-u}{u} \text{ is} -\frac{1}{2} \text{ for wrought iron and } -\frac{5}{11} \text{ for mild steel.}$$

Ex. 14. Find the proper sectional area of a bar of axle iron which has to carry loads varying from a maximum pull of 110,000 lbs. to a minimum pull of 44,000 lbs.

$$\text{The working stress} = 10000\left(1+\frac{1}{2}\,\frac{44000}{110000}\right) = 12000 \text{ lbs.}$$

Therefore

$$\text{the required sectional area} = \frac{110000}{12000} = 9\tfrac{1}{6} \text{ sq. in.}$$

Ex. 15. Find the working stress of a mild-steel girder which has to carry a dead load of D lbs. and a maximum live load of L lbs., and apply your result to the case of a lattice girder when the live is 3 times the dead load.

$$\begin{aligned} \text{The working stress} &= 16000\left(1+\frac{9}{11}\,\frac{D}{L}\right) \\ &= 16000\left(1+\frac{9}{11}\,\frac{1}{3}\right) \\ &= 20364 \text{ lbs.} \end{aligned}$$

Ex. 16. A mild-steel bar has to carry loads which vary between a maximum tension of 56,000 lbs. and a maximum compression of 42,000 lbs. Find its sectional area (disregarding buckling).

The working stress in lbs./sq. in.

$$\begin{aligned} &= 16000\left(1-\frac{9}{11}\,\frac{42000}{56000}\right) \\ &= 16000\times\frac{29}{44}. \end{aligned}$$

Therefore

$$\text{the sectional area} = \frac{56000}{16000\times\frac{29}{44}} = 5.31 \text{ sq. inches.}$$

Unwin's Formula.—Unwin has proposed to include all cases of fluctuating stress in the formula

$$a_1 = \frac{d}{2} + \sqrt{t(t-nd)},$$

n being a coefficient to be determined by experiment, while the other symbols are the same as before.

Then $$d = a_1 \pm a_2.$$

If $d=0$, the load is steady and $a_1 = t$.
If $d=a_1$, $a_2=0$, and the load alternates with entire relief.

In this case $$a_1 = \frac{a_1}{2} + \sqrt{t^2 - na_1t}$$

and therefore $$a_1 = 2t(\sqrt{1+n^2} - n).$$

If $d=a_1$, $a_1 = -a_2$, and the stresses are alternately tensile and compressive, but of equal intensity.

Therefore $$a_1 = \frac{2a_1}{2} + \sqrt{t(t-2a_1n)}$$

and $$a_1 = \frac{t}{2n}.$$

In these extreme cases if n is put equal to 1.42 for wrought iron and to 1.66 for steel, results are obtained almost identical with those given by Launhardt and Weyrauch. The formula may therefore be assumed to be approximately correct for intermediate cases.

A mean value of n for iron and steel seems to be about $\frac{3}{2}$, so that the formula may be written

$$\mathbf{a}_1 = \frac{\mathbf{d}}{2} + \sqrt{\mathbf{t}\left(\mathbf{t} - \frac{3}{2}\mathbf{d}\right)}.$$

Ex. 17. *A diagonal of a bowstring truss has a sectional area of* 3 *sq. ins. and carries loads which vary between a maximum tension of* 14 *tons and a maximum compression of* 6 *tons. Find the statical strength* (t) *of the material.*

$$a_1 = \frac{14}{3} \quad \text{and} \quad d = \frac{14+6}{3} = \frac{20}{3}.$$

Therefore $$\frac{14}{3}=\frac{1}{2}\cdot\frac{20}{3}+\sqrt{t\left(t-\frac{3}{2}\cdot\frac{20}{3}\right)}$$

and $t=10.175$ tons.

6. Remarks upon the Values of t, u, s, and f.—As yet the value of u in compression has not been satisfactorily determined, and for the present its value may be assumed to be the same both in tension and compression.

If, as Wöhler states, "repeated stresses" are detrimental to the strength of a material, then the values of u and s diminish as the repetitions increase in number, and are minima in structures designed for a practically unlimited life.

Only a very few of Wöhler's experiments give the values of t, u, s, and a, so that Launhardt's and Weyrauch's assumptions for the value of f must be regarded as tentative only and require to be verified by further experiments. The close agreement of Wöhler's results from tests upon untempered cast steel (Krupp), with those given by Launhard'ts formula may be seen from the following:

For $t=1100$ centners * per square zoll Wöhler found that $u=500$ centners per square zoll. Thus (2) becomes

$$a_1=500\left(1+\frac{6}{5}\frac{a_2}{a_1}\right),$$

and therefore $$a_1^2-500a_1-600a_2=0.$$

Hence for $a_2=0,\quad 250,\quad 400,\quad 600,\quad 1100,$

Launhardt's formula gives

$$a_1=500,\quad 710,\quad 800,\quad 900,\quad 1100,$$

w Wöhler's experiments gave

$$a_1=500,\quad 700,\quad 800,\quad 900,\quad 1100.$$

Again, with Phœnix iron, for $t=500$ centners per sq. zoll, u was found to be 300 centners per square zoll, and therefore

$$a_1=300\left(1+\frac{5}{6}\frac{a_2}{a_1}\right)$$

---

* A centner = 110.23 pounds. A square zoll = 1.0603 square inches.

or $$a_1^2-300a_1-250a_2=0.$$

If $a_2=240$, $a_1=436.8$, which almost exactly agrees with the result given by the tension experiments.

In general, the admissible stress per square unit of sectional area may be expressed in the form

$$f=v(1\pm m\phi),$$

v and m being certain coefficients which depend upon the nature of the material and also upon the manner of the loading. Consider three cases, the material in each case being wrought iron:

(*a*) Let the stresses vary between a maximum tension and an equal maximum compression; then

$$\phi=1,$$

and therefore $$f=700(1-\tfrac{1}{2})=350^k \text{ per cent}^2.$$

(*b*) Let the material be subjected to stresses which are either tensile or compressive, and let it always return to the original unstrained condition; then

$$a_1=0, \quad \text{or} \quad a_2=0, \quad \text{and} \quad \phi=0;$$

therefore $$f=700(1\pm0)=700^k \text{ per cent}^2.$$

(*c*) Let the material be continually subjected to the same dead load; then

$$a_1=a_2,$$

and therefore $f=700(1+\frac{1}{2})=1050^k$ per cent$^2=14{,}934$ lbs. per square inch, which is about one third of the ultimate breaking strength.

Thus in these three cases the admissible stresses are in the ratios of 1:2:3, ratios which have been already adopted in machine construction as the result of experience.

Wöhler, from his experiments upon untempered cast steel (Krupp), concluded that for alterations between an unloaded condition and either a tension or a compression, $f=1100^k$ per cent^2, and for alternations between equal compressive and tensile stresses, $f=580^k$ per cent².

It has not been unusual to take

$$A=\frac{a_1+a_2}{700}$$

for stresses alternately tensile and compressive, it being assumed that if the stresses are tensile only, their admissible values may vary from 0^k to 700^k per cent².

$$\text{Since } \phi=\frac{a_2}{a_1}, \quad a_1=\frac{700A}{1+\phi} \quad \text{and} \quad f=\frac{a_1}{A}=\frac{700}{1+\phi}. \quad . \quad \text{(A)}$$

Again, taking $u=2100$ k./cm.², $\frac{u-s}{u}=\frac{1}{2}$, and 3 as a factor of safety, Weyrauch's formula becomes

$$f=\frac{a_1}{A}=700(1-\tfrac{1}{2}\phi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B)}$$

Comparing the results of (A) and (B)

	for $\phi=$	0,	$\frac{1}{4}$,	$\frac{1}{2}$,	$\frac{3}{4}$,	1,
	(A) gives $f=$	700,	560,	467,	400,	350,
and	(B) gives $f=$	700,	612,	525,	437,	350.

7. Flow of Solids.—When a ductile body is strained beyond the elastic limit, it approaches a purely plastic condition in which a sufficiently great force will deform the body indefinitely. Under such a force the elasticity disappears and the material is said to be in a *fluid* state, behaving precisely like a fluid. For example, it flows through orifices and shows a contracted section. The stress developed in the material is called the *fluid pressure* or *coefficient of fluidity*.

The general principle of the flow of solids, deduced by Tresca, may be enunciated as follows

A pressure upon a solid body creates a tendency to the relative motion of the particles in the direction of least resistance.

This gives an explanation of the various effects produced in materials by the operations of wire-drawing, punching, shearing, rolling, etc., and in the manufacture of lead pipes. Probably it also explains the anomalous behavior of solids under certain extreme conditions.

Rails which have been in use for some time are found to have acquired an elongated lip at the edge. This is doubtless due to the flow of the metal under the great pressures to which the rails are continually subjected. Other examples of the flow of solids are to be observed in the contraction of stretched bars and in the swelling of blocks under compression. The period of fluidity is greater for the more ductile materials, and may disappear altogether for certain vitreous and brittle substances.

In punching a piece of wrought iron or steel, the metal is at first compressed and *flows inwards,* while the *shearing* only commences when the opposite surface begins to open. A case brought under the notice of the author may be mentioned in illustration of this. The thickness of a cold-punched nut was 1.75 ins., the nut-hole was .3125 inch in diameter, and the length of the piece punched out was only .75 inch. Thus the flow must have taken place through a depth of 1 in., and the shearing through a depth of .75 inch. Hence the surface really shorn was $\pi \times .3125 \times .75 = .736$ square inch in area, and a *measure* of the shearing action is the product of this surface area and the *fluid pressure.* The nature of the flow may be observed by splitting a cold-punched nut in half and treating the fractured surfaces with acid after having planed them and given them a bright polish. The metal bordering the core will be found curved downwards, the curvature increasing from the bottom to the top, and well-defined curves will mark the separating planes of the plates which were originally used in piling and rolling the iron.

In experimenting upon lead, Tresca placed a number of plates, one above the other, in a strong cylinder (Fig. 319) with a hole in the bottom. Upon applying pressure the lead was always found to flow when the *coefficient of fluidity* was about 2844 lbs. per square inch, the *difference of stress* being double this amount.

The separating planes assumed curved forms analogous to the corresponding surfaces of flow when water is substituted in the cylinder for the lead.

The flow of ductile metals, e.g., copper, lead, wrought iron, and soft steel, commences as soon as the elastic limit is exceeded, and in order that the flow may be continuous the distorting stress must constantly increase. On the other hand, in the case of truly *plastic* bodies, flow commences and continues under the same constant stress. It evidently depends upon the hardness of the material, and has been called the *coefficient of hardness*. The *longer* the stress acts the greater is the deformation, which gradually increases indefinitely or at a diminishing rate.

Experiment shows that there is very little alteration in the density of a ductile body during its plastic deformation, and Tresca's analytical investigations are based on the assumption that the body is deformed without sensible change of volume.

Consider a prismatic bar undergoing plastic deformation.

Let L be the length and A the section of the bar at commencement of deformation.

Let $L \pm x$ be the length and a the section of the bar at a subsequent period.

Let p be the intensity of the fluid pressure.

Since the volume remains unchanged,

$$LA = (L \pm x)a, \quad \dots\dots\dots \quad (1)$$

the positive or negative sign being taken according as the bar is in tension or compression.

Let P_1 be initial force on bar.

Let P be force on bar when its length is $L \pm x$. Then

$$P_1 = pA,$$

$$P = pa,$$

and hence
$$\frac{P}{P_1} = \frac{a}{A} = \frac{L}{L \pm x}. \quad \dots\dots\dots \quad (2)$$

Hence
$$P(L \pm x) = P_1 L = \text{a constant}, \quad \dots\dots\dots \quad (3)$$

and the force *diminishes* as the bar stretches and *increases* as the bar contracts under pressure. If equation (3) be referred to rectangular axes, the ordinates representing different values of P and the abscissæ the corresponding values of x, the stress-strain diagrams, tt in tension and cc in compression, are hyperbolic curves, having as asymptotes the axis of x, XOX, and a line parallel to the axis of y at a distance from it equal to the length L of the bar.

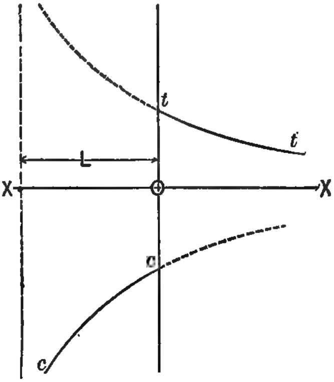


FIG. 318.

Second.—Consider a metallic mass (e.g. lead) resting upon the end CD of a cylinder of radius R and filling up a space of depth D. A hole of radius r is made at the centre of the face CD, through which the mass flows under the pressure of *fluidity* exerted by a piston. When the mass has been compressed to the thickness $DO=x$, let y be the corresponding length KE of the "jet."

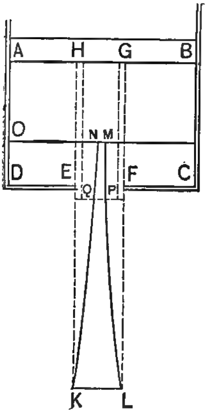


FIG. 319.

First, assume that the specific weight of the mass remains constant.

If dx be the diminution in the thickness DO corresponding to an increase dy in the length of the jet, then

$$\pi R^2 dx + \pi r^2 dy = 0. \quad . \quad . \quad . \quad (4)$$

Integrating, and remembering that $y=0$ when $x=D$,

$$R^2(D-x) - r^2 y = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Second, assume that the cylindrical portion $EFGH$ is *gradually* transformed into $NMPLKQN$, of which the part $PMNQ$ is cylindrical, while the diameter of the part $PLKQ$ *gradually* increases from the face of the cylinder to $KL(=EF)$ at the end of the jet. Then $\pi(R^2-r^2)dx$ = amount of metal which flows into the central cylinder

$$=2\pi r dr x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

dr being the depth to which the metal penetrates.

Third, assume that the diminution of the diameter of the cylindrical portion $PMNQ$ is directly proportional to the said diameter.

Then, if z be the radius of the cylinder $PQNM$,

$$\frac{dr}{r}=\frac{dz}{z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By eqs. (6) and (7), and therefore

$$(R^2-r^2)\frac{dx}{x}=2r^2\frac{dz}{z}.$$

Integrating,

$$(R^2-r^2)\log_e x=2r^2\log_e z+c,$$

c being constant of integration.

When $x=D$, $z=r$,

and therefore $$(R^2-r^2)\log_e\frac{x}{D}=2r^2\log_e\frac{z}{r},$$

or $$\frac{z}{r}=\left(\frac{x}{D}\right)^{\frac{R^2-r^2}{2r^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

By eqs. (5) and (8),

$$\frac{z}{r}=\left(1-\frac{r^2}{R^2}\frac{y}{D}\right)^{\frac{R^2-r^2}{2r^2}},$$

which is the equation to the profile PL or QK.

If $R^2=3r^2$, eq. (8) represents a straight line.

If $R^2=2r^2$, " " " " parabola.

8. Thin Hollow Cylinders; Boilers; Pipes.

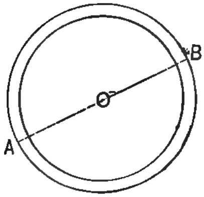


FIG. 320.

Let r be the radius of the cylinder.

Let t be the thickness of the metal.

Let p be the fluid pressure upon each unit of surface.

Let f be the tensile or compressive unit stress, according as p is an internal or an external pressure.

Assume (1) that the metal is homogeneous and free from initial strain;

(2) that t is small as compared with r;

(3) that the pressures are uniformly distributed over the internal and external surfaces;

(4) that the ends are kept perfectly flat and rigid;

(5) that the stress in the metal is *uniformly* distributed over the thickness.

The last assumption is equivalent to supposing that it is the *mean* circumferential stress which is governed by the strength of the metal, while in reality it is the *internal* or maximum circumferential stress which is so governed.

The figure represents a cross-section of the cylinder of thickness unity.

A section made by any diametral plane, as AB, must develope a total resistance of $2tf$, and this must be equal and opposite to the resultant of the fluid pressure upon each half, i.e., to $2pr$. Hence

$$2tf=2pr, \quad \text{or} \quad \mathbf{tf=pr}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This formula may be employed to determine the *bursting*, *proof*, or *working pressure* in a cylindrical or approximately cylindrical boiler, provided that f, instead of being the tensile or compressive unit stress, is some suitable coefficient which has been determined by experiment. If η is the efficiency of a riveted joint, the formula

$$\eta\mathbf{tf=pr}$$

may be employed to determine the working pressure in a cylindrical or approximately cylindrical boiler.

In ordinary practice the values of η and f are given by the following table:

Material.	Joint.	η	f in Pounds per Square Inch.
Wrought iron.	Single-riveted	.55	8,000 to 9,000
" "	Double-riveted	.7	8,000 " 9,000
" "	Treble-riveted	.8 to .85	8,000 " 9,000
Steel.	Single-riveted	.55	12,000 to 13,000
"	Double-riveted	.7	12,000 " 13,000
"	Treble-riveted	.8 to .85	12,000 " 13,000

For cast-iron cylinders the working value of f may be taken at about 2000 lbs. per square inch.

The total pressure upon each of the flat ends of the cylinder

$$=\pi r^2 p.$$

The longitudinal tension in a thin hollow cylinder

$$=\frac{\pi r^2 p}{2\pi rt}=\frac{pr}{2t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and is one half of the circumferential stress f.

Let the cylinder be subjected to an external pressure p', as well as to an internal pressure p. Then

$$\mathbf{ft=pr-p'r'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

r' being the radius of the outside surface of the cylinder. f is a tension or a pressure according as $pr \gtrless p'r'$.

Generally the difference between r and r' is very small, and eq. (3) may be written

$$ft=r(p-p')$$

9. Spherical Shells. — Let the data be the same as before. The section made by any diametral plane must develop a total resistance of $2\pi rtf$. Then

$$2\pi rtf=\pi r^2 p,$$

or

$$\mathbf{2tf=pr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence a spherical shell is *twice* as strong as a cylindrical shell of the same diameter and thickness of metal, so that the strongest parts of *egg-ended* boilers are the ends.

Let the shell be subjected to an external pressure p', as well as to an internal pressure p. Then

$$2\pi\frac{r'+r}{2}tf=\pi rp^2-\pi r'^2p'$$

and

$$f(r'+r)t=r^2p-r'^2p'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

f is a tension or a pressure according as $r^2p \gtrless r'^2p'$.

Generally $r'-r$ is very small, and the relation (2) may be written

$$ft=\frac{r}{2}(p-p'). \qquad (3)$$

For a thick hollow sphere Rankine obtained

$$p=2f\frac{r'^3-r^3}{r'^3+2r^3} \text{ approximately.} \qquad (4)$$

10. Practical Remarks. — A common rule requires that the working pressure in fresh-water boilers should not exceed one sixth of the bursting pressure, and in the case of marine boilers that it should not exceed one seventh.

An English Board of Trade rule is that the tensile working stress in the boiler-plate is not to exceed 6000 lbs. per square inch of gross section, and French law fixes this limit at 4250 lbs. per square inch.

The thickness to be given to the wrought-iron plates of a cylindrical boiler is, according to French law,

$$t=.0036nr+.1 \text{ in.};$$

according to Prussian law,

$$t=(e^{.003n}-1)r+1 \text{ in.}=.003nr+.1 \text{ in., approximately,}$$

r being the radius in inches, and n the excess of the internal above the external pressure in atmospheres.

The thickness given to cast-iron cylindrical boiler-tubes is, according to French law, five times the thickness of equivalent wrought-iron tubes; according to Prussian law

$$t=(e^{.01n}-1)r+\tfrac{1}{3} \text{ in.}=.01nr+\tfrac{1}{3} \text{ in., approximately.}$$

Steam-boilers before being used should be subjected to a hydrostatic test varying from $1\frac{1}{2}$ to 3 times the pressure at which they are to be worked.

Fairbairn conducted an extensive series of experiments upon the collapsing strength of riveted plate-iron flues, by enclosing the flues in larger cylinders and subjecting them to hydraulic pressure. From these experiments he deduced the following formula for a *wrought-iron* cylindrical flue or tube:

$$\left.\begin{array}{c}\text{Collapsing pressure}\\ \text{in pounds per square inch of surface}\end{array}\right\} = p = 403150\frac{t^{2.19}}{lr},$$

t being the thickness, r the radius in inches, and l the length in feet.

This formula cannot be relied upon in extreme cases and when the thickness of the tube is less than $\frac{3}{8}$ in.

In practice t^2 *may be generally used instead of* $t^{2.19}$. The experiments also showed that the strength of an elliptical tube is almost the same as that of a circular tube of which the radius is the radius of curvature at the ends of the minor axis. Hence, if a and b are the major and minor axes of the ellipse, the above formula becomes

$$p = 403150\frac{b}{a^2}\frac{t^{2.19}}{lr}.$$

By riveting angle or T irons around a tube its length is virtually diminished and its strength is therefore increased, as it varies inversely as the length.

The thickness of tubes subjected to external pressure is, according to French law, twice the thickness of tubes subjected to interior pressure but under otherwise similar conditions; according to Prussian law the thickness of heating-pipes is

$$t = .0067d\sqrt[3]{n} + .05 \text{ in., if of sheet iron,}$$

and $$t = .01d\sqrt[3]{n} + .07 \text{ in., if of brass.}$$

According to Reuleaux, the thickness (t) of a round flat plate of radius r, subjected to a normal pressure, uniformly distributed and of intensity p, is given by the formula

$$\frac{t}{r} = \sqrt{\frac{p}{f}}, \quad \text{or} \quad \frac{t}{r} = \sqrt{\frac{2}{3}\frac{p}{f}},$$

according as the plate is merely supported around the rim or is rigidly fixed around the rim, as, e.g., the end plates of a cylindrical boiler; f, as before, is the coefficient of strength. The corresponding deflections of the plate are

$$\frac{5}{6}\left(\frac{r}{t}\right)^4\frac{pt}{E} \quad \text{and} \quad \frac{1}{6}\left(\frac{r}{t}\right)^4\frac{pt}{E}.$$

A practical rule for the thickness t ins. of a loam-moulded cast-iron water-main under a head of h ft. of water, and of diameter d ins., is

$$t=\frac{1}{8}+\frac{hd}{13000}.$$

Formerly guns and cylinders were cast round chills for the purpose of equalizing the stress over a cross-section. The inside hot metal was at once cooled and was subjected to compression by the contraction of the more slowly cooling outside metal. With the same object in view it is now a common practice either to shrink ring upon ring or to wind wire around an internal tube.

TABLES.

THE STRENGTHS, ELASTICITIES, AND WEIGHTS OF VARIOUS ALLOYS, ETC.

Material	Weight in Pounds per Cu. Ft.	Tensile Strength in Thousands of Pounds per Sq. In.	Compressive Strength in Thousands of Pounds per Sq. In.	Modulus of Elasticity in Millions of Pounds per Sq. In. (E).	Modulus of Rigidity in Millions of Pounds per Sq. In. (G).
Aluminium	162	28 to 33	....	9 to 10	3.4 to 4.8
“ annealed	165 to 170	13.5			
“ cast		9 to 13			
“ rolled		13 to 23			
“ bronze (90% Cu, 10% Al)	....	90	....	14.5	5.6
Brass, cast	490 to 530	20	....	9 to 13.5	
“ ordinary yellow, cast (66% Cu, 34% Zn)		20 to 27	10.5		4.5 to 5.5
“ ordinary yellow, rolled		33 to 54	....		
“ sheet	490 to 530	30	....	9 to 13.5	5.5
“ wire	533	45 to 56	....	14.2	
Copper, cast	....	18 to 27	....	11 to 14	4.2 to 5
“ rolled or wrought	550	29 to 36	5.8	12 to 7	4.7 to 6.5
“ with .2 to .4% P	....	45 to 49			
“ wire, annealed	....	40 to 47			
“ “ hard drawn	555	58 to 68	....	17	
German-silver wire	....	67			
Gold, drawn	1200	38 to 41	....	12	4.7
Gun-metal (90% Cu, 10% Sn)	540	27 to 38	....	10 to 11.2	4.3
Lead	710	1.9	7.1	.72	5.6 to 6.7
“ cast	700	3			
“ wire	....	3.1	....	1	
Platinum wire	1340	50	....	24	
Phosphor-bronze	....	36 to 40	....	13.5	5.25
“ cast	....	55	....	14	
“ wire (hard)	....	100 to 150			
“ “ (annealed)	....	50 to 60			
Silver, drawn	655	42	....	11	
Soft solder	....	6.7			
Tin	465	2.3 to 5.6	....	6	
Zinc, cast	436	2.3 to 6.7			
“ rolled	450	30			
Hemp rope	....	9 to 11			
Leather belting	....	4.5			

THE STRENGTHS, ELASTICITIES, AND WEIGHTS OF IRON AND STEEL.

Material.	Weight in Pounds per Cu. Ft.	Tensile Strength in Thousands of Pounds per Sq. In.	Compressive Strength in Thousands of Pounds per Sq. In.	Shearing Strength in Thousands of Pounds per Sq. In.	Modulus of Elasticity in Millions of Pounds per Sq. In. (*E*).	Modulus of Rigidity in Millions of Pounds per Sq. In. (*G*).
Cast iron	430 to 470	11 to 33	56 to 145	13 to 29	10 to 24	3.8 to 7.6
Cast iron, average	450	18	90	20	16	6.3
Bars, Lowmoor and Yorkshire.	480	53 to 65				
“ average	480	56	36 to 45	45	28 to 30	10.5
“ plates, and shapes for structures	480	49	50	40	28 tc 30	10 to 12
Plates, finest Lowmoor, Yorkshire, or Staffordshire (with fibre)	480	58 to 65				
Plates, finest Lowmoor, Yorkshire, or Staffordshire (across fibre)	480	53				
Plates, average (with fibre)	480	52		18 to 27	25	9.5
“ “ (across fibre)	480	45		36 to 45	27	
Soft iron		45				10.8 to 11.3
Wire, charcoal (hard drawn)		78 to 90				
“ “ (annealed)		67				
Bars, special	490	100 to 130			29 to 42	
Cast steel, untempered	490	90 to 150			30	12
“ “ drawn	480	120				
“ “ tempered	490				36	14
Castings		33 to 100				
“ annealed		56 to 78				
Chrome steel		180				
Manganese steel, cast		85				
Mild (very) steel		54 to 80				
“ steel bars and plates (.2% C)	490	63 to 72			28 to 31	13 to 13.5
“ steel bars, plates, and shapes for structures	490	52 to 68		47 to 56	32	
Nickel steel, unhardened)		75				
“ “ (5% C) annealed		90				
“ “ (12% C) “		200				
Soft steel, unhardened	490	60 to 100			30	
“ “ hardened	490	120			30	
Steel, high carbon (hardened by submersion)			266 to 400			
Steel rails (.4% C)		78 to 100				
Tungsten steel		160				
Wire, ordinary		155				
“ “ tempered		224 to 330			28	
“ pianoforte		270 to 336			26	

STRENGTH OF WIRE ROPES.

Breaking strength in pounds (for working strength factor of safety=5).

Diameter, Inches.	Swedish Iron, Hemp Centre; 6 Strands, 19 Wires to Strand.	Cast Steel, Hemp Centre; 6 Strands, 19 Wires to Strand.	Crucible-steel, Hemp Centre; 6 Strands, 19 Wires to Strand.	Plough-steel, Hemp Centre; 6 Strands, 19 Wires to Strand.	Crucible-steel 7 wires to Strand.
[illegible]	5,000	10,000	11,560	13,100	11,160
½	8.800	17.600	20.200	22,800	19,400
¾	19,400	39.800	44.000	50.000	42,000
1	34,000	68,000	78.000	88,000	74.000
1¼	50.000	100.000	116,000	134 000	112.000
1½	72,000	144,000	168.000	192,000	158,000
1¾	96,000	192.000	224 000	256 000	
2	124,000	248.000	288,000	330 000	
2¼	156,000	312 000	364,000	416.000	
2½	190.000	380 000	444.000	508 000	
2¾	228,000	456,000	532.000	610 000	

TABLE GIVING STRENGTH AND DUCTILITY TESTS OF HARD-DRAWN COPPER WIRE FOR TELEGRAPHIC PURPOSES.

B.&S. Gauge No.	Diameter, Mils.			Weight per Mile.			Breaking Weight.		Twists.	
	Average Required	Minimum.	Maximum.	Average Required	Minimum.	Maximum.	Average Required	Minimum.	Average Required	Minimum.
12	80	79.3	81.2	102.6	100.8	105.7	334	327	40	36
10	101.9	100.9	102.9	165.9	162.7	169.2	535	516	34	29
9½	104.	103.	105.	172.8	169.5	176.2	555	536	33	28
9	114.4	113.4	115.4	209.1	205.5	212.8	670	640	30	25
8	128.5	127.5	129.5	263.9	259.8	268.	840	811	27	22
7½	137.	136.	138.	300.	295.6	304.3	950	917	25	21
7	144.3	143.3	145.3	332.8	328.2	337.4	1013	1013	24	20

The ductility test is made by properly gripping the sample wire at the ends by two vises whose jaws are 6 inches apart. and causing one vise to revolve uniformly at right angles to the wire at a uniform speed of about one revolution per second. The tests are reckoned by the number of complete revolutions made by the revolving vise at the moment the wire breaks. The wire should be capable of being wound in six close turns around wire of its own diameter and unwound, without breaking.

IRON TELEGRAPH-WIRE.

The wire should be soft and pliable and capable of elongating 15 per cent without breaking after being galvanized.

The breaking strength should not be less than 2½ times the weight of the wire in pounds per mile.

The ductility test should be made as follows: The wire is gripped by two vises, whose jaws are 6 inches apart, and one vise is caused to revolve uniformly at right angles to the wire at a uniform speed of one revolution per second. The number of twists in 6 inches should not be less than 15.

STRENGTH OF MANILA ROPE.

Circumference, Inches.	Breaking Strength, Pounds.	Circumference, Inches.	Breaking Strength, Pounds.	Circumference, Inches.	Breaking Strength, Pounds.
¾	560	4½	14,800	8	39,900
1	784	5	18,400	9	47,000
1½	1,560	5½	21,900	10	54,200
2	2,700	6	25,500	11	61,400
2½	4,300	6½	29,100	12	68,500
3	6,100	7	32,700	13	75,700
3½	8,500	7½	36,300	14	82,900
4	11,500				

THE BREAKING WEIGHTS AND COEFFICIENTS OF BENDING STRENGTH IN TONS (OF 2240 LBS.) OF VARIOUS RECTANGULAR BEAMS, THE WEIGHTS HAVING BEEN UNIFORMLY DISTRIBUTED.

Material.	Clear Span between Supports in Inches.	Breadth in Inches.	Depth in Inches.	Mean Breaking Weight of each Joist or Beam.	Coefficient of Bending Strength.
Yellow pine (Quebec), 11 joists.	142	3	9	5.66	2.48
" " " " "	142	3½	11	7.89	1.98
" " " 2 beams.	126	14	15	60.97	1.83
Fir (Baltic), 2 beams.	126	14	14	46.6	1.6
" " 11 joists.	142	3½	11	8.29	2.08
" (Swedish), 2 joists.	142	3	9	5.7	2.49
Pine (Baltic), 2 beams.	126	13½	13½	58.43	2.24
Baltic redwood deal (Wyberg), 2 joists..	142	3	9	5.75	2.52
Spruce deals (St. John), 3 pairs with bridging-pieces..................	142	3	9	6.81	2.98

THE STRENGTHS, ELASTICITIES, AND WEIGHTS OF TIMBERS.

This table contains the results of the most recent and most reliable experiments, but, generally speaking, only small specimens of the material have been tested. It is found that the strength, elasticity, and weight of a timber are affected by the soil, age, seasoning, per cent of moisture, position in the log, etc., and hence it is not surprising that specimens even when cut out of the same log show results which often differ very widely from the mean. Additional experiments on large timbers are needed, and in each case should be accompanied by a complete history of the specimen from the time of felling

Description of Timber.	Tensile Strength in Tons per Sq. In.	Compressive Strength in Tons per Sq. In. along Fibres.	Shearing Strength in Tons per Sq. In. along Fibres.	Young's Modulus E (in tons).	Coefficient of of Rigidity, G	Coefficient of Bending Strength in Tons per Sq. In.	Weight in Pounds per Cu. ft.
Acacia........	....	7.1			$G=\frac{2}{5}E$, approximately.		
Alder.........	4.5 to 6.3	3.1	....	....		....	50
Apple.........	8.8	....	....	....		....	50
Ash, Canadian .	2.45	2.5	.2 to .312	620		....	47
Ash, English...	5.35 to 7.58	4	....	723		....	43 to 53
Beech.........	4.9 to 9.8	3.67	....	607		....	43 to 53
Birch.........	6.69	1.47 to 2.27	.25 to .364	734		....	45 to 49
Box..........	9	4.6	....	803		....	64
Blue gum......	2.7	3.078	....	509		5.86	
Cedar.........	2.23 to 4.9	2.56	....	217		....	35 to 47
Chestnut......	....	....	.308	509		....	35 to 41
Ebony........	....	8.48					
Elm, Canadian.	4.3	2.9	....	1100		....	47
Elm, English...	5.89	4.6	....	....		....	34 to 37
Greenheart.. .	2.7 to 4.1	4.46 to 6.5	....	759		....	58 to 72
Hawthorn.....	4.68						
Hazel.........	8.48						
Hornbeam.....	9.1	2.6	....	....		....	47½
Ironbark......	7.12	4.54	....	....		8.15	
Ironwood......	4.31	5.21					
Jarrah.	1.31	3.2	....	....		4.13	
Lancewood....	3.6 to 6.7	....	....	....		....	42 to 63
Larch.	3.92 to 4.55	1.42 to 2.45	....	....		....	32 to 38
Lignum vitæ. ..	5.26	4.46	....	446		7.18	41 to 83
Locust........	4.5 to 6.7	1.33	.533				
Mahogany,Span	1.7 to 7.3	3.3	3.3	....		560 to 1339	53
" Hond.	1.3 to 3.6		....	....		712 to 879	35
Maple.........	4.7 to 7.7	2.23	....	....		....	49
Mora.........	4.1	4.4	....	830		....	57 to 68
* Oak, Am.	....	....	....	577		2.712	
Oak, Am. red.	4.46	1.89 to 2.6	.324 to .446	....		....	61
" white.....	8.8	2.84	.335 to .431	....		4.55	61
" Eng......	5.4	4.4	....	664		....	49 to 58
Pine, Dantzic. .	3.5	....	....	1025		....	36
" Memel. ..	4.5	....	....	....		....	34
" pitch.....	4.6	3.5	....	....		....	41 to 58
*" red......	....	....	....	670		2.93	34
" red.	1.7 to 6.67	2.4 to 3	....	962		3.71	34
*" yellow ...	....	....	....	779		3.255	
" yellow ...	2.2 to 6.87	2.4 to 3.6	.227	900		4.51	32
*" white. ..	....	....		484		2.146	
" white....	1.3 to 5.1	2.24	.119 to .164	....		3.03	30
Plane.........	5.4	....	....	604		....	40
Poplar........	2.94	1.8	....	340		....	23 to 26
* Spruce.......	....	....	....	594		2.18	
* "	2.99 to 5.97	....	....	438 to 737		1.63 to 2.86	
"	3	....	.113 to .167	700		....	29 to 32
Sycamore......	5.8	3.16	....	464		....	36 to 43
Teak.........	4.7 to 6.7	5.35	....	1000		....	41 to 52
Walnut	3.5	2.7	....	....		....	38 to 57
Willow........	4.6 to 6.25	1.5	....	629		....	24 to 35

* The results for these timbers are deduced from experiments carried out by Bauschinger Lanza, and others on comparatively large specimens.

THE BREAKING WEIGHTS AND COEFFICIENTS OF BENDING STRENGTH IN TONS (OF 2240 LBS.) OF VARIOUS RECTANGULAR BEAMS LOADED AT THE CENTRE.

Material.	Clear Span between Supports in Inches.	Breadth in Inches.	Depth in Inches.	Breaking Weight in Tons.	Coefficient of Bending Strength.	Remarks.
Yellow pine	129	14	15	38.15	2.34	
" "	129	14	15	34	2.09	
" "	45	5	7	5.9	1.62	
" "	45	5	7	5.7	1.57	Old timber
" "	45	6	5	3.1	1.67	" "
" "	45	5	5	3.05	1.64	
" "	45	2½	3½	.925	2.04	Old timber
" "	45	2½	3½	1.075	2.37	" "
Pitch-pine	129	14	15	59.25	3.64	
"	129	14	15	60.25	3.7	
"	45	5	7	7.8	2.14	
"	45	5	7	9.75	2.68	
"	45	5	7	10.65	2.92	
"	45	5	7	11	3.03	
"	45	2½	3½	1.6	3.52	
"	45	2½	3½	1.35	2.97	
Baltic pine	45	5	7	7	1.91	
" "	45	5	7	8.5	2.34	
" "	45	2½	3½	1.125	2.48	
" "	45	2½	3½	1.2	2.64	
American elm	45	5	7	14.9	4.1	Old timber
" "	45	5	7	15.6	4.29	" "
" "	45	2½	3½	2.65	5.84	" "
" "	45	2½	3½	2.6	5.73	" "
Greenheart	45	5	5	14	7.56	
"	45	2½	7	11.45	6.31	
"	45	3	3	3.85	9.625	
"	45	2½	3½	4.00	8.81	
"	45	2½	3½	3.55	7.82	
"	139	9	8	24.5	8.87	
Red pine	147	6	12	7.5	1.91	
" "	147	6	12	8.45	2.15	

N.B.—The results contained in the last two tables are mainly deduced from experiments carried out under the supervision of W. Le Mesurier, M. Inst. C. E., Dock Yard, Liverpool.

AVERAGE OF THE RESULTS OBTAINED BY THE AUTHOR WITH BEAMS OF LARGE SCANTLING IN THE TESTING LABORATORY, McGILL UNIVERSITY.*

Canadian Timbers.	Coefficient of Bending Strength in Pounds per Sq. In.	Coefficient of Elasticity in Pounds.	Weight per Cubic Foot in Pounds.
Douglas fir—specially selected, free from knots, and out of log at a distance from the heart	9,000	2,000,000	40
Douglas fir—ordinary first quality	6,000	1,430,000	34
Red pine	5,400	1,430,000	34

* For details of these tests the reader is referred to the Transactions of the Canadian Society of Civil Engineers, Vol. IX, 1895, and Vol. XII, 1898.

FURTHER RESULTS OBTAINED BY THE AUTHOR WITH BEAMS OF LARGE SCANTLING.

Timber.	Coefficients of Strength for			Shear.	Sp. Wt.
	Bending.	Tension.	Compression.		
White pine	4,800	10,000	3,500	340	26
Red pine	5,400	11,500	3,900	380	33
Hemlock	5,000	8,000	3,200	380	33
Spruce	5,000	9,000	3,200	300	30

Coefficients (a) of Linear Expansion per Unit of Length from 32° F. to 212° F.

Materials.	a.	Materials.	a.	Materials.	a.
Brass	.001868	Gun-metal	.00181	Steel, hardened	.001247
Bronze	.00182	Iron wire	.00144	Tin	.002173
Cast iron	.001075	Lead	.0002848	Wrought iron (bar)	.001235
Copper	.001718	Oak	.000746	Wrought iron (smith's)	.001182
Fir	.00352	Platinum	.000884	Zinc, cast	.002941
Glass	.00861	Silver	.001909	Zinc, hammered	.003108
Gold	.001466	Steel, unhardened	.001079		

The Weights and Crushing Weights of Rocks, etc.

Material.	Weight per Cubic Foot in Lbs.	Crushing Weight in Lbs. per Sq. In.
Asphalt	156	8,300
Basalt, Scotch	184	17,200
" Greenstone	181	16,800
" Welsh	172	
Béton		800 to 1,400
Brick, common	100 to 135	550 to 800
" stock (Eng.)		2,250
" Sydney, N. S.		2,200
" yellow-faced (Eng.)		1,440
" Staffordshire blue		7,200
" fire		1,700
" pressed (best)	150	10,200
Brickwork	112	
Cement, Portland	86 to 94	1,700 to 6,000
" Roman	100	
Clay	119	
Concrete, ordinary	119	460 to 775
" in cement	137	
Earth	77 to 125	
Firestone	112	19,600
Freestone	.....	3,000 to 3,500
Glass, flint	192	27,500
" crown	157	31,000
" common green	158	31,000
" plate	172	
Granite, Aberdeen gray	163	10,800
" " red	165	
" Cornish	166	14,000
" Sorrel	167	12,800
" Irish	.....	10,450
" U. S. (Quincy)	.....	15,000
" Argyll	.....	10,900
Gneiss	96 to 175	19,600
Limestone	154 to 162	7,500 to 9,000
Lime, quick	53	
Mortar	86 to 119	120 to 240
" (average)	106	
Masonry, common brick	116 to 144	500 to 800
" in cement		760
" rubble		$\frac{4}{10}$ of cut stone
Marble, statuary	170	3,200
miscellaneous	168 to 170	8,000 to 9,700
Oölite, Portland stone	151	4,100
" Bath stone	123	
Sand, quartz	177	
" river	117	
" pit	100	
" fine	95	
Sandstone, red (Eng.)	133	5,700
" Derby grit	150	3,100
" paving (Eng.)	156 to 157	5,700 to 6,000
" Scotch	153 to 155	5,300 to 7,800
" U. S.	.....	5,300
Scoria paving-blocks (on edge)	.....	10,000
" " " (on flat)	.....	11,250
Shingles	88	
Slate, Anglesea	179	10,000 to 24,000
" Cornish	157	
" Welsh	180	
Trap	170	

CRUSHING STRENGTH OF GRANITES, LIMESTONES, AND SANDSTONES.

Locality.	Material.	Authority.	Crushing Strength in Pounds per Sq. In.
Aberdeen	Granite	Haswell	10,769
Cornish	"	"	6,339
Dublin	"	"	10,450
Newry	"	"	12,850
Patapsco	"	"	5,340
Bay of Fundy	"	Gillmore	12,020
City Point	"	"	15,093
Dix Island	"	"	15,000
Duluth	"	"	19,000
Fox Island	"	"	15,062
Greenwick	"	"	11,700
Harbor Quarry	"	"	16,837
Hurricane Island	"	"	14,937
New Haven	"	"	9,750
Port Deposit	"	"	19,755
Quincy	"	"	17,750
Rockport	"	"	19,750
Vinalhaven	"	"	16,950
Westerly	"	"	17,750
Huron Island	"	Merrill	20,650
Keene	"	"	10,375
Mt. Raymond	"	"	5,970
Monsoon	"	"	15,390
New London	"	"	12,500
Richmond	"	"	19,104
Stony Creek	"	"	16,750
Mt. Johnson (dark gray)	"	Bovey	23,500
Nepean (bed vertical)	"	"	16,746
" (bed horizontal)	"	"	19,033
Stanstead	"	"	13,500
" (black and white)	"	"	13,526
Quincy, Mass. (bed vertical)	"	"	23,522
" " (bed horizontal)	"	"	26,860
St. Philip, Q. (reddish)	Laurentian	"	23,500
" " (bed vertical)	"	"	24,520
" " (bed horizontal)	"	"	30,950
St. Philip	"	"	26,000
Arbroath	Sandstone	Haswell	7,850
Aquia Creek	"	"	5,340
Yorkshire	"	"	5,710
Albion	"	Gillmore	13,500
Belleville	"	"	11,700
Berea	"	"	10,250
Cleveland	"	"	7,910
Craigleigh	"	"	12,000
Dorchester	"	"	9,412
Fond du Lac	"	"	6,250
Haverstraw	"	"	4,350
Kasota	"	"	11,675
Little Falls	"	"	9,850
Marquette	"	"	7,450
Massillon	"	"	8,750
Medina	"	"	17,725
Middleton	"	"	6,950
North Amherst	"	"	6,650
Seneca	"	"	10,500
Vermillon	"	"	8,850
Warrensburg	"	"	5,000
Altamount	"	Merrill	1,149
Edinburgh	"	"	12,000
Glencoe	"	"	12,752
Hummelstown	"	"	16,610
Jordan	"	"	3,750
Long Meadow	"	"	8,812
Manitou	"	"	13,046
Michigan	"	"	6,323
New Gunnison	"	"	9,903
Oswego	"	"	6,220
Rawlins	"	"	10,833
Taylor's Falls	"	"	5,500
Quebec (bed vertical)	"	Bovey	7,344
" (bed horizontal)	"	"	9,130

Crushing Strength of Limestones, Marbles, etc.

Locality.	Material.	Authority.	Crushing Strength in Pounds per Sq. In.
English Magnesian	Limestone	Haswell	3,130
Bardstown	"	Gillmore	16,250
Joliet	"	"	16,900
Marquette	"	"	8,050
Marblehead	"	"	12,600
North River	"	"	13,425
Williamsville	"	"	12,375
Bedford	"	Merrill	10,125
Quebec (bed vertical)	"	Bovey	12,597
" (bed horizontal)	"	"	12,330
Stockbridge	Marble	Haswell	10,382
Dorset	"	Gillmore	8,670
Quincy, Ill.	"	"	9,878
Tuckahoe	"	"	13,594
Italian	"	Merrill	12,156
Vermont	"	"	13,400
Scotland	Whinstone	Haswell	8,300
Connecticut	Freestone	"	3,319
Fairhaven	Slate	Merrill	12,870

Average Coefficients of Shearing Strength for Various Woods (Bovey).

White oak	690	Yellow pine	400	White ash	560
Douglas fir	380	Maple	700	Hemlock	400
Birch	880	Elm	590	Cottonwood	260

The shearing strengths are for planes parallel to the axis of the scantling. The tensile strength is liable to considerable variation. The direct tensile strength is much greater than the direct compressive strength, and the failure of a beam loaded transversely, by crippling on the compression side, is likely to take place under a load much less than the material could bear in tension.

Kiln-drying largely increases the direct compressive strength, but greatly diminishes the shearing strength, while the direct tensile strength does not appear to be much affected, although in the majority of cases it is diminished, and sometimes considerably.

Average Results obtained in Cement-testing Laboratory, McGill University.

Neat Portland cement, after 1 to 4 weeks, has a crushing strength of from 3000 to 6000 lbs per square inch and a tensile strength of from 400 to 600 lbs. per square inch.

Neat Natural cement, after 1 to 4 weeks, has a crushing strength of from 600 to 1200 lbs. per square inch and a tensile strength of from 100 to 200 lbs. per square inch.

Material.	Crushing Load, Lbs. per Sq. In.	Material.	Crushing Load, Lbs. per Sq. In.
Common brick piers (cement mortar)	800 to 1000	Brick piers (lime mortar)	250 to 400
Pressed brick piers (cement mortar)	1100 to 1200	Concrete (1 2 5) (145 to 150 lbs. per cu. ft.)	400 to 800
		Concrete (1 2 4, 3 to 6 months old)	2350

After 28 days' immersion the average compressive strengths of concrete 4-in. cubes, made of good Portland cement and

(a) sharp angular sandstone gravel in the ratio of 1 to 1, 1 to 2, 1 to 3, 1 to 4, were found to be 1600, 1400, 900, and 600 lbs. per square inch respectively:

(b) smooth water-worn Laurentian gravel in the ratio of 1 to 1, 1 to 2, 1 to 3, 1 to 4, were found to be 1350, 950, 800, and 500 lbs. per square inch, respectively

TABLE OF AVERAGE VALUES OF E, G, K, IN MILLIONS OF LBS./SQ. IN. AND OF σ.

	E.	G.	K.	σ.
Brass.	.....	5 to 5.5	15.3	3.1 to 3.3
Cast iron.	.....	5 to 7.6	.....	3 to 4.7
Cast steel, tempered.	.....	14		
Cast steel, untempered.	.....	12		
Copper.	.....	5.6 to 6.7	17.1 to 24.4	2.9 to 3
Delta metal, rolled.	.....	5.25	14.4	
Glass.	.....	3.3 to 3.9	.....	3.9
Gun-metal.	.....	3.7	5.8	
Iron, boiler-plate	.....	14	21.1	
Lead.	.....	.27	5.3	
Soft iron.	.....	10.8 to 11.3		
Soft steel, unhardened.	.....	11	21.6 to 26.7	
Soft steel, hardened.	.....	11		
Steel plates (¼ to 1 per cent C).	.....	13		
Steel, boiler-plate.	.....	13.5		3.6 to 4.6
Tin.	.....	22		
Wood.	.....	.1 to .17		
Wrought-iron bars.	11	10.5	21.1	3.6
Wrought-iron plates.	.....	9.5		
Water.	3.14	3.14	3.2	
India-rubber.	.....	.....	.....	2
Zinc.	.....	5.1 to 5.4		
Slate.	.....	3.2		
Granite.	.....	1.8		
Granites:				
* Stanstead.	5 to 6.05	2.22 to 2.46	3.68 to 4.1	4
* Peterhead.	7.9 to 8.07	3.2 to 3.3	4.56 to 4.96	4.2
* Lily Lake.	8.1	3.4	6.2	5.1
* Westerly.	7.2 to 7.6	2.98 to 3.16	4 to 4.75	4.3
* Quincy.	6.75 to 7.9	2.8 to 3.26	3.66 to 4.5	5.0
* Bowens.	6.65 to 6.95	2.64 to 2.92	4.55 to 4.9	3.9
Marble.	.....	1.7		
Marbles				
* Carrara.	8.4	2.8	5.21	3.64
* Tennessee.	9 to 9.5	3.6 to 3.8	5.8 to 6.32	4
* Esserite.	9.67 to 10	3.72 to 4.1	6.5 to 7	3.9

* These results have been obtained by Dr. Adams, F.R.S.C., of McGill University, in his experiments on the flow of rocks.

EXAMPLES.

1. How many square inches are there in the cross-section of an iron rail weighing 30 lbs. per lineal yard? How many in a yellow-pine beam of the same lineal weight? *Ans.* 3 sq. in.; 45 sq. in.

2. A vertical wrought-iron bar 60 ft. long and 1 in. in diameter is fixed at the upper end and carries a weight of 2000 lbs. at the lower end. Find the factors of safety for both ends, the ultimate strength of the iron being 50,000 lbs. per square inch. *Ans.* $19\frac{9}{14}$; $18.\frac{32}{151}$.

3. A vertical rod fixed at both ends is weighted with a load w at an intermediate point. How is the load distributed in the tension of the upper and compression of the lower portion of the rod? *Ans.* Inversely as the lengths.

4. Find the length of a steel bar of sp. gr. 7.8 which, when suspended vertically, would break by its own weight, the ultimate strength of the metal being 60,000 lbs. per square inch. *Ans.* 17,723 ft.

5. The iron composing the links of a chain is ⅓ in. in diameter; the chain is broken under a pull of 10,000 lbs. What is the corresponding tenacity per square inch? *Ans.* $57{,}272\frac{8}{11}$ lbs.

6. A vertical iron suspension-rod 90 ft. long carries a load of 20,000 lbs. at its lower end; the rod is made up of three equal lengths square in section.

Find the sectional area of each length, the ultimate tenacity of the iron being 50,000 lbs. per square inch, and 5 a factor of safety.

Ans. $\frac{200}{99}$ sq. in.; $\frac{2000}{99^2}$ sq. in.; $\frac{2000000}{99^3}$ sq. in.

7. If the rod in the previous question is of a conical form, what should be the area of the upper end? Also find the intensities of the tension at 30 and 60 ft. from the lower end.

Ans. 2.0609 sq. in.; 9999.35 lbs., 9999.4 lbs. per square inch.

8. The dead load of a bridge is 5 tons and the live load 10 tons per panel, the corresponding factors of safety being 3 and 6. If the two loads are taken together, making 9 tons per panel, what factor of safety would you use? *Ans.* 5.

9. The end of a beam 10 in. broad rests on a wall of masonry. If it be loaded with 10 tons, what length of bearing surface is necessary, the safe crushing stress for stone being 150 lbs. per square inch? *Ans.* $13\frac{1}{3}$ in.

10. Find diameter of bearing surface at the base of a column loaded with 20 tons, the same stress being allowed as in the preceding question. *Ans.* $\sqrt{380.12}$.

11. What should be the diameter of the stays of a boiler in which the pressure is 30 lbs. per square inch, allowing one stay to each $1\frac{1}{2}$ sq. ft. of surface and a stress of 3500 lbs. per square inch of section of iron? *Ans.* $1\frac{1}{2}$ in.

12. A short cast-iron post is to sustain a thrust of 64,000 lbs., the ultimate crushing strength of the iron being 80,000 lbs. per square inch and 10 a factor of safety. Find the dimensions of the post, which is rectangular in section with the sides in the ratio of 2 to 1. *Ans.* 4 in.; 2 in.

13. How many $\frac{7}{8}$-in. rivets must be used to join two wrought-iron plates, each 36 in. wide and $\frac{1}{2}$ in. thick, so that the rivets may be as strong as the riveted plates, the tensile and shearing strength of wrought-iron being in the ratio of 10 to 9? *Ans.* 19 rivets (18.3)

14. A horizontal cast-iron bar 1 ft. lòng exactly fits between two vertical plates of iron. How much should its temperature be raised so that it might remain supported between the plates by the friction, the coefficient of friction being $\frac{1}{6}$? *Ans.* $\frac{1}{10}$° F.

15. A brick wall 2 ft. thick, 12 ft. high, and weighing 112 lbs. per cubic foot is supported upon solid pitch-pine columns 9 in. in diameter, 10 ft. in length, and spaced 12 ft. centre to centre. Find the compression unit stress in the columns (1) at the head; (2) at the base. The timber weighs 50 lbs. per cubic foot.

If the crushing stress of pitch-pine is 5300 lbs. per square inch and the factor of safety 10, find the height to which the wall may be built.

Ans. 507.03 lbs.; 510.5 lbs.; 12.46 ft.

16. Determine the diameter of the wrought-iron columns which might be substituted for the timber columns in the preceding example, allowing a working stress in the metal of 7500 lbs. per square inch. *Ans.* 2.36 in.

17. A short piece of steel at a temperature of 62° F. just fits in between two parallel plates. If the steel be heated up to a temperature of 162° F, find the stress induced in the steel, the walls being assumed unyielding. You

may assume that the coefficient of expansion of steel per degree F. is $\frac{1}{150000}$ and that $E = 12{,}500$ tons per square inch.

If the coefficient of friction between the bar and plate is $\frac{1}{6}$, what vertical load will the bar just carry? *Ans.* $2\frac{7}{9}$ tons for each square inch of section.

18. There is a thrust of 105,000 lbs. along a strut (prevented from bending) made up of four $4'' + 4'' \times \frac{1}{2}''$ angle-irons and 40 ft. long. Find the load per square inch, and the amount by which the strut is shortened, E being 28,000,000 lbs. per square inch. *Ans.* 7000 lbs.; .12 in.

19. A force of 10 lbs. stretches a spiral spring 2 in. Find the work done in stretching it successively 1 in., 2 in., 3 in., up to 6 in.
Ans. $\frac{5}{2}$, $\frac{20}{2}$, $\frac{45}{2}$, $\frac{80}{2}$, $\frac{125}{2}$, $\frac{180}{2}$ in.-lbs.

20. A roof tie-rod 142 ft. in length and 4 sq. in. in sectional area is subjected to a stress of 80,000 lbs. If $E = 30{,}000{,}000$ lbs., find the elongation of the rod and the corresponding work. *Ans.* 1.136 in.; $3786\frac{2}{3}$ ft.-lbs.

21. An iron wire $\frac{1}{8}$ in. in diameter and 250 ft. in length is subjected to a tension of 600 lbs., the consequent strain being .$\frac{1}{800}$ Find E, and show by a diagram the amount of work done in stretching the wire within the limits of elasticity. *Ans.* $14{,}661{,}818\frac{2}{11}$ lbs.

22. A timber pillar 30 ft. in length has to support a beam at a point 30 ft. from the ground. If the greatest safe strain of the timber is $\frac{1}{800}$, what thickness of wedge should be driven between the head of the pillar and the beam?
Ans. $\frac{1}{10}$ ft.

23. During the test of a given piece of $\frac{3}{4}$-in. hard round steel bar in compression, a length of 20 ins. was shortened by .007 in. under a load of 2 long tons. Find the corresponding E. *Ans.* 28,961,616 lbs./sq. in.

24. An iron bar of uniform section and 10 ft. in length stretches .12 in. under a unit stress of 25,000 lbs. Find E, the bar being 1 sq. in. in section. If 25,000 lbs./sq. in. is the proof stress, find the modulus of resilience. Determine the work stored up in the bar in foot-pounds, and compare it with the work which would be stored up if for half its length the rod has its section increased to 4 sq. ins.
Ans. 25,000,000 lbs.; 25 in in.-lb. units; 125 ft.-lbs.; $78\frac{1}{8}$ ft.-lbs.

25. A ship at the end of a 600-ft. cable and one at the end of a 500-ft. cable stretch the cables 3 in. and $2\frac{1}{2}$ in. respectively. What are the corresponding strains? *Ans.* $\frac{1}{2400}$.

26. A rectangular timber tie is 12 in. deep and 40 ft. long. If $E = 1{,}200{,}000$ lbs., find the proper thickness of the tie so that its elongation under a pull of 270,000 lbs. may not exceed 1.2 in. *Ans.* $7\frac{1}{2}$ in.

27. A wrought-iron bar 60 ft. long is stretched 5 in. by a pull of 50,000 lbs. Find its diameter, E being 25,000,000 lbs. *Ans.* .6 in.

28. A wrought-iron rod 984 ft. long alternately exerts a thrust and a pull of 52,910 lbs.; its cross-section is 9.3 sq. in. Find the loss of stroke, E being 29,000,000 lbs. *Ans.* 4.632 in.

29. A wrought-iron bar 2 sq. in. in sectional area has its ends fixed between two immovable blocks when the temperature is at 32° F. If $E = 29{,}000{,}000$ lbs., what pressure will be exerted upon the blocks when the temperature is 100° F.? $\alpha = .00125$. *Ans.* $27388\frac{8}{9}$ lbs.

30. A wrought-iron rod 25 ft. in length and 1 sq. in. in sectional area is subjected to a steady stress of 5000 lbs. What amount of live load will instantaneously elongate the rod by $\frac{1}{8}$ in., E being 30,000,000 lbs.?

Ans. 6250 lbs.

31. The following examples *a* to *v* give the observations made in the actual tests of specimens of different materials. In each case, where possible, draw the stress-strain diagram and calculate the stress at the elastic limit, at the yield-point, and at fracture. Also, determine Young's modulus, the reduction of area, and the equivalent elongation.

(*a*) *Tensile test.* Distance between gauge-points = 10 ins.
Section = 1 in. × .735 in.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
100	0	800	.00438	1600	.00871
400	.00219	1200	.00652	11360	B.W.

Ans. B.W./sq. in. = 15,640 lbs.; E = 2,504,000 lbs./sq. in.

(*b*) *Tensile test.* Distance between gauge-points = 10 ins.
Section = .98 in. × .74 in.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
100	0	800	.00622	1500	.01237
200	.00092	900	.00713	1600	.01318
300	.00178	1000	.00799	1700	.01404
400	.00263	1100	.00892	1800	.01485
500	.00354	1200	.0098	1900	.01567
600	.00440	1300	.01069	2000	.01648
700	.00531	1400	.01145	5730	B.W.

Ans. B.W./sq. in. = 7900 lbs.; E = 1,595,000 lbs./sq. in.

(*c*) *Tensile test.* Distance between gauge-points = 10 ins.
Sectional area = .7425 sq. in.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
100	0	800	.00647	1200	.01032
400	.00281	1000	.0084	6200	B.W.
600	.00459				

Ans. B.W./sq. in. = 8350 lbs.; E = 1,435,000 lbs./sq. in.

(*d*) *Tensile test.* Distance between gauge-points = 18 ins.
Section = 1.97 in. × .5 in.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
1000	0	5000	.050	9000	.09
2000	.015	6000	.060	10,000	.105
3000	.025	7000	.07	19,830	B.W.
4000	.037	8000	.08		

Ans. B.W./sq. in. = 20,132 lbs.; E = 1,625,000 lbs./sq. in.

(*e*) *Tensile test.* Distance between gauge-points =10 ins.
Section =.98 in. ×.73 in.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
100	0	1100	.00762	2200	.01446
200	.00076	1200	.00907	2400	.01572
400	.00219	1400	.00907	2600	.01718
600	.00358	1600	.01051	6800	B.W.
800	.00492	1800	.01194		
1000	.00609	2000	.01332		

Ans. B.W./sq. in. =9500 lbs.; E =2,070,000 lbs./sq. in.

(*f*) *Tensile test.* Distance between gauge-points =8 ins.
Section =.98 in. ×.73 in.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
100	0	1000	.00672	2000	.01441
200	.00077	1200	.00831	2200	.01591
400	.00217	1400	.00990	2400	.01748
600	.00360	1600	.0115	7600	B.W.
800	.00513	1800	.01307		

Ans. B.W./sq. in. =10,600 lbs.; E =1,840,000 lbs./sq. in.

(*g*) *Tensile test.* Distance between gauge-points =16 ins.
Initial diameter =.29 in.
Final " =.213 in.
Equivalent extension =.14 in.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
200	.002	1600	.030	3000	.060
400	.009	1800	.033	3200	.066
600	.011	2000	.036	3400	.075
800	.015	2200	.040	3600	.086
1000	.018	2400	.044	37,950	Broke
1200	.022	2600	.049		
1400	.026	2800	.053		

(*h*) *Tensile test.* Distance between gauge-points =36 ins.
Initial diameter =.136 in.
Final " =.085 in.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
150	.01	600	.05	1000	.150
300	.02	700	.06	1110	B.W.
400	.03	800	.075		
500	.04	900	.095		

Ans. Load at fracture =76,035 lbs./sq. in.; E =24,752,000 lbs./sq. in.

(*i*) *Tensile test.* Distance between gauge-points =30 ins.
Sectional area =9.95 in.×.26 in.
Wt./lin. ft. =1.143 lbs.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
0	0	3500	2.28	6500	3.55
500	.67	4000	2.52	7000	3.71
1000	.91	4500	2.73	7500	3.86
1500	1.23	5000	2.94	8000	4.02
2000	1.52	5500	3.14	8500	4.22
2500	1.78	6000	3.32	8600	B.W.
3000	2.04				

Ans. Load at fracture =3320 lbs./sq. in.; E =235,000 lbs./sq. in.

(*j*) *Tensile test.* Distance between gauge-points =8 ins.
Initial diameter =.995 in.
Final " =.961 in.
Equivalent elongation =.405 in.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
2000	0	20,000	.00615	38,000	.01287
4000	.00075	24,000	.00735	42,000	.01445
6000	.00145	28,000	.00925	46,000	.01622
10,000	.00282	30,000	.00987	48,000	.01712
14,000	.00412	32,000	.01057	50,000	Yield Pt.
18,000	.00542	36,000	.01217	92,650	B.W.

Ans. Load in lbs./sq. in. at E.L. =61,780; at Y.P. =50,000; at fracture =119,240; red. of area =6.7 per cent; equiv. elong. =5.06 per cent; E =27,850,000 lbs./sq. in.

(*k*) *Tensile test.* Distance between gauge-points =8 ins.
Section =2.01 in.×.25 in. initial.
" =1.60 in.×.14 in. final.
Equivalent elongation =2 22 ins.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext in Ins.
500	0	10,000	.00544	20,000	.01098
2000	.00086	12,000	.00655	21,000	.01154
4000	.00197	14,000	.00766	22,000	.01209
6000	.00310	16,000	.00877	23,000	.01300
8000	.00431	18,000	.00988	29,340	B.W.

Ans. Load at E.L. =39,800 lbs./sq. in.; at Y.P. =45,800 lbs./sq. in.; at fracture =58,390 lbs./sq. in.; red. of area =25.22%; equiv. elong. =27.75%; E =28.320 000 lbs./sq. in.

(*l*) *Tensile test.* Distance between gauge-points = 8 ins.
Section = 2 ins. × 1 in. initial.
" = 1.37 in. × .62 in. final.
Equivalent elongation = 2.78 ins.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
30,000	0	52,000	.00613	66,000	.00822
40,000	.00463	56,000	.00662	68,000	.00860
44,000	.00513	60,000	.00717	70,000	.00910
48,000	.00561	64,000	.00786	119,900	B.W.

Ans. Load in lbs./sq. in. at E.L. = 31,000; at Y.P. = 35,000; at fracture = 59,950; red. of area = 57.5%; equiv. elong. = 34.75%; E = 31,500,000 lbs./sq. in.

(*m*) *Tensile test.* Distance between gauge-points = 6 ins.
Initial sectional area = .4359 sq. in.
Final " " = .1452 "
Equiv. elong. in 8 ins. = 2.05 ins.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
1000	0	11,000	.00461	16,000	.00691
7000	.00281	12,000	.00506	17,000	
8000	.00326	13,000	.00551	23,900	B.W.
9000	.00371	14,000	.00599		
10,000	.00416	15,000	.00644		

Ans. Load in lbs./sq. in. at E.L. = 29,820; at Y.P. = 39,000; at fracture = 54,830; red. of area = 66.7%; equiv. elong. in 8 ins. = 25.63% E = 30,590,000 lbs./sq. in.

(*n*) *Tensile test.* Distance between gauge-points = 8 ins.
Initial diameter = .757 ins.
Final " = .617 ins.
Equivalent elongation = 1.84 ins.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins	Load in Lbs.	Ext. in Ins.
500	0	9000	.00667	14,000	.01086
1000	.00039	10,000	.00747	15,000	.01175
3000	.00203	11,000	.00829	16,000	.01280
5000	.00358	12,000	.00912	24,060	B.W.
7000	.00511	13,000	.00997		

Ans. Load in lbs./sq. in. at E.L. = 24,445; at Y.P. = 35,556; at fracture = 53,467; red. of area = 33.5%; equiv. elong. = 23%; E = 22,880,000 lbs./sq. in.

(*o*) *Tensile test.* Distance between gauge-points = 8 ins.
Initial diameter = .875 in.
Final “ = .472 in.
Equivalent elongation = 2.33 ins.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
1000	0	11,000	.00439	17,000	.00757
2000	.00049	12,000	.00488	18,000	.00822
4000	.00141	13,000	.00536	19,000	.00929
6000	.00222	14,000	.00589	31,500	B.W.
8000	.00309	15,000	.00644		
10,000	.00392	16,000	.00699		

Ans. Load in lbs./ sq. in. at E. L. = 16,620; at Y.P. = 31,590; at fracture = 52,360; red. of area = 70.9%; equiv. elong. = 29.1%; E = 31,790,000 lbs./sq. in.

(*p*) *Tensile test.* Distance between gauge-points = 8 ins.
Diameter = .409 in.

	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
Copper	100	.001	400	.00218	600	.00299
	200	.00138	500	.00258	700	.00340
	300	.00178				

Ans. E = 15,220,000 lbs./sq. in.

(*q*) *Tensile test.* Distance between gauge-points = 8 ins.
Diameter = .369 in.

	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
Brass	100	.001	400	.00219	600	.00299
	200	.00143	500	.00258	700	.00341
	300	.00182				

Ans. E = 18,617,000 lbs./sq. in.

(*r*) *Tensile test.* Distance between gauge-points = 10 ins.
Initial diameter: external = 1.28 in., internal = 1.065 in.
Final “ “ = 1.205 in., “ = 1.03 in.
Equivalent elongation = .89 in. for 8 ins.
Pipe under internal pressure of 1000 lbs./sq. in.

Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.	Load in Lbs.	Ext. in Ins.
0	0	5500	.00501	9000	.00801
100	.00030	6000	.00544	9500	.00845
1000	.00107	6500	.00587	10,000	.00897
2000	.00193	7000	.00629	10,500	.00950
3000	.00284	7500	.00670	11,000	.01170
4000	.00371	8000	.00712	18,560	B.W.
5000	.00459	8500	.00755		

Ans. Load in lbs./sq. in. at E.L. = 29,000; at Y.P. = 33,200; at fracture = 52,300; red. of area = 23%; equiv. elong. = 11.1%; E = 29,600,000 lbs./sq. in.

(*s*) *Compression test.* Distance between gauge-points = 1.25 in.;
1 division = .1/250,000 in.
Section = 1 in. × .5 in.

Load in Lbs.	Comp. in Ins.	Load in Lbs.	Comp. in Ins.	Load in Lbs.	Comp. in Ins.
1000	.00362	6000	.00570	10,000	.00743
2000	.00402	8000	.00665	53,100	B.W.
4000	.00485				

Ans. Load in lbs./sq. in. at fracture = 106,200 lbs.; $E = 14,764,000$ lbs./sq. in.

(*t*) *Compression test.* Distance between gauge-points = 1.25 ins.
1 division = 1/250,000 in.
Section = 1 in. × ½ in.

Load in Lbs.	Comp. in Ins.	Load in Lbs.	Comp. in Ins.	Load in Lbs.	Comp. in Ins.
1000	.00288	6,000	.00398	12,000	.00528
2000	.00307	8,000	.00439	14,000	.00569
4000	.00351	10,000	.00485	43,400	B.W.

Ans. Load in lbs./sq. in. at fracture = 86,400; $E = 28,646,000$ lbs./sq. in.

(*u*) *Compression test.* Distance between gauge-points = 1.25 ins.
1 division = 1/250,000 in.
Section = 1 in. × ½ in.

Load in Lbs.	Comp. in Ins.	Load in Lbs.	Comp. in Ins.	Load in Lbs.	Comp. in Ins.
1000	.00425	6,000	.00532	12,000	.00655
2000	.00447	8,000	.00573	14,000	.00698
4000	.00490	10,000	.00613	68,000	B.W.

Ans. Load in lbs./sq. in. at fracture = 136,000; $E = 29,761,000$ lbs./sq. in.

(*v*) *Compression test.* Distance between gauge-points = 72 ins.
Section = 3.03 in. × 3.02 ins.

Load in Lbs.	Comp. in Ins.	Load in Lbs.	Comp. in Ins.	Load in Lbs.	Comp. in Ins.
1000	0	7,000	.021	30,000	.081
2000	.004	8,000	.024	35,000	.115
3000	.008	9,000	.028	40,000	.132
4000	.011	10,000	.031	45,000	.148
5000	.015	15,000	.049	52,000	B.W.
6000	.018	20,000	.065		

Ans. B.W. = 5683 lbs./sq. in.; $E = 2,336,000$ lbs./sq. in.

32. A steel rod 100 ft. in length has to bear a weight of 4000 lbs. If $E =$ 35,000,000 lbs., and if the safe strain is .0005, determine the sectional area of the rod (1) when the weight of the rod is neglected; (2) when the weight of the rod is taken into account. Also in the former case, determine the work done in stretching the rod $\frac{1}{10}$ in.. $\frac{2}{10}$ in., $\frac{3}{10}$ in., . . . $\frac{6}{10}$ in., successively.

Ans. $\frac{8}{35}$ sq. in.; $\frac{576}{2471}$ sq. in.; $33\frac{1}{3}$, $133\frac{1}{3}$, 300, . . . 1200 in.-lbs.

33. A line of rails is 10 miles in length when the temperature is at 32° F.

Determine the length when the temperature is at 105½ F., and the work stored up in the rails per square inch of section, E being 20,000,000 lbs. $\alpha=.0016$ per 180° F. *Ans.* 10.00653 miles; 338405.76 ft.-lbs.

34. A wrought-iron bar 25 ft. in length and 1 sq. in. in sectional area stretches .0001745 ft. for each increase of 1° F. in the temperature. If $E=29,000,000$ lbs., determine the work done by an increase of 20° F.

How may this property of extension under heat be utilized in straightening walls that have fallen out of plumb? *Ans.* 7.064 ft.-lbs.

35. Three bars, each 10 ft. long, one of brass, one of copper, and one of mild steel, have sectional areas of ⅝, ¾, and 1½ sq. ins. respectively. Compare the forces required to stretch the bars by the same amount, and also compare the amounts of work done, E in tons per square inch being 4800 for brass, 8000 for copper, and 12,000 for steel. *Ans.* 1:2:6.

36. A load P, gradually applied, induces an intensity of stress f in a bar of length l. If the limit of elasticity is not exceeded, show that the same load falling a height h before extending the bar induces the stress

$$f\left(1+\sqrt{1+\frac{2Eh}{fl}}\right),$$

E being Young's modulus.

37. A ½-in. copper bar and a ¾-in. mild-steel bar of the same length are to be stretched the same amount. Compare the forces to do this and also the amounts of work expended in each case, E per square inch being 8500 tons for the copper and 15,000 tons for the steel. *Ans.* 34:135.

38. A steel bar of 3 sq. ins. sectional area stores up 50 ft.-lbs. of energy when subjected to a direct pull of 2½ tons/sq. in. Find the length of the bar E being 15,000 tons/sq. in. *Ans.* 40 ft.

39. Two vertical bars, the one of brass, the other of steel, are fixed at their upper ends and carry at their lower ends a rigid cross-bar which supports a weight at its middle point. When the cross-bar is horizontal the stress induced in each of the vertical bars is 1500 lbs./sq. in. Find the ratio of the initial lengths of the bars, the E being 4500 tons/sq. in. for brass and 13,500 tons/sq. in. for steel. *Ans.* 5 to 6.

40. Find the greatest length of an iron suspension rod which will carry its own weight. the stress being limited to 4 tons per square inch. What will be the extension under this load, E being 12,500 tons? *Ans.* 2688 ft.; .860 ft.

41. An iron bar 20 ft. long and 2 ins. in diameter is stretched $\frac{1}{20}$ of an inch by a load of 7 tons applied along the axis. Find the intensity of stress on a cross-section, and the coefficient of elasticity of the material.

Ans. Stress $=2.23$ tons; $E=10,700$ tons/sq. in.

42. The length of a cast-iron pillar is diminished from 20 ft. to 19.97 ft. under a given load. Find the strain and the compressive unit stress, E being 17,000,000 lbs. *Ans.* .0015; 25,500 lbs. per square inch.

43. A rectangular timber strut 24 sq. in. in sectional area and 6 ft. in length is subjected to a compression of 14,400 lbs. Determine the diminution of the length, E being 1,200,000 lbs. *Ans.* .003 ft.

44. Find the height from which a weight of 200 lbs. may be dropped so that the maximum admissible stress produced in a bar of 1 sq. in. section and 5 ft. long may not exceed 20,000 lbs. per square inch., the coefficient of elasticity being 27,000,000 lbs. *Ans.* $\frac{5}{27}$ ft., or, more accurately, $\frac{49}{270}$ ft.

45. A bar of steel 4″×1″ is rigidly attached at each end to a bar of brass 4″×$\frac{3}{8}$″; the combined bar is then subjected to a load of 20 tons. Find the load taken by each bar. E for steel $=13{,}000$ tons per square inch; for brass $=4000$ tons per square inch.

Ans. 17.93 tons on steel; 2.07 tons on brass.

46. An hydraulic hoist-rod 50 ft. in length and 1 in. in diameter is attached to a plunger 4 in. in diameter, upon which the pressure is 800 lbs. per square inch. Determine the altered length of the rod, E being 30,000,000 lbs.

Ans. .0213 ft.

47. Work equivalent to 50 ft.-lbs. is done upon a bar of constant sectional area, and produces in it a uniform tensile stress of 10,000 lbs. per square inch. Find the cubic content of the bar, E being 30,000,000. *Ans.* 360 cu. in.

48. A bar 2 sq. ins. in area and 11 ft. long is suspended vertically and has a collar at the bottom end. It is provided with a sliding weight of 0.5 ton. Find the height from which the weight must be dropped in order to just give a permanent set to the bar, the elastic limit of the material being 14 tons per square inch. Young's modulus $=12{,}000$ tons per square inch.

Ans. .3465 ft.

49. What would be the resilience of a steel tie-bar 1 in. in diameter and 48 ins. in length if it became permanently stretched under a load exceeding 10 tons, E being 16,000 tons per square inch? *Ans.* 191 in.-ton.

50. The dead load upon a short hollow cast-iron pillar with a sectional area of 20 sq. ins. is 50 tons (of 2000 lbs.). If the strain in the metal is not to exceed .0015, find the greatest live load to which the pillar might be subjected, E being 17,000,000 lbs. *Ans.* 205,000 lbs.

51. A steel suspension rod 30 ft. in length and $\frac{1}{2}$ sq. in. in sectional area carries 3500 lbs. of the roadway and 3000 lbs. of the live load. Determine the *gross* load and also the extension of the rod, E being 35,000,000 lbs.

Ans. $\frac{57}{3500}$ ft.

52. A steel rod 10 ft. in length and $\frac{1}{2}$ sq. in. in sectional area is strained to the proof by a tension of 25,000 lbs. Find the resilience of the rod, E being 35,000,000 lbs. *Ans.* $178\frac{4}{7}$ ft.-lbs.

53. The resilience of an iron bar 1 sq. in. in section and 20 ft. long is 300 ft.-lbs. What would be the resilience if for 19 ft. of its length it was composed of iron 2 sq. in. in section, the remaining foot being the same size as before?

Ans. 157.5 ft.-lbs.

54. Determine the shortest length of a metal bar a sq. in. in sectional area that will safely resist the shock of a weight of W lbs. falling a distance of h ft. Apply the result to the case of a steel bar 1 sq. in. in sectional area, the weight being 50 lbs., the distance 16 ft., the proof-strain $\frac{1}{700}$, and $E=35{,}000{,}000$ lbs.

Ans. $\frac{2EWh}{af^2-2Wf}$, f being the safe unit stress; $\frac{11200}{499}$ ft.

55. A pitch-pine pile 14 in. square is 20 ft. above ground, and is being

driven by a falling weight of 112 lbs. If $E=1{,}500{,}000$ lbs., find the fall so that the inch-stress at the head of the pile may be less than 800 lbs.

Supposing that the pile sinks 2 in. into the ground, by how much would it be safe to increase the fall? *Ans.* 7.456 ft.; 116.5 ft.

56. An $8'' \times 8''$ pile 18 ft. above ground is driven by a falling weight of 100 lbs. Determine (*a*) the strain; (*b*) the amount of the compression; (*c*) the distance through which the weight falls so that the stress per square inch in the pile may not exceed 800 lbs. per square inch. Take $E=1{,}600{,}000$ lbs.

If the pile sinks 4 in. into the ground, by how much would it be safe to increase the fall?

57. A rod 1 sq. in. in section and 5 ft. long carries a weight of 200 lbs. which drops through 2 in. before commencing to stretch the rod. Assuming all the energy of the blow to be stored up in the rod and that the limits of elasticity are not exceeded, estimate the intensity of stress induced in the rod. $E=30 \times 10^6$ lbs. per square inch.

58. A chain l ft. in length and a sq. in. in sectional area has one end securely anchored, and suddenly checks a weight of W lbs. attached to the outer end, and moving with a velocity of V ft. per second away from the anchorage. Find the greatest pull upon the chain. *Ans.* $\text{Pull} = V\sqrt{\frac{aEW}{lg}}$.

59. Apply this result to the case of a wagon weighing 4 tons and worked from a stationary engine by a rope 3 sq. in. in sectional area. The wagon is running down an incline at the rate of 4 miles an hour and, after 600 ft. of rope have been paid out, is suddenly checked by the stoppage or reversal of the engine. $E=15{,}000{,}000$ lbs. *Ans.* 26,884 lbs.

60. A chain l ft. in length and a sq. in. in sectional area has one end attached to a weight of W lbs. at rest, and at the other end is a weight of nW lbs. moving with a velocity of V ft. per second and away from the first. Find the greatest pull on the chain. *Ans.* $\text{Pull} = V\sqrt{\frac{aEWn}{lg(n+1)}}$.

61. A dead weight of 10 tons is to act as a drag upon a ship to which it is attached by a wire rope 150 ft. in length and having an effective sectional area of 8 sq. in. If the velocity of the floating ship is 20 ft. per second, and if its inertia is equivalent to a mass of 390 tons, find the greatest pull on the chain. $E=15{,}000{,}000$ lbs. *Ans.* 208 tons.

62. A coal-wagon weighing 3200 lbs., when running down an incline at a rate of 6 miles an hour, is suddenly checked by the stoppage of the engine after 500 ft. of cable have been paid out. Find the maximum pull on the cable, its sectional area being 3 sq. in. and its $E=15{,}000{,}000$ lbs. per square inch. *Ans.* 26,400 lbs.

63. A square steel bar 10 ft. long has one end fixed; a *sudden* pull of 40,000 lbs. is exerted at the other end. Find the sectional area of the bar consistent with the condition that the strain is not to exceed $\frac{1}{750}$. $E=30{,}000{,}000$ lbs. Find the resilience of the bar. *Ans.* 2 sq. in.; $533\frac{1}{3}$ ft.-lbs.

64. How much work is done in subjecting a cube of 125 cu. in. of iron to a tensile stress of 8000 lbs. per square inch? $E=30,000,000$ lbs.

Ans. $11\frac{1}{9}$ ft.-lbs.

65. A signal-wire 2000 ft. in length and $\frac{1}{8}$ in. in diameter is subjected to a steady stress of 300 lbs. The lever is suddenly pulled back, and the corresponding end of the wire moves through a distance of 4 in. Determine the instantaneous increase of stress. $E=25,000,000$ lbs. *Ans.* $51\frac{103}{672}$ lbs.

66. If the total back-weight is 350 lbs., what is the range of the signal end of the wire? *Ans.* $\frac{124}{16500}$ ft.

67. A steel rod of length L and sectional area A has its upper end fixed and hangs vertically. The rod is tested by means of a ring weighing 60 lbs. which slides along the rod and is checked by a collar screwed to the lower end. A scale is marked upon the rod with the zero at the fixed end. If the strain in the steel is not to exceed $\frac{1}{700}$, what is the reading from which the weight is to be dropped? $E=35,000,000$ lbs.

Ans. Distance from point of suspension $=(\frac{701}{700}-\frac{25}{42}A)L$.

68. A bar 1 sq. in. sectional area and 32 in. long is subjected to a tensile pull of 10 tons. Calculate the work stored up in the bar. $E=30,000,000$ lbs. per sq. inch. *Ans.* $\frac{16}{9}$ ft.-lbs.

69. A load of 1000 lbs. falls 1 in. before commencing to stretch a suspending rod by which it is carried. If the sectional area of the rod is 2 sq. in., length 100 in., and $E=30,000,000$ lbs., find the stress produced.

If the rod carries a load of 5000 lbs., and an additional load of 2000 lbs is suddenly applied, what is the stress produced?

Ans. $17,827\frac{1}{2}$ lbs.; 4500 lbs./sq. in.

70. Steam at a pressure of 50 lbs. per square inch is suddenly admitted upon a piston 32 in. in diameter. The steel piston-rod is 48 in. in length and 2 in. in diameter, E being 35,000,000 lbs. Find the work done upon the rod.

What should be the pressure of admission to strain the rod to a proof of .001? *Ans.* 117.69 ft.-lbs.; $68\frac{23}{64}$ lbs. per square inch.

71. A boulder-grappler is raised and lowered by a wire rope 1 in. in diameter, hanging in double sheaves. On one occasion a length of 150 ft. of rope was in operation, the distance from the winch to the upper block being 30 ft. The grappler laid hold of a boulder weighing 20,000 lbs. What was the extension of the rope, E being 15,000,000 lbs.?

The boulder suddenly slipped and fell a distance of 6 in. before it was again held. Find the maximum stress upon the rope.

What weight of boulder may be lifted if the proof-stress in the rope is not to exceed 25,000 lbs. per square inch of *gross* sectional area?

Ans. $\frac{1}{1500}$ ft.; $50,452\frac{1}{2}$ lbs./sq. in.; $78,571\frac{3}{7}$ lbs.

72. A steel bar stretches $\frac{7}{110}$ of its original length under a stress of 20,000 lbs. per square inch. Find the change of volume and the work done per cubic inch. *Ans.* $\frac{1}{3000}$; $\frac{5}{9}$ ft.-lb. per cubic inch.

73. The steady thrust or pull upon a prismatic bar is suddenly reversed. Show that its effect is trebled.

74. A shock of N ft.-lbs. is safely borne by a bar l ft. in length and a sq. in.

in sectional area. Determine the increased shock which the bar will bear when the sectional area of the last mth of its length is increased to ra.

Ans. $N\left(1-\frac{1}{m}+\frac{1}{rm}\right)$.

75. The pull on one of the tension-bars of a lattice girder fluctuates from 12.8 tons to 4 tons. If 24 tons is the statical breaking strength of the metal, 15 tons the primitive strength, determine the sectional area of the bar, 3 being a factor of safety. *Ans.* 2.15 sq. in. (Launhardt); 1.87 sq. in. (Unwin).

76. The stress in a diagonal of a steel bowstring girder fluctuates from a tension of 15.15 tons to a compression of 7.65 tons. If the primitive strength of the metal is 24 tons and the vibration strength 12 tons, find the proper sectional area of the diagonal, 3 being a factor of safety.

Ans. 2.53 sq. in. (Weyrauch); 1.7 sq. in. (Unwin); 40 tons per square inch being statical strength.

77. A member of a truss is subjected to tensile stresses varying from a maximum of 150,000 lbs. to a minimum of 50,000 lbs. Find the proper sectional area of the member, 1st, if of steel, 2d, if of wrought iron, the factor of safety being 3. *Ans.* 7.523 sq. in.; 17.5 sq. in.

78. The diagonal of a truss is subjected to stresses varying from a maximum of 120,000 lbs. in tension to a minimum of 80,000 lbs. in compression. Find the proper sectional area of the diagonal, 1st, if of steel, 2d, if of wrought iron, 3 being the factor of safety. *Ans.* 8 sq. in.; 12.41 sq. in.

79. A steel diagonal is subjected to stresses which vary from a maximum compression of 10,000 lbs. to a maximum tension of 10,000 lbs. Find its sectional area, taking 3 as the factor of safety and $u=2s=60{,}000$ lbs.

Ans. 1 sq. in.

80. A wrought-iron screw-shaft is driven by a pair of cranks set at right angles. Neglecting the obliquity of the connecting-rods, and assuming that the pull on the crank-pin is constant, compare the coefficients of strength (a' and t) to be used in calculating the diameter of the shaft. How is the result affected by the stopping of the engine? *Ans.* $a'=.904$; $a'=\frac{3}{5}t$.

81. Show that the change of a unit of volume of a solid body under a longitudinal stress is $\lambda\left(1-\frac{2}{m}\right)$, which becomes $\frac{\lambda}{2}$ if $m=4$, as in metals, and *nil* when $m=2$, as in India-rubber.

82. Taking $f=E\lambda$ as the ordinary analytical expression of Hooke's Law, find the value of the modulus of elasticity when calculated (1) from the *actual* stress and the elongation per unit of *initial* length; (2) from the actual stress and the elongation per unit of stretched length.

Ans. (1) $E+f$; (2) $E+f(1+\lambda)^2=E+f(1+2\lambda)$, if λ is small.

83. During the plastic deformation of a prismatic bar, show that the change in sectional area is proportional to the deformation calculated on the altered length of the bar.

84. A prismatic bar of volume V changes in length from L to $L\pm x$ under the "fluid pressure" p. Find the corresponding work.

Ans. $pV\log_e(L\pm x)$.

85. A pyramid weighing 125 lbs. per cu. ft. has a height of 60 ft. and rests upon a square base of 25 sq. feet. Find the amount and work of compression, E being the coefficient of elasticity.

86. In an indicator the area of a piston is $\frac{1}{2}$ sq. in., the inertia of the moving parts attached to the piston is equivalent to a weight of .33 lbs. moving with the same velocity, and the pencil moves 4 times as fast as the piston, 1 inch on the diagram corresponding to a pressure of 80 lbs. on the piston. Find the time of a complete oscillation. *Ans.* .01453 sec.

87. A right cone of weight W and height h rests upon its base of radius r. Find the amount and work of the compression.

$$\textit{Ans.}\ \text{Comp.}=\frac{Wh}{2\pi Er^2};\ \text{work}=\frac{1}{5}\frac{W^2h}{\pi Er^2}.$$

88. An elastic trapezoidal lamina $ABCD$, of natural length l and thickness unity, has its upper edge AB $(2a)$ fixed and hangs vertically. If a weight W is suspended from the lower edge CD $(2b)$, show that, neglecting the weight of the lamina, the consequent elongation $=\frac{1}{2}\frac{W}{E}\frac{l}{a-b}\log_e\frac{a}{b}$. If an additional weight is placed upon W and then suddenly removed, show that the oscillation set up is isochronous and that the time of a complete oscillation

$$=\pi\left\{\frac{Wl\log_e\frac{a}{b}}{2gE(a-b)}\right\}^{\frac{1}{2}}.$$ Examine the case when $a=b$.

$$\textit{Ans.}\ \text{Ext.}=\frac{1}{2}\frac{Wl}{aE};\ \text{time of oscillation}=\pi\sqrt{\frac{Wl}{2aEg}}.$$

89. If the specific weight of the lamina in the preceding question is w, find how much it will stretch under its own weight, and also the work of extension. Determine the result when $a=b$.

$$\textit{Ans.}\ \frac{1}{2E}\frac{wb^2l^2}{(a-b)^2}\log_e\frac{b}{a}+\frac{wl^2}{4E}\frac{a+b}{a-b};\ \frac{wl^2}{2E}.$$

$$\text{Work}=\frac{w^2l^3}{4E(a-b)^3}\left\{\frac{a^4-b^4}{4}-b^2(a^2-b^2)+b^4\log_e\frac{a}{b}\right\};\ \frac{w^2al^3}{3E}.$$

90. A hollow tower, of height h, is in the form of a solid of revolution about a vertical axis. The hollow portion is a right cylinder of radius R, and the radius of the base of the tower is a. If the specific weight of the tower is w, find the curve of the generating line so that the stress at every point of the tower may be f. Also find the load on the top of the tower.

Ans. $y^2-R^2=(a^2R^2)e^{-\frac{wx}{f}}$; $\pi f(b^2-R^2)$, b being the radius of the top of the tower.

91. An elastic lamina in the form of an isosceles triangle ABC has its base AB $(=2a)$ fixed and hangs vertically. If its weight is W, find its elongation. Take coefficient of elasticity $=E$, thickness of lamina $=$ *unity*, and $L=$ the distance of C from AB.

$$\textit{Ans.}\ \frac{WL}{4aE}.$$

92. A weight of 1 ton depresses its supports .01 ft. Neglecting the weight of the supports, find the time of a complete oscillation. *Ans.* .1111 second.

93. A weight of 10 lbs. at the end of a spiral spring stretches it 3 ins. The spring is then stretched an additional 3 in. and suddenly released. Find the time of a complete oscillation. *Ans.* .555 second.

94. The weight of the piston of an indicator and its attached parts is equivalent to 1 lb. on the piston. The pencil moves six times as fast as the piston. The area of the piston is .5 sq. in. Find the period of oscillation with a 100-lb. spring. *Ans.* .00586 second.

95. A body symmetrical with respect to a vertical plane is slightly displaced vertically from its position of stable equilibrium. Find the period of an oscillation.

Ans. $2\pi\sqrt{\frac{V}{Ag}}$, V being the volume of the water displaced and A the sectional area of the body at the water-line.

96. If the body in the preceding example receives a slight angular displacement, what will be the period of an oscillation?

Ans. $2\pi\sqrt{\frac{I}{Wzg}}$, I being the moment of inertia, W the weight, and z the distance between the metacentre and the C. of G.

97. A uniform circular plate weighing 4 lbs. and 1 ft. in diameter is hung in a horizontal plane by three parallel cords from the ceiling, and when set into small torsional oscillation is found to have a period of 3 seconds. A body weighing 6 lbs. is laid on it, and the period is then found to be 5 seconds. Find the moment of inertia of the body.

98. A revolving weight of w lbs. is carried on elastic bearings which yield at the rate of 1 ft. to b lbs. Find the extent of the forced vibration when the number of revolutions is n per second; find also the value of n so that free vibration might be set up.

Ans. $\frac{w}{g}\frac{r}{b}(2\pi n)^2$; period (neglecting wt. of bearing) $=2\pi\sqrt{\frac{w}{gb}}=\frac{1}{2}$, and

therefore $$n=\frac{1}{2\pi}\sqrt{\frac{gb}{w}}.$$

99. A U tube a ft. in length contains water. If the water is slightly displaced from its mean position of equilibrium, show then the periodic time is $\frac{\pi}{4}\sqrt{a}$, if $g=32$ ft.

100. A weight of 5 lbs. is supported by a spring. The stiffness of the spring is such that putting on or taking off a weight of 1 lb. produces a downward or upward motion of 0.04 ft. What is the time of a complete oscilliation, neglecting the mass of the spring?

101. A weight W is suspended by a spring, which it stretches. The weight

is further depressed 1 ft., when it is suddenly released and allowed to oscillate. Find its velocity at a distance x from the position of equilibrium.

Ans. $\sqrt{E(1-x^2)\frac{g}{W}}$.

102. If a spring deflects .001 ft. under a load of 1 lb., what will be the period of oscillation of a weight of 14 lbs. upon the spring?

103. A mass of fifty pounds is attached to the free end of a spring, made of wire $\frac{3}{10}$ of an inch diameter and having ten free coils. The mean diameter of the helix is 3 in. Find the period of vibration if the spring is extended axially and is then let go.

104. A vertical elastic rod of natural length L and of which the mass may be neglected is fixed at its upper end and carries a weight W_1 at the lower end. A weight W_2 falls from a height h upon W_1. Find the velocity and extension of the rod at any time t.

$$Ans.\ v^2=\frac{g}{W_1+W_2}\left(2W_2h-\frac{EA}{L}x^2\right)=\left(\frac{dx}{dt}\right)^2,$$

x being measured from mean position of (W_1+W_2).

105. Determine the functions F and f in Art. 4 when P_1 is zero, and also when the rod is perfectly free; i.e., when $P_0=0$ and $P_1=0$.

106. The centre of mass of a body (supposed to be small) weighing 25 lbs. is 20 in. from the fulcrum of a massless horizontal lever. It is supported, also, by a vertical massless spring acting on the lever at a distance of 5 in. from the fulcrum. The stiffness of the spring is such that a pull of 1 lb. elongates it 0.001 ft. What is the natural time of vibration of the system?

If the point of support of the spring gets a vertical simple harmonic motion of amplitude 0.1 ft. with a frequency of three complete oscillations per second, what is the nature of the motion of the weight, supposing that the natural vibrations are stilled?

107. A metal rod $\frac{1}{4}$ sq. in. in area and 5 ft. long hangs vertically with its upper end fixed and carries a weight of 18 lbs. at the lower end. On striking the rod it emitted a musical note of 264 vibrations per second (middle C of piano-forte). Find the coefficient of elasticity, the weight of the rod being neglected. *Ans.* 30,979,160 lbs.

108. What should be the thickness of the plates of a cylindrical boiler 6 ft. in diameter and worked to a pressure of 50 lbs. per square inch, in order that the working tensile stress may not exceed 1.67 tons per square inch of gross section? *Ans.* .42 in.

109. A cylindrical boiler with hemispherical ends is 4 ft. in diameter and 22 ft. in length. Determine the thickness of the plates for a steam-pressure of 4 atmospheres, the working strength of the material being 4000 lbs./sq. in.

110. Find the thickness of plates required for a boiler which is to work at a pressure of 160 lbs./sq. in., the diameter of the shell being 8 ft., the efficiency of the riveted joint 89%, and the working stress in the metal, 5 tons/sq. in. *Ans.* .77 in.

111. A 7-ft. boiler of $\frac{7}{16}$-in. plates was burst at a longitudinal double-riveted joint by a pressure of 310 lbs. per square inch. Find the coefficient of ultimate strength. *Ans.* 29,760 lbs.

112. A 50-in. cylindrical boiler of $\frac{5}{16}$-in. plates is made of wrought iron whose safe coefficient of strength is 4000 lbs. per square inch. Find the working pressure. If $E=30{,}000{,}000$ lbs., also find the thickness and deflection of the end plates. *Ans.* 50 lbs./sq. in.

113. Determine the thickness of the plates used in the construction of a boiler 10 ft. in diameter working under a pressure of 120 lbs. per square inch, taking the efficiency of the joints to be 70 per cent. and the safe stress at 10,000 lbs. per square inch. *Ans.* 1.03 in.

114. Find the thickness of plates required for a boiler-shell to work at a pressure of 200 lbs. per square inch, the diameter of the shell being 5 ft. 6 in.; the allowable stress on the plate may be taken as 12,000 lbs. per square inch. *Ans.* .55 in.

115. The plates of a cylindrical boiler 5 ft. in diameter are $\frac{1}{2}$ in. thick. Find to what pressure the boiler may be worked so that the tensile stress in the plates may not exceed $1\frac{1}{2}$ tons per square inch of gross section.
Ans. 50 lbs./sq. in.

116. Cylindrical boiler of mild steel 5 ft. 6 in. in diameter at elastic limit; pressure 300 lbs. per square inch. What is the thickness? Joints supposed to be of 60 per cent of strength of unhurt plate. Replace this boiler with tubes 5 in. diameter of same length. How many tubes are needed to make up the same volume? What will be their thickness (no seams)? Their weight? *Ans.* .47 in.; 174; .021 in.; 193 lbs. per ft. length.

117. The pressure in a boiler of 1.2 m. diameter is 7 atmospheres. What must be the thickness of the metal if the safe working stress is $3k/\text{cm.}^2$?
Ans. 12.4 mm.

118. What must be the thickness of the metal in a cast-iron cylinder, of 400 mm. diameter, which has to bear an internal pressure of 7 atmospheres, the safe working strength of the metal being 1.5 k/cm.? *Ans.* 8.26 mm.

119. A copper pipe is 4 in. diameter and $\frac{3}{8}$ in. thick. What is the working pressure? Take $f=2000$ lbs. per square inch. *Ans.* 375 lbs./sq. in.

120. What is the collapsing pressure of a flue 10 ft. long, 36 in. in diameter, and composed of $\frac{1}{2}$-in. plates? Also of a flue 30 ft. long, 48 in. in diameter, and $\frac{7}{16}$-in. thick? *Ans.* 490.84 lbs.; 91.59 lbs.

121. Determine the thickness of a 2-in. locomotive fire-tube to support an external pressure of 5 atmospheres.

122. A copper steam-pipe is 4 in. in diameter and $\frac{1}{4}$ in. thick. Find the working pressure, the safe coefficient of strength for copper being 1000 lbs. per square inch. *Ans.* 125 lbs. per square inch.

123. Find the proper head of water for a pipe of 3 ft. diameter and 1 in. thickness. Also find the corresponding pressure in pounds per square inch.

Determine what value of the working stress will cause the ordinary rule for thin cylinders to give the same answer.

Ans. 316 ft.; 136.7 lbs./sq. in.; 2460 lbs./sq. in.

124. A 10-in. cast-iron water-pipe is subjected to a pressure of 250 lbs. per square inch. Find its thickness, the coefficient of working strength being 2000 lbs. per square inch. *Ans.* $1\frac{1}{3}$ in.

125. A cast-iron main is 36 in. internal diameter and 1 in. thick, the safe tensile stress being $1\frac{1}{2}$ tons per square inch. What is the greatest head of water to which it should be subjected?

If the pipes are of wrought iron, what should be the thickness of the metal, the safe strength per square inch being 5 tons and the strength of the longitudinal seams being 60 per cent that of the solid plate?

126. Given a wrought-iron pipe 2 ft. diameter, $\frac{1}{2}$ in. thick; its working stress is 5 tons to the square inch, but strength of plate is diminished 30 per cent because of riveted joint. What is the working pressure?

Ans. 326 lbs./sq. in.

127. A cylindrical boiler, 12 ft. diameter, is constructed of $\frac{29}{32}$-in. steel plate. The test pressure applied is 245 lbs. per square inch. Find the stress produced in the plate, and hence deduce the stress in the plate between the rivet-holes, the sectional area being there reduced to .77 of the solid.

Ans. 19,500 lbs./sq. in.; 25,300 lbs./sq. in.

128. Sixty tubes of wrought iron, 4 in. inside diameter, 10 ft. long, $\frac{1}{4}$ in. thick. Find volume, weight, internal area in square inches, and working pressure, if working stress = 10,000 lbs. per square inch. Neglect the ends. How many tubes 22 in. diameter, 10 ft. long, will have the same volume? And find the thickness suitable for the working pressure. Find also the area and weight.

Ans. 52.36 cu. ft.; 6729 lbs. weight; 12.54 sq. in.; 1250 lbs./sq. in.; 2; 1.375 in.; 380 sq. in.; 6525 lbs.

129. A vertical cast-iron pipe is 4 in. in internal diameter. The pressure at a certain place is 50 lbs. per square inch. At this place, and at a place 300 feet lower in level, find the proper thickness of the metal if the working stress is taken as 3000 lbs. per square inch. *Ans.* .033 in.; .149 in.

130. A steel spherical shell 36 in. in diameter and $\frac{3}{8}$ in. thick is subjected to an internal fluid pressure of 300 lbs. per square inch. Find its coefficient of strength. *Ans.* 7200 lbs.

131. A thin, hollow, spherical elastic envelope, whose internal radius is R, was subjected to a fluid pressure which caused it to expand gradually until its radius became R_1. Determine the work done.

CHAPTER V.

STRESSES, STRAINS, EARTHWORK, AND RETAINING-WALLS.

1. Internal Stresses.—The application of external forces to a material body will strain or deform it, and the particles of the body will be in a state of mutual stress.

In the following calculations it is assumed

(*a*) That the stresses under consideration are parallel to one and the same plane, viz., the plane of the paper:

(*b*) That the stresses normal to this plane are constant in direction and magnitude:

(*c*) That the thickness of the plane is unity.

Def. The angle between the direction of a given stress and the normal to the plane on which it acts is called the obliquity of the stress.

2. Compound Strain.—(*a*) First consider an indefinitely small rectangular element $OACB$ (Fig. 321) of a strained body, kept in equilibrium by stresses acting as in the figure.

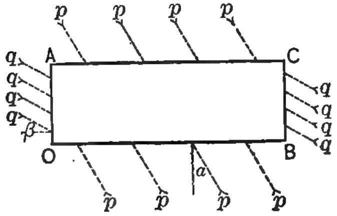


FIG. 321.

p is the intensity of stress on the faces OB, AC, and α its obliquity.

q is the intensity of stress on the faces OA, BC, and β its obliquity.

$OBp \cos \alpha$, the total normal stress on OB, is balanced by $ACp \cos \alpha$, the total normal stress on AC.

$OBp \sin \alpha$, the total shear on OB, is equal in magnitude but opposite in direction to $ACp \sin \alpha$, the total shear on AC.

These two forces, therefore, form a couple of moment

$$OBp \sin \alpha \, OA.$$

Similarly, the total normal stresses on the faces OA, BC balance and the total shears form a couple of moment $OA \cdot q \sin \beta \cdot OB$.

In order that equilibrium may be maintained the two couples must balance.

Therefore $$OBp \sin \alpha \, OA = OAq \sin \beta \, OB,$$

or $$\mathbf{p} \sin \alpha = \mathbf{q} \sin \beta = \mathbf{t}, \text{ suppose.}$$

Hence *at any point of a strained body the intensities of the shears on any two planes at right angles to each other are equal.*

(*b*) Second, let it be required to find the resultant stress upon any plane MN at a point O in a strained body. Consider the equilibrium of an indefinitely small *triangular* element OAB, bounded by a plane AB parallel to MN and two planes OA, OB, at right angles to each other (Fig. 322).

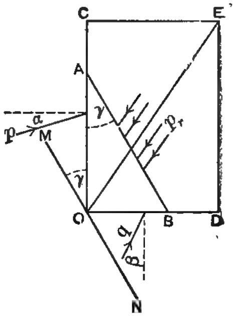


FIG. 322.

Let p be the stress on OB, α its obliquity.

Let q be the stress on OA, β its obliquity.

Let t be the shearing stress on each of the planes OA, OB. Then

$$t = p \sin \alpha = q \sin \beta.$$

p_n, the normal component of p, $= p \cos \alpha$.

q_n, " " " " q, $= q \cos \beta$.

Produce OA and take $OC = p_n OB + tOA =$ the total force on OB in the direction of OA.

Produce OB, and take $OD = q_n OA + tOB =$ the total force on OA in the direction of OB.

Complete the rectangle CD.

OE represents in direction and magnitude the resultant of the two forces OC, OD, and must, therefore, be equal in magnitude and opposite in direction to the total force on AB.

Let p_r be the stress on AB. Then

$$(p_r AB)^2 = OE^2 = OC^2 + OD^2 = (p_n OB + tOA)^2 + (q_n OA + tOB)^2,$$

and $$p_r^2 = (p_n \sin \gamma + t \cos \gamma)^2 + (q_n \cos \gamma + t \sin \gamma)^2$$

$$= p_n^2 \sin^2 \gamma + q_n^2 \cos^2 \gamma + t \sin^2 \gamma (p_n + q_n) + t^2.$$

In the limit *MN* and *AB* coincide, and p_r is the resultant stress upon the plane *MN* at *O* and acts in the direction *OE*.

If $t=0$, then α and β are each $=0$, and there is no shearing stress upon *OA* or upon *OB* at *O*. The stress upon each of these planes at *O* is wholly normal, and such stresses are called *principal stresses*, while the two planes upon which they act are called *planes of principal stress.*

If *OA* and *OB* are planes of principal stress and if p_1 and p_2 are the principal stresses along *OA* and *OB* respectively, then the resultant stress p_r on any plane *MN* is given by

$$p_r^2 = p_1^2 \sin^2 \gamma + p_2^2 \cos^2 \gamma.$$

3. Constant Components of p_r.—Equal Principal Stresses.—Let *OA*, *OB* be the *principal* axes at any point *O* in a strained mass. It is required to determine the resultant stress on any plane *MON* when the two principal stresses are *like and equal.*

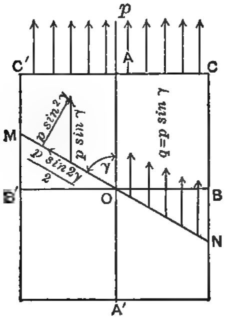


FIG. 323.

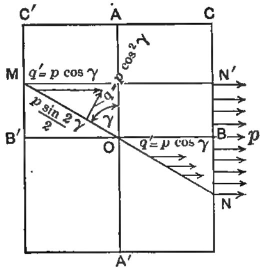


FIG. 324.

Let each principal stress $= p$.

Let the angle $MOA = \gamma$.

Let the point *O* lie within an indefinitely small rectangular element having its sides parallel to *OA* and *OB*.

First. Consider the effect of the stress *p* acting in a direction parallel to *OA*, Fig. 323. (This stress may be due either to a *pull* or a *push.*)

Let q be the corresponding parallel resultant stress developed in the plane MON. Then

$$qMN = pCC' = pBB',$$

and therefore $$q = p \sin \gamma.$$

The shear stress along $OM = q \cos \gamma = p \dfrac{\sin 2\gamma}{2}$.

The normal stress upon $OM = q \sin \gamma = p \sin^2 \gamma$.

Second. Let the stress p act in a direction parallel to OB, Fig. 324, and let q' be the corresponding parallel resultant stress developed in MN. Then

$$q'MN = pNN'.$$

Therefore $$q' = p \cos \gamma.$$

The shear stress along $ON = q' \sin \gamma = p \dfrac{\sin 2\gamma}{2}$.

The normal stress upon $ON = q' \cos \gamma = p \cos^2 \gamma$.

Third. Let the *two like* principal stresses now act simultaneously.

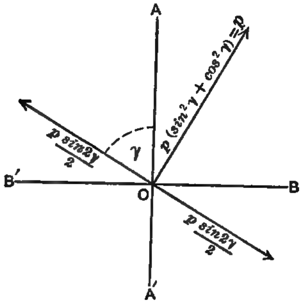

A
$p(\sin^2\gamma + \cos^2\gamma) = p$
$p \frac{\sin 2\gamma}{2}$
γ
B'
O
B
$p \frac{\sin 2\gamma}{2}$
A'

FIG. 325.

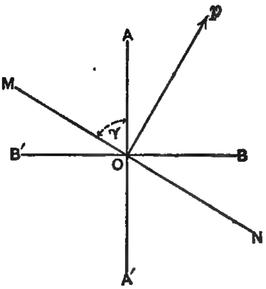


FIG. 326.

The resultant effect is the algebraic sum of the separate effects, and therefore the resultant stress at O on the plane MON is equivalent to

(*a*) two shear stresses each equal to $\dfrac{p \sin 2\gamma}{2}$, but acting in *opposite* directions so that they neutralize each other, and (*b*) two normal stresses, $p \sin^2 \gamma$ and $p \cos^2 \gamma$, acting in the *same* direction so that they are together equal to a normal stress

$$p \sin^2 \gamma + p \cos^2 \gamma = p.$$

Hence, if the two principal stresses at a point in a strained mass are like and equal, the resultant stress on any plane through the point is also a like and equal stress and is normal to the plane, Fig. 326.

Again, let the two principal stresses be equal but *unlike.*

Suppose, for example, that the stress parallel to OA is due to a push, so that it acts in the direction shown by Fig. 327. The resultant effect on the plane MON is now a shear stress $\frac{p \sin 2\gamma}{2}$ acting in the *same* direction as the shear stress due to the stress p in Fig. 324, and a normal stress $p \sin^2 \gamma$ acting in a direction *opposite* to that of the corresponding stress in Fig. 323.

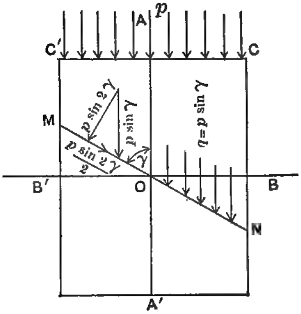


FIG. 327.

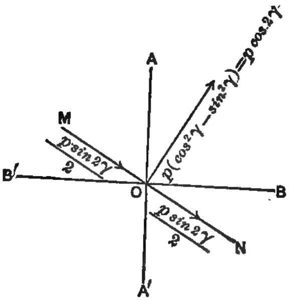


FIG. 328.

Hence, if the two *unlike* but equal principal stresses act simultaneously, they are equivalent to (Fig. 328)

(*a*) a resultant shear stress $\frac{p \sin 2\gamma}{2} + \frac{p \sin 2\gamma}{2} = p \sin 2\gamma$ in the plane MN, and

(*b*) a resultant normal stress $p \cos^2 \gamma - p \sin^2 \gamma = p \cos 2\gamma$. This normal stress will be a push or a pull according as $\gamma >$ or $< 45°$.

Take OK, Fig. 329, in the plane $MN = p \sin 2\gamma$ and OL normal to the plane $= p \cos 2\gamma$.

Complete the rectangle $OKHL$. Then OH is the resultant stress on the plane MN at O, and its value is given by

$$OH=\sqrt{OL^2+OK^2}=\sqrt{p^2\cos^2 2\gamma+p^2\sin^2 2\gamma}=p.$$

Also, if the angle $LOH=\beta$,

$$\tan\beta=\frac{OK}{OL}=\frac{p\sin 2\gamma}{p\cos 2\gamma}=\tan 2\gamma,$$

and therefore $\beta=2\gamma$ or $180^\circ+2\gamma$.

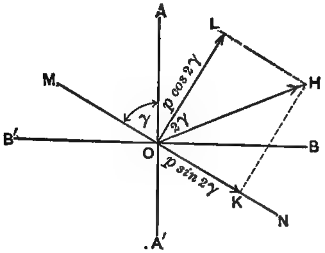


FIG. 329.

Hence, if the two principal stresses at a point in a strained mass are equal but unlike, the resultant stress upon any plane through the point is an equal stress acting in a direction inclined at an angle 2γ *to the normal.*

4. **Unequal Principal Stresses.**—Let the principal stresses at any point be p_1 and p_2. Then

$$p_1=\frac{p_1+p_2}{2}+\frac{p_1-p_2}{2}=m+n, \text{ suppose,}$$

and

$$p_2=\frac{p_1+p_2}{2}-\frac{p_1-p_2}{2}=m-n.$$

The principal stresses may therefore be considered as composed of two *like and equal* stresses $m\left(=\frac{p_1+p_2}{2}\right)$, and of two *unlike and equal* stresses $n\left(=\frac{p_1-p_2}{2}\right)$. The effects of these like and unlike stresses may be obtained separately and their results superposed.

The resultant of the two *like* stresses is a stress m along the normal OE to the plane MN, Figs. 325 and 326, and the resultant of the two *unlike* stresses is a stress n along OF making an angle 2γ (Fig. 330) or $180^\circ+2\gamma$ (Fig. 331) with the normal. Take $OE=m$ and $OF=n$. Complete the parallelogram $OEGF$. The diagonal OG is the resultant stress on MN at O in direction and magnitude.

Its value by Fig. 330 is given by

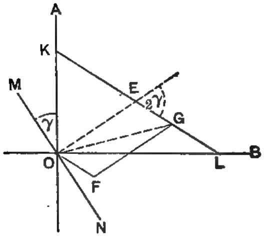


FIG. 330.

$$OG^2=m^2+n^2-2mn\cos(180^\circ-2\gamma)$$

$$=\frac{p_1^2+p_2^2}{2}+\frac{p_1^2-p_2^2}{2}\cos 2\gamma$$

$$=p_1^2\cos^2\gamma+p_2^2\sin^2\gamma.$$

Its value by Fig. 331 is given by

$$OG^2=m^2+n^2-2mn\cos 2\gamma$$

$$=\frac{p_1^2+p_2^2}{2}-\frac{p_1^2-p_2^2}{2}\cos 2\gamma$$

$$=p_1^2\sin^2\gamma+p_2^2\cos^2\gamma,$$

as already obtained in Art. 2.

Thus, in the triangle *OEG*, the angle *OEG* is either 2γ or $180^\circ-2\gamma$.

Let *GE* be produced to meet *OA* and *OB* in *K* and *L*. Then, evidently,

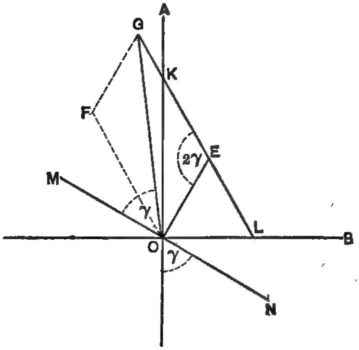


FIG. 331.

$$EK=EO=EL=m,$$

and *E* is the middle point of *KL*.

Therefore, by Fig. 330,

$$GK=GE+EK=+n+m=p_1,$$

$$GL=-GE+EL=-n+m=p_2,$$

and, by Fig. 331,

$$-GK=-GE+EK=-n+m=p_2,$$

and $$GL=+GE+EK=+n+m=p_1.$$

Let ε be the angle between *GE* and the plane *MN*. Then

$GE\cos\varepsilon=\frac{p_1-p_2}{2}\cos\varepsilon$ is the shear stress in *MN* at *O*.

This is greatest when $\cos \varepsilon = 1$, i.e., when $\varepsilon = 0$ or GE is parallel to MN. In this case OE is at right angles to EG, and therefore $2\gamma = 90°$ or $\gamma = 45°$. Hence the shear stress in MN at O is a maximum and equal to $\frac{p_1 - p_2}{2}$ when MN is inclined to the axis of principal stress at an angle of 45°.

5. Planes of Principal Stress.—Ellipse of Stress.—Let MON be a plane of principal stress, i.e., a plane in which there is no shearing stress at O. The resultant OE, Fig. 332, must be normal to MN and AB, and therefore

$$\tan \gamma = \cot COE = \frac{OC}{OD} = \frac{p_n OB + tOA}{q_n OA + tOB} = \frac{p_n \tan \gamma + t}{q_n + t \tan \gamma},$$

from which
$$\frac{2t}{q_n - p_n} = \frac{2 \tan \gamma}{1 - \tan^2 \gamma} = \tan 2\gamma.$$

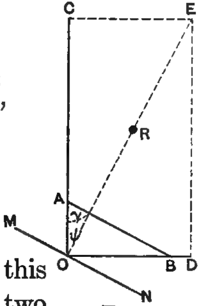


FIG. 332.

Two values of γ, viz., γ and $\gamma + 90°$, satisfy this equation so that at every point O there are two planes, at right angles to each other, upon which the stress is wholly normal, i.e., there are two principal planes.

Again, take the principal planes as the planes of reference.

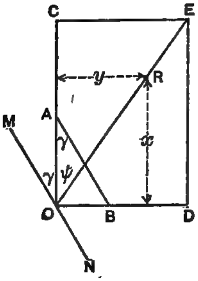


FIG. 333.

Let p_1, p_2 be the principal stresses along OA, OB, Fig. 333, respectively. Consider as before the equilibrium of a triangular element OAB, AB being parallel to a plane MON inclined at an angle γ to OA at O.

Take $OC = p_1 OB$, $OD = p_2 OA$, and complete the rectangle CD. Then $OE = pAB$ is the resultant of OC and OD.

Take $OR = p_r$, and let x, y be the coordinates, with respect to O, of the point R. Then

$$x = p_r \cos \psi = p_r \frac{OC}{OE} = p_r \frac{p_1 OB}{p_r AB} = p_1 \sin \gamma,$$

and
$$y = p_r \sin \psi = p_r \frac{OD}{OE} = p_r \frac{p_2 OA}{p_r AB} = p_2 \cos \gamma.$$

Therefore $\frac{x^2}{p_1^2}+\frac{y^2}{p_2^2}=\sin^2\gamma+\cos^2\gamma=1$, and $\tan\gamma\tan\psi=\frac{p_2}{p_1}$.

Hence the locus of R is an ellipse, called the *ellipse of stress,* with its axes ($2p_1$ and $2p_2$) lying in the planes of principal stress. The stress upon any plane MON at O inclined at γ to the major axis is the semi-diameter of the ellipse drawn in a direction making an angle ψ with the major axis, ψ being given by the last equation.

If, further, the two principal stresses are of equal magnitude but of opposite sign, i.e., if the one is a push and the other a pull, the normal component $\frac{p_1+p_2}{2}$ becomes *nil,* so that there is no normal stress, while the maximum shear stress on each of two planes making an angle of 45° with the axes of principal stress is $\frac{p_1-p_2}{2}=p_1=-p_2$.

This is defined to be a state of *simple shear.* It has also already been proved, Art. 2, that if there is a shear stress on any given plane at a point in a strained solid, there must be an equal shear stress on a plane through the point at right angles to the first.

In simple shear an elementary cube becomes distorted (Fig. 334) without change of volume. One diagonal is lengthened and the other shortened by the same amount, while its angles are changed by a small quantity ϕ to $90°+\phi$ and $90°-\phi$. This small angle ϕ is taken as a measure of the shear strain and is called the *angle of shear.* Hence, by Hooke's law, which applies to shear as to other stresses,

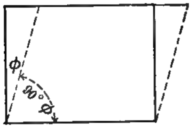


FIG. 334.

$$q\text{, the intensity of shear} \propto \phi$$

$$\text{or}\quad q=G\phi,$$

where G is a coefficient called the *modulus of rigidity.* The value of G is generally determined by torsion experiments and is usually found to be about $\frac{2}{5}E$, E being Young's modulus.

Again,

$$\text{the work stored up by the distortion}=\frac{q}{2}\phi=\frac{q^2}{2G}.$$

6. Conjugate Stresses.—The *obliquity* of a stress is the angle between the direction of the stress and the normal to the plane upon which it acts. This angle will be designated in the following calculations by the symbol θ.

At any point in a strained mass two stresses are said to be *conjugate* when each stress acts upon a plane parallel to the direction of the other.

In Fig. 335 the stress acting upon the plane MON at O is parallel to the plane $M'ON'$, and its obliquity is the angle $M'OE'$, OE' being the normal to MON. If the stress at O upon the plane $M'ON'$ is parallel to MON, its obliquity, viz., the angle MOE between OM and the normal OE to the plane $M'ON'$, is evidently equal to $M'OE'$. *Hence the conjugate stresses at any point in a strained mass have equal obliquities.*

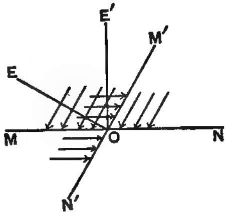


FIG. 335.

Let OA, OB be the planes of principal stress at any point O of a strained mass, and let MON be any other plane inclined at an angle γ to OA, Fig. 336.

Upon the normal to MN take $OE = m = \frac{p_1 + p_2}{2}$, and with E as centre and a radius $= n = \frac{p_1 - p_2}{2}$, describe a circle intersecting OE at P and Q. Draw any line OSR intersecting the circle at S and R. Join ES and ER. Then OR and OS are each the *third* side of a triangle of which the other two sides are the constant components m and n, while the angle ROE is common to the two triangles. Thus OR and OS are conjugate stresses, and the obliquity θ of each is the angle ROE.

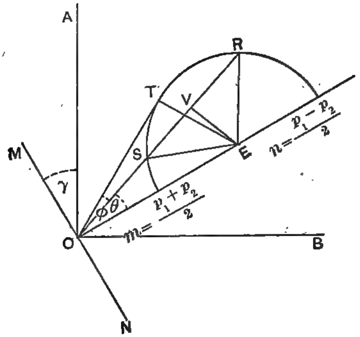


FIG. 336.

Draw the tangent OT and join ET. The angle EOT is the maximum value which the obliquity θ can have, and corresponds to the case in which the two conjugate stresses are equal.

Let ϕ be the maximum value of θ, i.e., the angle TOE. Then

$$\sin\phi=\frac{ET}{OE}=\frac{n}{m}=\frac{\mathrm{p}_1-\mathrm{p}_2}{\mathrm{p}_1+\mathrm{p}_2}.$$

Let $r=OR$, the *greater* of the two conjugate stresses.
Let $s=OS$, " *lesser* " " " " "
Draw EV at right angles to and bisecting SR.
Then

$$\mathrm{r}+\mathrm{s}=OR+OS=2OV=2m\cos\theta=(\mathrm{p}_1+\mathrm{p}_2)\cos\theta,$$

and

$$\mathrm{rs}=OR\cdot OS=OT^2=OE^2-ET^2=m^2-n^2=\mathrm{p}_1\mathrm{p}_2.$$

Again, $EV=m\sin\theta$, and therefore

$$VR=VS=\sqrt{ER^2-EV^2}=\sqrt{n^2-m^2\sin^2\theta}$$
$$=m\sqrt{\frac{n^2}{m^2}-\sin^2\theta}=m\sqrt{\sin^2\phi-\sin^2\theta}$$
$$=m\sqrt{\cos^2\theta-\cos^2\phi}$$

Hence $$r=OV+VR=m(\cos\theta+\sqrt{\cos^2\theta-\cos^2\phi})$$

and $$s=OV-VS=m(\cos\theta-\sqrt{\cos^2\theta-\cos^2\phi}).$$

Therefore $$\frac{\mathrm{r}}{\mathrm{s}}=\frac{\cos\theta+\sqrt{\cos^2\theta-\cos^2\phi}}{\cos\theta-\sqrt{\cos^2\theta-\cos^2\phi}},$$

r being > s.

Putting $\sin\alpha=\frac{\cos\phi}{\cos\theta}$, the last equation may be written

$$\frac{\mathrm{r}}{\mathrm{s}}=\frac{1+\cos\alpha}{1-\cos\alpha}=\cot^2\frac{\alpha}{2}.$$

Ex. 1. *The principal stresses at a point in a strained mass are tensions of* 255 *lbs. and* 171 *lbs. per square inch. Find the resultant stress and its obliquity on a plane through the point inclined at* 27° *to the plane of greatest principal stress.*

$$m=\frac{255+171}{2}=213,$$

$$n=\frac{255-171}{2}=42.$$

Therefore $(\text{resultant stress})^2 = m^2 + n^2 - 2mn \cos(180° - 2\gamma)$

$$= 213^2 + 42^2 + 2.213.42 \cdot \cos 54°,$$

and resultant stress = 240 lbs. sq. inch.

Again, $$\frac{\sin\theta}{\sin 2\gamma} = \frac{n}{240} = \frac{42}{240} = \frac{\sin\theta}{\sin 54°}.$$

Therefore $\sin\theta = .1417$ and $\theta = 8°9'$.

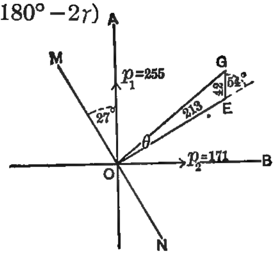


FIG. 337.

Ex. 2. *The principal stresses at a point in a strained solid are a tension of* 300 *lbs./sq. in. and a compression of* 200 *lbs./sq. in. Find (a) the resultant stress on a plane inclined at* 30° *to the axis of greatest principal stress and its obliquity; (b) the plane upon which there is only a shearing stress and the magnitude of the stress.*

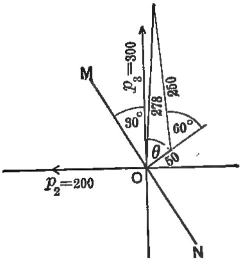


FIG. 338.

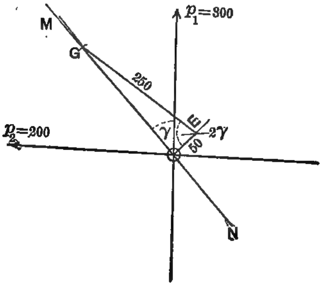


FIG. 339.

$$(a)\quad m = \frac{300 - 200}{2} = 50,$$

$$n = \frac{300 + 200}{2} = 250.$$

Then by Fig. 338 $OG^2 = m^2 + n^2 - 2mn \cos(180° - 60°)$,

$$= 2500 + 62{,}500 + 12{,}500$$

$$= 77{,}500$$

and $OG = 278.4$ lbs. sq. inch.

Also, $$\frac{\sin(60° - \theta)}{\sin\theta} = \frac{50}{250} = \frac{1}{5} = \sin 60° \cdot \cot\theta - \cos 60°,$$

or $\cot\theta = .8083$ and $\theta = 51°\ 3'$.

(*b*) Let the plane make an angle γ with the axis of greatest principal stress. The resultant stress OG lies wholly in this plane.

Therefore by Fig. 339 $OG^2 = n^2 - m^2 = 60{,}000,$

and $OG = 245$ lbs.

Again, $\cos 2\gamma = \dfrac{m}{n} = .2$ and $2\gamma = 78° \; 28',$

or $\gamma = 39° \; 14'$

Ex. 3. *At a point within a strained solid there are two like conjugate stresses of* 70 *lbs. and* 10 *lbs., the common obliquity being* $\cos^{-1} .8$. *Find the principal stresses and the angle between the two planes on which the conjugate stresses act.*

$$70 + 10 = r + s = (p_1 + p_2)\tfrac{4}{5}.$$

Therefore $p_1 + p_2 = 100.$

Also, $p_1 p_2 = rs = 700.$

Hence $p_1 - p_2 = 85$ very nearly,

so that $p_1 = 92\tfrac{1}{2}$ and $p_2 = 7\tfrac{1}{2}.$

Again, $OG = \dfrac{p_1 + p_2}{2} = 50 = m,$

and $GR = \dfrac{p_1 - p_2}{2} = 30\sqrt{2} = n.$

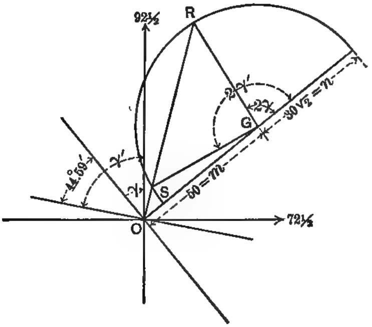


FIG. 340.

Therefore

$$70^2 = OR^2 = 50^2 + (30\sqrt{2})^2 - 2 \cdot 50 \cdot 30\sqrt{2} \cdot \cos(180° - 2\gamma),$$

or $\cos 2\gamma = .1414$ and $\gamma = 40° \; 56'.$

Also, $10^2 = OS^2 = 50^2 + (30\sqrt{2})^2 - 2 \cdot 50 \cdot 30\sqrt{2} \cdot \cos(180° - 2\gamma'),$

or $\cos(180° - 2\gamma') = .9998$ and $\gamma' = 85° \; 55'$

Hence $\gamma' - \gamma = 44° \; 59'.$

7. Shafting.—Let a shaft be subjected to the action of a couple of moment M_t in a plane *at right angles* to the direction of the shaft and also to a load which develops stresses *in the direction* of its length.

A particle at any cross-section of the shaft is acted upon by

(*a*) a stress p normal to the section, (*b*) two equal shear stresses q in the plane of the section and at right angles to each other.

Combining one of these shear stresses with p, and taking $OF=p$ and $OH=q$, the resultant stress is OG $(=\sqrt{p^2+q^2})$, the diagonal of the rectangle $OFGH$. Take $OE=m=\frac{p_1+p_2}{2}$ and join EG. Then

$$EG=n=\frac{p_1-p_2}{2}.$$

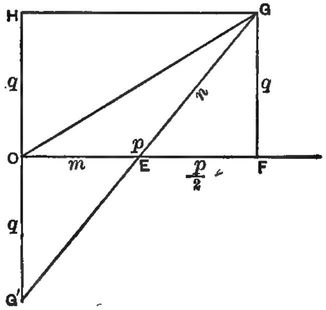


FIG. 341.

The remaining shear stress is normal to the plane OFG. Rotate the plane containing this stress and p until it is coincident with the plane OFG, and take $OG'=q$. Then OE and EG' are the constant components of OG', and GEG' is evidently a straight line. Therefore E is the middle point of OF. Hence

$$\text{the maximum shear stress}=\frac{p_1-p_2}{2}=EG=\sqrt{\frac{p^2}{4}+q^2},$$

and

$$\text{the maximum principal stress}=p_1=OE+EG=\frac{p}{2}+\sqrt{\frac{p^2}{4}+q^2}.$$

Also, as proved in Chap. IX, if the shaft is round and of radius r,

$$q=\frac{2M_t}{\pi r^3}.$$

Ex. 4. *Suppose that a force P acts along the shaft, its line of action coinciding with the axis of the shaft.*

Then $$p=\frac{P}{\pi r^2}.$$

Therefore $$\frac{p_1-p_2}{2}=\frac{1}{2\pi r^2}\sqrt{P^2+\frac{16M_t^2}{r^2}},$$

and $$p_1=\frac{1}{2\pi r^2}\left(P+\sqrt{P^2+\frac{16M_t^2}{r^2}}\right).$$

Ex. 5. Suppose that p is caused by a bending action due to a force acting at right angles to the shaft, as, e.g., when a horizontal shaft carries a heavy pulley between its bearings.

Let M_b be the bending moment at any point due to the force. Then

$$p=\frac{2M_b}{\pi r^2}.$$

Therefore

$$\frac{p_1-p_2}{2}=\frac{2}{\pi r^3}\sqrt{M_b^2+M_t^2}$$

and

$$p_1=\frac{2}{\pi r^3}(M_b+\sqrt{M_b^2+M_t^2}).$$

By this and the preceding example, if the values of the working principal and maximum shear stresses are given, the corresponding values of r can be at once calculated, the greater value being adopted for the shaft in question.

Ex. 6. *A steel shaft, in which the working stress is not to exceed* 11,200 *lbs./sq. in., is* 40 *ft. between bearings and carries a* 30-*in. pulley weighing* 200 *lbs. The effective tangential force on the pulley is* 600 *lbs./sq. in. Find the diameter of the shaft.*

$$11200=p_1=\frac{2}{\pi r^3}\left\{M_b+\sqrt{M_b^2+M_t^2}\right\},$$

$$M_b=\frac{(600+200)40}{4}\times 12 \text{ in.-lbs.}=96000 \text{ in.-lbs.},$$

$$M_t=600\times 15=9000 \text{ in.-lbs.}$$

Hence

$$r^3=\frac{1}{17600}(96000+\sqrt{96000^2+9000^2})$$

and

$$r=2.22 \text{ in., or dia.}=4.44 \text{ in.}$$

Ex. 7. Perhaps the most important example of the application of the principles just enunciated is the case of a shaft acted upon by a crank (Fig. 342).

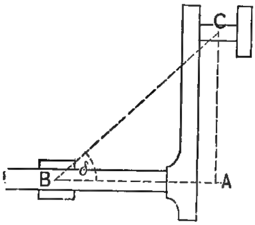


FIG. 342.

A force P applied to the centre C of the crank-pin is resisted by an equal and opposite force at the bearing B, forming a couple of moment $P\cdot CB=M$.

This couple may be resolved into a bending couple of moment $M_b=P\cdot AB=P\cdot BC\cos\delta=M\cos\delta$, and a twisting couple of moment $M_t=P\cdot AC=P\cdot BC\sin\delta=M\sin\delta$; δ being the angle ABC.

Therefore

$$p_1=\frac{2}{\pi r^3}\{M\cos\delta+M\}=\frac{4M}{\pi r^3}\cos^2\frac{\delta}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and the max. shear

$$=\frac{2M}{\pi r^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

8. Rankine's Earthwork Theory.—A mass of earth gives way by the sliding or slipping of the particles over each other, and its stability depends partly upon friction and partly upon the cohesion of the particles. This cohesive power may be considerable in certain soils, such as clay, and especially when they are moist, but it is eventually destroyed by the action of the air and by changes of temperature, so that the stability of the mass must be considered as depending upon the *frictional resistance* only. When a mass of loose soil is piled upon the ground it will be in equilibrium so long as the surface slopes at an angle which does exceed a certain angle, called the *angle of repose* and usually designated by the symbol ϕ. Tables have been prepared giving the value of ϕ for different soils, and for other soils experience may enable the engineer to interpolate suitable values for the angle of repose.

In Fig. 343 two particles are pressed together by a normal stress p.

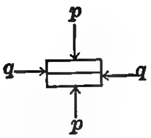


FIG. 343.

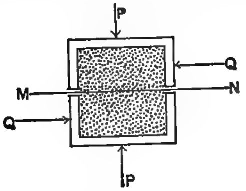


FIG. 344.

If q is the shear stress which just makes the one particle slide over the other, then $\frac{q}{p}=$ the coefficient of friction $=\mu$.

In Fig. 344 two masses of earth, confined as shown, but with freedom of movement along the plane MN, are pressed together by normal forces P. If the two equal forces Q, acting in opposite directions and at right angles to P, are just sufficient to make the one mass slide over the other along MN, then again

$$\frac{Q}{P}=\text{coefficient of friction}=\mu.$$

In Fig. 345 a stress p, having an obliquity θ, is developed on the faces AB and CD by the pressures P. Equilibrium requires the development of a stress q on the faces AC and BD, having the same tangential component as p, i.e., $p \sin \theta$. But the normal components of q on the two faces neutralize each other, and the tangential components are at right angles to MN. Thus the stability with respect to the plane MN is unaffected by q, and if slipping is just about to take place, so that $\theta = \phi$, then

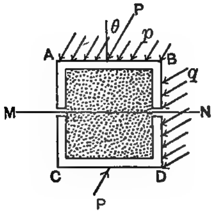


FIG. 345.

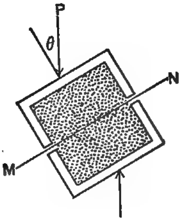


FIG. 346.

$$\mu = \frac{P \sin \phi}{P \cos \phi} = \tan \phi.$$

If Fig. 345 is now turned round until P is vertical, and if $\theta = \phi$, then MN, Fig. 346, is inclined to the horizontal at the *angle of repose*, and may be called the plane of sliding.

If $\theta < \phi$, and if the soil is dry and non-cohesive, sliding will not take place however great p may be; but if $\theta > \phi$, sliding will occur although p may be ever so small.

Hence at any point in a mass of earth there is a tendency to slide along every plane through the point, excepting along the planes of principal stress. This tendency *increases* with the obliquity of the resultant stress but is unaffected by its magnitude. It has been shown that if p_1 and p_2 are the principal stresses at the point,

$$\sin \phi = \frac{p_1 - p_2}{p_1 + p_2}, \quad \text{and therefore} \quad \frac{p_1}{p_2} = \frac{1 + \sin \phi}{1 - \sin \phi}.$$

Hence at every point the condition of equilibrium requires that the ratio of the greater to the lesser principal stress shall not exceed

$$\boldsymbol{\frac{1 + \sin \phi}{1 - \sin \phi}}.$$

9. Pressure on a Vertical Plane.—Suppose that the earth behind a wall is spread out in layers having the same uniform slope θ. This angle must be less than the angle of repose ϕ, or the earth will run over the wall. The stress on a vertical plane CD, Fig. 347, at any point D is parallel to the ground surface and is also *conjugate* to the vertical stress at D upon the plane DE which is parallel to the ground surface. This follows because the vertical stress is balanced independently of the shear stresses on the sides, and therefore for any layer the stresses parallel to the ground surface must also balance each other independently and must be of equal mean intensity throughout the layer in question.

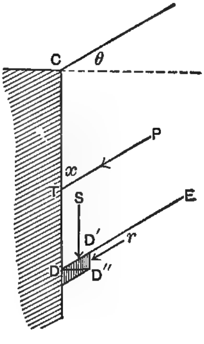


FIG. 347.

Take the thickness of the wall and of the earth, perpendicular to the plane of the paper, to be *unity*. Let w be the specific weight of the earth, and take $CD = x$. Then if r is the vertical stress at D, and if s is the stress conjugate to and $< r$,

$$rDD' = \text{weight upon element } DD' = wxDD'',$$

or

$$r = wx\frac{DD''}{DD'} = wx\cos\theta,$$

and also

$$\frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}} = \frac{s}{r} = \frac{s}{wx\cos\theta}.$$

Therefore

$$s = wx\cos\theta\frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}},$$

and the total pressure on CD

$$= \int_0^x s\,dx = \frac{wx^2}{2}\cos\theta\frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}} = P, \text{ suppose.}$$

The line of action of the resultant pressure is necessarily parallel to the ground surface and intersects CD in the point T, where

$$CT = \tfrac{2}{3}CD = \tfrac{2}{3}x.$$

If the ground surface is *horizontal*, i.e., if $\theta=0$,

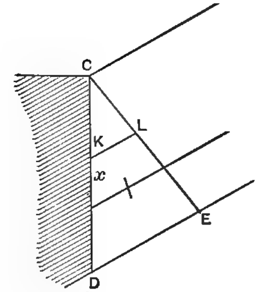


FIG. 348.

$$s=wx\frac{1-\sin\phi}{1+\sin\phi}$$

and $$\textbf{total pressure}=\frac{wx^2}{2}\,\frac{1-\sin\phi}{1+\sin\phi}.$$

If the substance retained by the wall has *no angle of repose*, i.e., if it is a fluid, $\phi=0$. Then

$$r=wx=s$$

and $$\textbf{total pressure}=\frac{wx^2}{2}.$$

These results may also be obtained in the following manner: Take DE so that $DE:x::s:r$ (Fig. 347). The pressure at D upon CD is then wDE, and if from any other point K in CD, KL is drawn parallel to DE, and therefore to the ground surface, wKL is the pressure at K upon CD.

Hence the total pressure upon CD

= the weight of the triangular prism CDE

$$=\frac{w}{2}DC\cdot DE\cos\theta=\frac{wx^2}{2}\cos\theta\frac{s}{r}=\text{etc.}$$

The resultant of all the pressures upon CD parallel to ED evidently passes through the C. of G. of the triangular prism and must therefore intersect CD at a point *two thirds* of the total depth CD below C.

10. Foundations in Earth.—CASE I.—Let the weight of the superstructure be uniformly distributed over the base, and let p_0 be the intensity of the pressure produced by it (Fig. 349). When the superstructure has just stopped subsiding and the earth on each side is on the point of heaving up, the ratio of the vertical stress p_0 to the horizontal stress p_h at points along the bottom of the superstructure must have its greatest value. Therefore

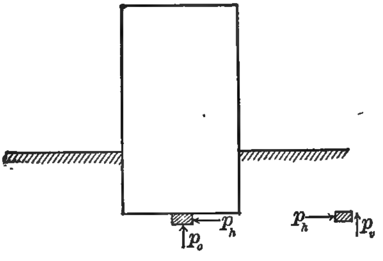


FIG. 349.

$$\frac{p_o}{p_h}\leqq\frac{1+\sin\phi}{1-\sin\phi}.$$

At the same level but at points in the natural soil, clear of the superstructure, the ratio of the horizontal stress p_h to the vertical stress p_v has its greatest possible value. Therefore

$$\frac{p_h}{p_v} \leqq \frac{1+\sin\phi}{1-\sin\phi}.$$

Hence $$\frac{p_o}{p_v} \leqq \left(\frac{1+\sin\phi}{1-\sin\phi}\right)^2.$$

If x is the depth of the foundation, and w the weight of a cubic foot of the earth,

$$p_v = wx,$$

and therefore $$\frac{p_o}{wx} \leqq \left(\frac{1+\sin\phi}{1-\sin\phi}\right)^2.$$

Let $h+x$ be the height of the superstructure, and let a cubic foot of it weigh w'. Then

$$p_o = w'(x+h).$$

Hence a minimum value of x is given by

$$\frac{w'(h+x)}{wx} = \left(\frac{1+\sin\phi}{1-\sin\phi}\right)^2 = \frac{1}{k^2}, \text{ suppose;}$$

and therefore $$\mathbf{x} = \frac{\mathbf{w'hk^2}}{\mathbf{w - w'k'^2}}.$$

Case II.—Let the superstructure produce on the base a uniformly varying pressure of maximum intensity p_1 and minimum intensity p_2 By Case I,

$$\frac{p_1}{wx} \leqq \left(\frac{1+\sin\phi}{1-\sin\phi}\right)^2.$$

In the natural ground the minimum horizontal intensity of pressure is

$$p_h = wx\frac{1-\sin\phi}{1+\sin\phi}.$$

When the foundation trench is excavated, this pressure tends to raise the bottom and push in the sides. The weight of the superstructure should therefore be at least equal to the weight of the material excavated in order to develope a horizontal pressure of an intensity equal to p_h. Therefore

$$\frac{p_h}{p_2} \leqq \frac{1-\sin\phi}{1+\sin\phi}.$$

Hence

$$\frac{p_2}{wx} \geqq 1$$

and

$$\frac{p_1}{p_2} \leqq \left(\frac{1+\sin\phi}{1-\sin\phi}\right)^2.$$

Ex. 7. *Earth weighing* 120 *lbs./cu. ft. and having an angle of repose* (ϕ) *of* 30° *is laid in horizontal layers behind the vertical face of a retaining-wall* 12 *ft. high. The ground surface is level with the top of the wall. Find the total pressure on the vertical face and the overturning moment with respect to any point in the base of the wall.*

The horizontal pressure at the bottom of the face $= 120 \times 12 \frac{1-\sin 30^\circ}{1+\sin 30^\circ}$

$= 480$ lbs.

" average pressure on the face $= \frac{1}{2}(480) = 240$ lbs.

" total pressure $= 12 \times 240 = 2880$ lbs.

" overturning moment $= 2880 \times \frac{12}{3} = 11{,}520$ ft.-lbs.

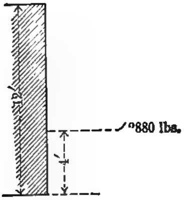


FIG. 350.

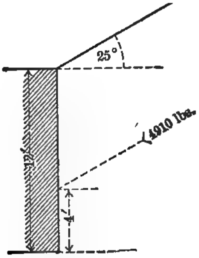


FIG. 351.

Ex. 8. *In the preceding example find the total pressure on the face when the ground surface slopes from the top of the wall at an angle of* 25°.

The total pressure parallel to the ground surface

$$= \frac{120}{2} \times 12^2 \times \cos 25^\circ \frac{\cos 25^\circ - \sqrt{\cos^2 25^\circ - \cos^2 30^\circ}}{\cos 25^\circ + \sqrt{\cos^2 25^\circ - \cos^2 30^\circ}}$$

$$= 60 \times 144 \times .9063 \frac{.6986}{1.114} = 4910 \text{ lbs.}$$

Ex. 9. *A wall* 20 *ft. high and* 8 *ft. thick weighs* 125 *lbs./cu. ft., and is built in earth weighing* 100 *lbs./cu. ft. and having an angle of repose of* 30°. *How deep must the foundation be sunk consistent with the equilibrium of the earth?*

p_0 = vertical intensity of pressure at base of wall $=20\times125=2500$ lbs./sq. ft.

p_h, the horizontal intensity of the pressure at the base of the wall, is given by

$$\frac{2500}{p_h}=\frac{1+\sin 30^\circ}{1-\sin 30^\circ}=3, \quad \text{or} \quad p_h=833\tfrac{1}{3} \text{ lbs./sq. ft.}$$

p_v, the vertical intensity at the same depth in the natural ground clear of the wall, is given by

$$\frac{p_h}{p_v}=\frac{1+\sin 30^\circ}{1-\sin 30^\circ}=3.$$

But $\qquad p_v=100x,$

x being the depth of the foundation.

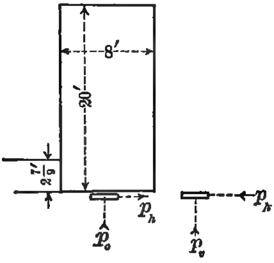


FIG. 351.

Therefore $\qquad p_h=3p_v=300x;$

Hence $\qquad \dfrac{2500}{300x}=3 \quad \text{or} \quad x=2\tfrac{7}{9}$ ft.

11. Line of Rupture.—Another expression for the pressure on *AB* may be obtained as follows:

If the whole mass in front of *AB* (Fig. 353) were suddenly removed some of the earthwork behind *AB* would fall away.

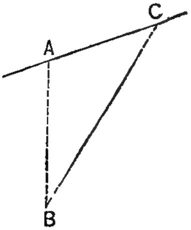


FIG. 353.

Suppose that the volume *ABC* would slip along the plane *CB*.

The stability of *ABC* is maintained by the reaction *P* on *AB*, the weight *W* of *ABC*, and the frictional resistance along *BC*.

Let the direction of *P* make an angle β with the horizon.

Let the angle $CBA=i$.

Let *R* be the mutual pressure on the plane *BC*.

Resolving along and perpendicular to *BC*,

$$-P\cos(90^\circ-i-\beta)+W\cos i=R\tan\phi$$

and $$P\sin(90^\circ-i-\beta)+W\sin i=R.$$

Therefore $$-P\sin(\beta+i)+W\cos i=\tan\phi\{P\cos(\beta+i)+W\sin i\}$$

and $$P=W\frac{\cos i-\sin i\tan\phi}{\sin(\beta+i)+\cos(\beta+i)\tan\phi}=W\frac{\cos(i+\phi)}{\sin(\beta+i+\phi)}.$$

But $$W=w\frac{BA\cdot BC}{2}\sin i=\frac{wx^2}{2}\frac{\cos\theta\sin i}{\cos(\theta+i)};$$

therefore $$P=\frac{wx^2}{2}\frac{\cos\theta\sin i}{\cos(\theta+i)}\frac{\cos(i+\phi)}{\sin(\beta+i+\phi)}.$$

The only variable upon which P depends is the angle i.

Differentiating the right-hand side of the last equation with respect to i and putting the result equal to zero, a value of i is found in terms of β, θ, and ϕ which will make P a maximum.

The line inclined at this angle to the vertical is called the *line of rupture.*

If the ground-surface is horizontal, $\theta=0$.

If the face retaining the earth is vertical, *and if it is also assumed that the friction between the face and the earthwork is nil,* P is horizontal and $\beta=0$. Hence

$$P=\frac{wx^2}{2}\tan i\cot(i+\phi).$$

This is a maximum when $2i=90°-\phi$, and then

$$P=\frac{wx^2}{2}\tan\left(45°-\frac{\phi}{2}\right)\cot\left(45°+\frac{\phi}{2}\right)=\frac{wx^2}{2}\left(\frac{1-\tan\frac{\phi}{2}}{1+\tan\frac{\phi}{2}}\right)^2=\frac{wx^2}{2}\frac{1-\sin\phi}{1+\sin\phi};$$

the same result as that obtained by Rankine's theory.

The following is an easy geometrical proof of the result $2i=90°-\phi$:

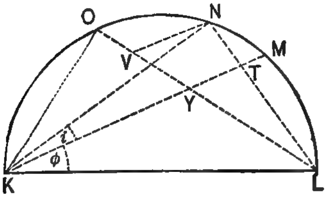


FIG. 354.

On any line KL (Fig. 354) describe a semicircle.

Draw KM inclined at the angle ϕ to KL, and KN inclined at the angle i to KM.

Join NL, cutting KM in T.

Let O be the middle point of the arc KM.

Join OL, cutting KM in Y.

Draw NV parallel to KM. Then

$$\tan i \cot (i+\phi)=\frac{NT}{KN}\times\frac{KN}{NL}=\frac{NT}{NL}=\frac{VY}{VL}.$$

The ratio $\frac{VY}{VL}$ is evidently a maximum when N coincides with O, and hence $\tan i \cot (i+\phi)$ is a maximum when KN coincides with KO.

Now the arc $OK=$ the arc OM, and therefore the angle $OKM=$ the angle OLK.

Hence if $OKM=i$, OLK must also $=i$.

But $OKL+OLK=90^\circ=i+\phi+i=2i+\phi$.

Therefore $$i=45^\circ-\frac{\phi}{2}.$$

12. Retaining walls.—Consider the equilibrium of the section $ABMN$ of a wall of unit thickness, acted upon at D by the force P, which tends to overturn the portion under consideration. This tendency is resisted by the weight W upon MN and by the stresses developed along the division surface MN.

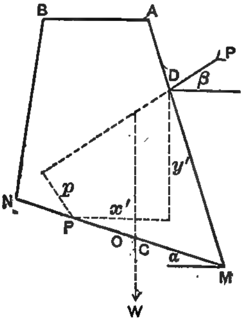


FIG. 355.

Let the resultant of these stresess intersect the surface MN in the point F. This point is called the *centre of resistance* (or *centre of pressure*) and its distance FO from the middle point O of MN is always taken to be qt, t, being the width MN and q a coefficient whose value is less than *unity* and must be determined by experience.

In ordinary practice q varies from $\frac{1}{4}$ to $\frac{3}{8}$.

Let the line of action of W intersect MN in C, and take $CO=rt$. The coefficient r is less than unity and depends upon the form of the section $ABMN$.

Let α and β be the inclinations of MN and P, respectively, to the horizontal.

Let p be the perpendicular from F upon the direction of P.

Three conditions of equilibrium must be fulfilled.

First condition. *The line of resistance, i.e., the locus of F, must intersect every bed joint well within the outer edges M.*

The position of P is determined by the condition that the moment of W with respect to F

$$\geqq \text{the moment of } P \text{ with respect to } F,$$

or

$$W(qt \mp rt) \geqq Pp,$$

the upper or lower signs being taken according as C is on the left or right of O.

$W(qt \mp rt)$ is called the *moment of stability*

and Pp is called the *overturning moment.*

If x', y' are the horizontal and vertical coördinates of D with respect to F,

$$p = y' \cos \beta - x' \sin \beta.$$

The line of resistance of a dam, Fig. 356, may be easily determined graphically in the following manner:

Let $ABFG$ be the cross-section, the thickness perpendicular to the plane of the paper being unity. Divide the figure into convenient portions $ABCL$, etc., and find the centres of gravity G_1, G_2, G_3, G_4 of these portions. Calculate the weights W_1, W_2, W_3, W_4 of the several portions and lay off $a1$, 12, 23, 34, Fig. 357, on a vertical line to represent these weights. Take any pole O and draw the polygon $bcdf$, Fig. 358. Then R_1, the resultant of W_1, W_2, passes through g; R_2, the resultant of W_1, W_2, W_3, through h; and R_3, the total resultant, through k. The intersections of W_1, R_1, R_2, R_3 with the planes LC, RD, HE, and GF determine the line of resistance $V'Q'R'S'T'$ when there is no water pressing on the dam.

Let the water now rise to the top of the dam. The water-pressures on the several portions are represented in magnitude by the areas AML, $M'LKN$, etc., and in direction by the lines P_1, P_2, P_3, P_4 drawn normally to the faces AL, LK, etc., and through the centres of gravity of the above figures. Lay off al, lm, mn, np parallel to P_1, P_2, P_3, P_4, respectively, and proportional to the areas ALM, $M'LKN$, etc. Draw the polygon $qrst$, Fig. 359. Then R_1' drawn through u parallel to MQ gives the line of action of the resultant of P_1 and P_2. Similarly R_2' and R_3' give the lines of action of the resultants of P_1, P_2, P_3, and of P_1, P_2, P_3, P_4, respectively.

Now $l1$ is the resultant of P_1 and W_1. Through the intersection of P_1 and W_1 draw a line parallel to $l1$ cutting LC in Q. Similarly

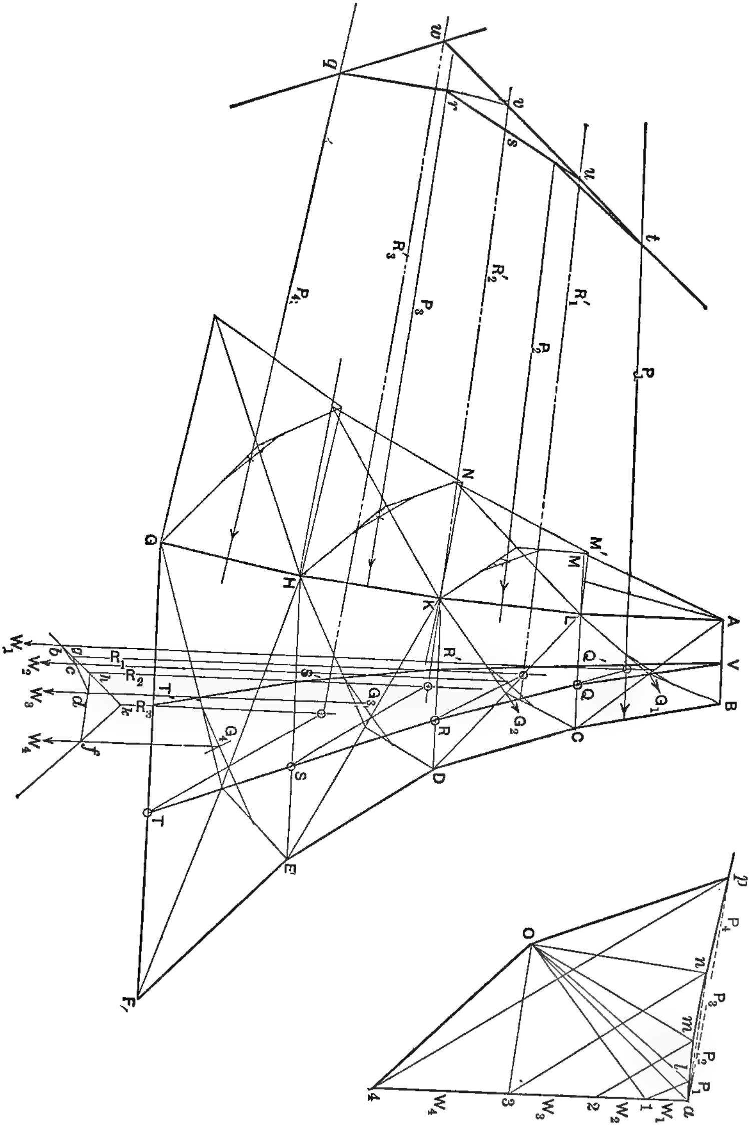


FIG. 359.

FIG. 358.

FIG. 356.

FIG. 357.

through the intersection of R_1 and R_1' draw a line parallel to $m2$ cutting KD in R. Then Q and R are the points at which the resultants of the forces acting above their respective planes cut those planes. The points S and T are obtained in the same manner and $QRST$ is the line of resistance required.

Second condition. *The angle between the directions of the resultant pressure on MN and a normal to MN must be less than the angle of friction, Fig.* 355.

Let ϕ be the angle of friction and R the mutual *normal* pressure.

Then $$R \tan \phi > P \cos \overline{\alpha+\beta} - W \sin \alpha$$

$$> \{P \cos \beta - (P \sin \beta + W) \tan \alpha\} \cos \alpha,$$

and $$R = P \sin \overline{\alpha+\beta} + W \cos \alpha$$

$$= (P \cos \beta \tan \alpha + P \sin \beta + W) \cos \alpha.$$

Therefore

$$\tan \phi > \frac{P \cos \beta - (P \sin \beta + W) \tan \alpha}{P \cos \beta \tan \alpha + P \sin \beta + W},$$

or $$\frac{\tan \phi + \tan \alpha}{1 - \tan \phi \tan \alpha} > \frac{P \cos \beta}{P \sin \beta + W}.$$

Hence $$\tan (\alpha + \phi) > \frac{P \cos \beta}{P \sin \beta + W}.$$

If MN is horizontal,

$$\alpha = 0,$$

and then $$\tan \phi > \frac{P \cos \beta}{P \sin \beta + W}.$$

If the ground surface is also horizontal,

$$\beta = 0,$$

and then $$\tan \phi > \frac{P}{W}.$$

Four fifths of ϕ is usually taken as the limiting value of the angle of friction.

Third condition. *The maximum intensity of pressure on the surface MN must not exceed the safe working strength of the material.*

The load upon any surface (or *bed*) MN cannot be said to be ever uniformly distributed, but this third condition is practically ensured by the assumption that the intensity of the pressure normal to the surface diminishes uniformly from the outer or most compressed edge inwards.

Let R be the total pressure normal to MN.

Let f be the maximum normal intensity of pressure at M.

Draw MG at right angles to MN, Fig. 360, and take $MG=f$.

There are *three* cases to be considered.

CASE *a*.—Let the normal intensity vary from $MG=f$ at M to *nil* at N. Join GN.

The intensity of the pressure at any point in MN is the intercept between MN and GN of a line drawn through the point parallel to MG.

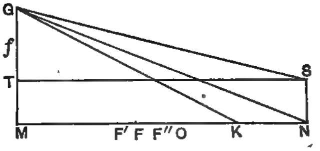


FIG. 360.

Hence $R=$ area of triangular prism $MGN=\frac{ft}{2}$.

Also, the resultant of all the stresses parallel to MG passes through the C. of G. of MGN and intersects MN in the point F, where $MF=\frac{1}{3}t$.

Therefore $$qt=OF=\frac{t}{2}-\frac{t}{3}=\frac{t}{6}$$

and $$q=\frac{1}{6}.$$

Thus F is one of the limiting points of the *middle third* of the surface MN.

CASE *b*.—Let the normal intensity vary from $MG=f$ at M to *nil* at a point K lying inside the edge N. Join GK.

The normal intensity at any point in MK is the intercept between MK and GK of the line drawn through the point parallel to MG.

The resultant of all the stresses on MK passes through the C. of G. of the triangular prism GMK and intersects MN in the point F', where $MF'=\frac{1}{3}MK$.

In this case $$OF'=qt>OF>\frac{t}{6},$$

and therefore $$q>\frac{1}{6}.$$

Also, $R=$ area of triangular prism GMK

$$=\frac{1}{2}fMK=\frac{1}{2}fBMF'=\frac{3}{2}f(OM-OF')$$

$$=\frac{3}{2}f\left(\frac{t}{2}-qt\right).$$

Hence
$$q=\frac{1}{2}-\frac{2}{3}\frac{R}{ft}.$$

Over the distance KN the stress is either *nil* or negative, i.e., a tension.

CASE *c*.—Let the normal intensity vary from $MG=f$ at M to a minimum $NS=f_0$ at N. Join GS.

The normal intensity at any point of MN is now the intercept between MN and GS of a line drawn through the point parallel to MG.

The resultant of all the stresses on MN passes through the C. of G. of the trapezoidal prism $GMNS$ and intersects MN in a point F'' between F and O, so that in this case

$$qt=OF''<OF<\frac{t}{6},$$

and therefore
$$q<\frac{1}{6}.$$

Draw ST parallel to MN. Then

$$\text{area } GMNS\times OF''=\text{rectangle } TMNS\times 0$$

$$+\text{triangle } GTS\left(\frac{t}{2}-\frac{t}{3}\right),$$

or
$$\frac{f+f_0}{2}qt=\frac{f-f_0}{2}\frac{t^2}{6},$$

and
$$f+f_0=\frac{2ft}{6q+t}.$$

But
$$R=\frac{f+f_0}{2}t=\frac{ft^2}{6q+1}.$$

Hence
$$\mathrm{q}=\frac{1}{6}\left(\frac{\mathrm{ft}}{\mathrm{R}}-1\right).$$

Ex. 10. *A masonry wall, Fig.* 361, *of rectangular section, x ft. high,* 4 *ft. wide, and weighing* 125 *lbs./cu. ft. is built upon a horizontal bed and retains water on one side, level with the top of the wall. Find x* (*a*) *when $q=\frac{1}{3}$;* (*b*) *if the safe working strength of the material is* 10,000 *lbs./sq. ft. How much* (*c*) *of the wall can be removed in case* (*a*) *without changing the moment of stability?*

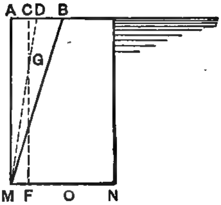


FIG. 361.

The horizontal water-pressure on wall $=62\frac{1}{2}\frac{x^2}{2}$ lbs., acting $\frac{x}{3}$ ft. above the bed.

Therefore the *overturning moment* $=62\frac{1}{2}\cdot\frac{x^2}{2}\cdot\frac{x}{3}=\frac{125}{12}x^3$ ft.-lbs.

The weight of the wall $=4x\cdot125=500x$ lbs., and the distance of its line of action from the centre of resistance $F=4q$.

Therefore the *moment of stability* $=500x\times4q=2000qx$ ft.-lbs. Hence for equilibrium

$$\frac{125}{12}x^3\leqq2000qx,$$

$$\text{or } x^2\leqq192q.$$

(*a*) If $q=\frac{1}{3}$, $x^2\leqq64$, and therefore $x\leqq8$ ft., so that the height of the wall must not exceed 8 ft.

(*b*) Again, if $f=10{,}000$ lbs.,

$$\textit{first} \text{ assume } q=\frac{1}{6}\left(\frac{ft}{R}-1\right)=\frac{1}{6}\left(\frac{10000\times4}{500x}-1\right)=\frac{1}{6}\left(\frac{80}{x}-1\right).$$

Then
$$x^2\leqq\frac{192}{6}\left(\frac{80}{x}-1\right)\leqq32\left(\frac{80}{x}-1\right)$$

and x lies between 12 and 13 ft.

Each of these values makes $q>\frac{1}{6}$, and therefore the assumption made for the value of q is incorrect.

$$\textit{Second.} \text{ Assume } q=\frac{1}{2}-\frac{2}{3}\frac{R}{ft}=\frac{1}{2}-\frac{2}{3}\frac{500x}{10000\times4q}=\frac{1}{2}-\frac{x}{120}.$$

Then
$$x^2\leqq192\left(\frac{1}{2}-\frac{x}{120}\right)$$

or
$$x^2-\tfrac{8}{5}x\leqq96.$$

Therefore
$$x\leqq9.03 \text{ ft.},$$

so that the height of the wall must not exceed 9.03 ft.

Also, $$q=\frac{1}{2}-\frac{9.03}{120}=.4247 \text{ and is} >\frac{1}{6},$$

showing that the value assumed for q is justified.

(*c*) The triangular portion MAB, Fig. 361, may be removed if its C. of G. is vertically above the centre of resistance F. Let the vertical FG intersect AB in C. Also let MG meet AB in D; D is the middle point of AB. Then

$$AC=\tfrac{2}{3}AD=\tfrac{1}{3}AB.$$

Therefore $$AB=3AC=3\times\tfrac{2}{3}=2 \text{ ft.},$$

so that B is the middle point of AF, and thus one fourth of the wall may be removed.

Again, let the resultant of the horizontal water-pressure (P) and the weight upon the base make with the normal to the base an angle θ before the portion MAB has been removed and an angle θ' after its removal.

If W is the weight on the base in the *first* case,

$$\tan\theta=\frac{P}{W},$$

and in the second case,

$$\tan\theta'=\frac{P}{\tfrac{3}{4}W}.$$

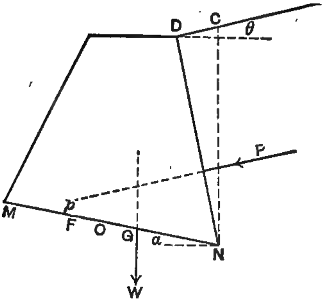


Fig. 362.

Therefore $\tan\theta:\tan\theta'::4:3$.

Ex. 11. *Fig.* 362 *represents a slice of a wall of unit thickness, retaining earth which has an angle of repose* ϕ *and slopes away from the top of the wall at an angle* θ. *Consider the equilibrium of the whole mass in front of a vertical plane through the inner edge* N.

The weight W on the base is then made up of the weight of the slice of the wall and the weight of the triangular prism, also of unit thickness, of earth, viz., NCD.

Let the line of action of W intersect the base in G.

Let F be the centre of resistance in the base and let O be the middle point of MN. Then

$$\text{the } \textit{moment of stability} = W(OF\mp OG)\cos\alpha,$$
$$=W(q\mp r)\,t\cos\alpha,$$

α being the slope of the base.

Again, if x is the vertical distance between N and the ground surface, i.e., NC, then

$$\text{the resultant pressure, } P, \text{ on } CN=\frac{wx^2}{2}\cos\theta\,\frac{\cos\theta-\sqrt{\cos^2\theta-\cos^2\phi}}{\cos\theta+\sqrt{\cos^2\theta-\cos^2\phi}},$$

and acts *parallel* to the ground surface at a distance $\frac{x}{3}$ above N.

Therefore the *overturning moment* is Pp, p being the perpendicular from F upon the direction of P. Hence

$$W(q \mp r)t \cos \alpha \geqq Pp.$$

For example, if $\theta = \phi$, i.e., if the ground slopes at the natural angle of repose,

$$P = \frac{wx^2}{2} \cos \phi.$$

If $\theta = 0$, i.e., if the ground surface is horizontal,

$$P = \frac{wx^2}{2} \frac{1 - \sin \phi}{1 + \sin \phi} \quad \text{and} \quad p = \frac{x}{3},$$

since P is also horizontal. Therefore

$$W(q \mp r) \tan \alpha \geqq \frac{wx^3}{6} \frac{1 - \sin \phi}{1 + \sin \phi}.$$

If $\theta = 0 = \phi$, i.e., if the substance retained by the wall has *no angle of repose* the case becomes one of fluid pressure, and

$$W(q \mp r)t \cos \alpha \geqq \frac{wx^3}{6}.$$

Ex. 12. *Gravelly earth with an angle of repose of* 37° *and weighing* 96 *lbs./cu. ft. is spread out in horizontal layers and is retained by a wall of rectangular section* 6 *ft. high and weighing* 120 *lbs./cu. ft. The earth rises to the top of the wall. Find the width t of the wall, q being* $\frac{1}{6}$.

$$\text{Moment of stability} = 6t \cdot 120 \cdot \frac{t}{6} = 120t^2 \text{ ft.-lbs.}$$

$$\text{Overturning moment} \quad = 96 \cdot \frac{6^2}{2} \cdot \frac{1 - \sin 37^\circ}{1 + \sin 37} \times \frac{6}{3} = 859 \text{ ft.-lbs.}$$

Therefore $$120t^2 \geqq 859,$$

or $$t^2 \geqq 7.16$$

and $$t \geqq 2.68 \text{ ft.}$$

Ex. 13. *The water-face AC of a wall, Fig.* 363, *has a batter of* 1 *in* 10; *the width of the wall AD at the top is* 6 *ft.*; *the rear of the wall DEF has two slopes, DE, having a batter of* 2 *in* 10, *and EF, a batter of* 78 *in* 100; *the masonry weighs* 125 *lbs. per cubic foot, and the maximum compression must not exceed* 85 *lbs. per square inch. Find the safe heights of the two portions AE and EC.*

First. To find the height of the upper portion:

Consider the equilibrium of the whole mass of masonry and water in front of a vertical plane through the edge N'.

Taking $10x$ ft. as the height, the other dimensions of the section are as shown in the figure.

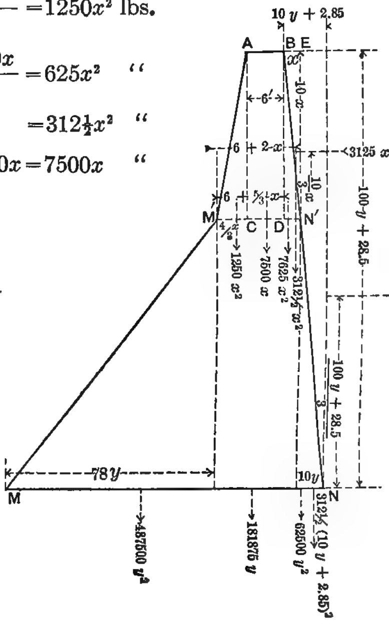


Fig. 363

Wt. of triangle $AM'C=125\dfrac{2x\cdot10x}{2}=1250x^2$ lbs.

" " " $BN'D=125\dfrac{x\cdot10x}{2}=625x^2$ "

" " " $BN'E$ of water $=312\frac{1}{2}x^2$ "

" " rectangle $ABDC=125\cdot6\cdot10x=7500x$ "

Total R on base $=2187\frac{1}{2}x^2+7500x$.

Horizontal pressure on vertical face $N'E=62\frac{1}{2}\dfrac{(10x)^2}{2}=3125x^2$, acting $\dfrac{10}{3}x$ ft. above the base.

Let $\bar{x}$ ft. be the distance between M' and the line of action of the vertical resultant pressure on $M'N'$.

Assume that $q=\dfrac{1}{2}-\dfrac{2}{3}\dfrac{R}{ft}$, where

t = width $M'N'$ of base $=3(x+2)$ ft.

Then also $\dfrac{2}{3}\dfrac{R}{f}$ is the distance between M' and the centre of resistance F in the base.

Taking moments about F,

$$3125x^2\cdot\frac{10}{3}x=R\left(\bar{x}-\frac{2}{3}\frac{R}{f}\right)$$

$$=R\bar{x}-\frac{2}{3}\frac{R^2}{f}$$

$$=1250x^2\cdot\tfrac{4}{3}x+7500x(3+2x)$$

$$+625x^2(6+\tfrac{7}{3}x)+312\tfrac{1}{2}x^2(6+\tfrac{8}{3}x)$$

$$-\frac{2}{3}\cdot\frac{1}{144\times85}(2187\tfrac{1}{2}x^2+7500x)^2,$$

which reduces to

$$193776x^2+6125x^3=412704x+528768,$$

and x lies between $28\frac{1}{2}$ and 29 ft.

Take $x=28\frac{1}{2}$ ft.

Then $R = 39142.97$ lbs.; $t =$ width of base $= 14.55$ ft.;

$$\text{Wt. of masonry/lin. ft. of thickness} = \tfrac{1}{2}(6 + 14.55)28\tfrac{1}{2} \times 125$$

$$= 36605 \text{ lbs.}$$

Also,

$$q = \frac{1}{2} - \frac{2}{3}\,\frac{39142.97}{144 \times 85 \times 14.55} = .353 > \tfrac{1}{6},$$

and therefore the assumption $q = \dfrac{1}{2} - \dfrac{2}{3}\,\dfrac{R}{ft}$ is justified.

If x' is the distance between M' and the vertical through the C. of G. of the wall,

$$x'(36605) = 1250x^2 + \tfrac{4}{3}x + 7500x(2x+3) + 625x^2(\tfrac{7}{3}x + 6).$$

Substituting in this equation the value of x,

$$x' = 7.89 \text{ ft.}$$

Second. To find the height of the lower portion:

Consider the equilibrium of the whole mass in front of the vertical through the edge N.

Take $100y$ as the height of the lower portion; the other dimensions are as shown in the figure.

The total R' on the base

$$= 36605 + 3900y^2 \times 125 + 14.55 \times 100y \times 125 + 500y^2 \times 125$$

$$+ \frac{62\frac{1}{2}}{2}(10y + 2.85)(10y + 28.5).$$

Assuming, as before, that

$$q' = \frac{1}{2} - \frac{2}{3}\,\frac{R'}{ft'},$$

and using similar symbols,

$$\frac{62\frac{1}{2}}{2}(100y + 28.5)^2\,\frac{10y + 28.5}{3} = R'\left(\bar{x}' - \frac{2}{3}\,\frac{R'}{144 \times 85}\right)$$

$$= 3900y \times 125 \times 52y + 1455y \times 125(7.275 + 78y)$$

$$+ 500y^2 \times 125\left(78y + 14.55 + \frac{10}{3}y\right)$$

$$+ \frac{125}{4}(10y + 2.85)(100y + 28.5)\left(78y + 14.55 + \frac{2}{3}\overline{10y + 2.85}\right)$$

$$+ 36605(7.89 + 787) - \frac{2}{3}\,\frac{R'^2}{144 \times 85},$$

which reduces to

$$y^4 - .27047^3 + .1846y^2 - .1226y - .000146 = 0.$$

Hence y lies between .45 and .46 and is approximately .457. Thus the height of the lower portion is about 45.7 ft.

Also, $$R' = 235314 \text{ lbs.},$$

and $$\frac{2}{3}\frac{R'}{ft'} = \frac{235314}{991440} = .237,$$

and therefore $q' = .5 - .237 = .267$ is $> \frac{1}{6}$, and the assumption $q' = \frac{1}{2} - \frac{2}{3}\frac{R'}{ft'}$ is justified. The width of $MN = 78y + 14.55 + 10y = 54.77$ ft.

13. **Practical Rules.**—When the surface of the earthwork is horizontal and the face of the wall against which it abuts vertical, the pressure on the wall according to Rankine's theory is

$$P = \frac{wx^2}{2}\frac{1-\sin\phi}{1+\sin\phi},$$

and the direction of P is horizontal.

This result is also identical with that obtained in Art. 11, on the assumption of Coulomb's wedge of maximum pressure (Poncelet's Theory).

Experience has conclusively proved that this theoretical value of P is very much greater than its real value, so that the thickness of a wall designed in accordance with theory will be in excess of what is required in practice. In the deduction of the formula, indeed, the altogether inadmissible assumption is made that there is no friction between the earthwork and the face of the wall. This is equivalent to the supposition that the face is perfectly smooth and that therefore the pressure acts normally to it. Boussinesque, Levy, and St. Venant have demonstrated that the hypothesis of a normal pressure only holds true,

either, first, if the ground-surface is horizontal and the wall-face inclined at an angle of $45° - \frac{\phi}{2}$ to the vertical,

or, second, if the wall-face is vertical and the ground-surface inclined at an angle ϕ to the horizon.

When the surface of the ground is horizontal and the face of the wall vertical, and when $\phi = 45°$, the above formula gives the correct *magnitude* of P. Its direction, however, is not horizontal, but makes an angle with the vertical equal to the angle of friction between the earth and the wall. The wall-face is generally sufficiently rough to hold fast a layer of earth, and in all probability Boussinesque's assumption that the friction between the wall and the earth is equal to that inherent in the earth is a near approximation to the truth. The *direction* of P will thus be considerably modified, leading to a smaller overturning moment and a corresponding diminution in the necessary thickness of the wall.

In practice the thrust P may always be made small by carrying up the backing in well-punned horizontal layers.

In order to neutralize the very great thrust often induced by alternate freezing and thawing and the consequent swelling, a most effective expedient is to give a batter of about 1 in 1 to the rear line of the wall extending below the line to which frost penetrates.

The greatest difficulty in formulating a table of earth thrusts arises from the fact that there is an infinite variety of earth work. As an example of this, Airy states that he has found the cohesive power of clay to vary from 168 to 800 pounds per square foot, the corresponding coefficients of friction varying from 1.15 to .36, and that even this wide range is less than might be found in practice.

A correct theory for the design of retaining-walls is as yet wanting. According to Baker, experience has shown that with good backing and a good foundation the stability of a wall will be insured by making its thickness one fourth the height, and giving it a front batter of 1 or 2 in. per foot, and that under no conditions of ordinary surcharge or heavy backing need its thickness exceed one half the height. Baker's usual practice in ground of average character is to make the thickness one third the height from the top of the footings, and if any material is taken out to form a face panel, three fourths of it is put back in the form of a pilaster.

General Fanshawe's rule for brick walls of rectangular section retaining ordinary material is to make the thickness

24% of the height for a batter of 1 in 5;
25% " " " " " 1 in 6;
26% " " " " " 1 in 8;
27% " " " " " 1 in 10;
28% " " " " " 1 in 12;
30% " " " " " 1 in 24;
32% " " " " for a vertical wall.

As a general rule, a batter of 1 in 24 to 1 in 12 is usually adopted in practice, and should not exceed *one sixth* of the height of the wall. The stability of a wall with a straight or a curved batter in compression is always greater than that of a wall with a vertical face. This is especially the case if the joints are at right angles to the face, when the tendency to slide forward is largely diminished.

Again, if the wall turns round the toe ever so little, the wall with a vertical face overhangs and gives an idea of instability, while it also offends the eye. When the face has a batter a slight rotation is not noticed.

By the use of counterforts it can be shown that there is a small saving of masonry in a vertical wall of rectangular section. Brunel curved the face of a retaining-wall and made its thickness *one fifth* or *one sixth* of the height. He also used counterforts 30 ins. wide at intervals of 10 ft.

Experience indicates that counterforts should be stepped. The width of the top of a wall should be as small as possible consistent with sound construction.

The thickness at the footing adopted by Vauban for walls with a front batter of 1 in 5 or 1 in 6 and plumb at the rear is approximately given by the empirical formula

$$\text{thickness} = .19H + 4 \text{ ft.},$$

H being the height of the wall above the footing. Counterforts were introduced at intervals of 15 ft. for walls above 35 ft. in height and at intervals of 12 ft. for walls of less height. The counterfort projects from the wall a distance of $\frac{H}{5}+3$ ft. approximately, and the approximate width of the counterfort is $\frac{H}{10}+3$ ft., diminishing to $\frac{H}{15}+2$ ft.

The great importance of the foundation will be better appreciated by bearing in mind that the great majority of failures have been due to defective foundations. If water can percolate to the foundation, a softening action begins and a consequent settlement takes place, which is most rapid in the region subjected to the greatest pressure, viz., the toe. In order to counteract this tendency to settle, the toe may be supported by raking piles, the rake being given to diminish the bending action of the thrust on the piles. It is also advisable to distribute the weight as uniformly as possible over the base, a condition which is not compatible with large front batters and deep offsets, as they tend to concentrate weight on isolated points. In the case of dock walls, too, a large front batter will keep a ship farther away from the coping and will necessitate thicker fenders, as well as cranes with wider throws. As an objection to offsets Bernays urges that, in settling, the backing is liable to hang upon them, forming large holes underneath. He therefore favors the substitution of a batter for the offsets. On the other hand, if water stands on both sides of the walls, the hydrostatic pressure on the offsets will greatly increase its stability.

Dock walls are liable to far greater variations of thrust than ordinary retaining-walls. The water in a dock with an impermeable bottom may stand at a much higher level than the water at the back of the wall, and its pressure may thus even more than neutralize the thrust due to the backing. With a porous bottom the stability of a wall may be greatly diminished by an upward pressure on the base. The experience of dock-wall failures has led to the conclusion that a large moment of stability is not of so much importance as "weight with a good grip on the ground." Many authorities, both practical and theoretical, have urged the great advantages in economy and strength attending the employment of counterforts. The use of Portland cement, or cement concrete, will guard against the breaking away of the counterforts from the main body of the wall, as has often happened in the case of the older walls. But a uniform distribution of pressure as well as of weight is important, and it therefore seems more desirable to introduce the extra weight of the counterforts into the main wall. Besides, the building of the counterforts entails of itself an increased expense.

14. Reservoir Walls.—Let f be the maximum safe pressure per square foot of horizontal base, at inner face of a full reservoir, at outer face when empty.

Let w be the weight of a cubic foot of the masonry.

Assume that the wall is to be of *uniform strength*, i.e., that the section of the wall is of such form that in passing from any horizontal section to the consecutive one below, the ratio of the increment of the weight to the increment of the surface is constant and equal to f.

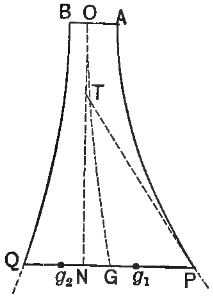


FIG. 364.

Let AB, Fig. 364, be the top of the wall. Take any point O as origin, and the vertical through O as the axis of x.

Let $OA=t_1$, $OB=t_2$, and let

$$T=t_1+t_2=AB.$$

For the profile AP consider a layer of thickness dx at a depth x. Then

$$\frac{wydx}{dy}=f, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$dx=\frac{f}{w}\frac{dy}{y};$$

therefore

$$x=\frac{f}{w}\log_e y+c,$$

c being a constant of integration.

When $x=0$, $y=t_1$;

therefore

$$0=\frac{f}{w}\log_e t_1+c,$$

and hence

$$x=\frac{f}{w}\log_e \frac{y}{t_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is the equation to AP and is the logarithmic curve.

It may be similarly shown that the equation to BQ is

$$x=\frac{f}{w}\log_e\frac{y}{t_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Equations (2) and (3) may also be written in the forms

$$y=t_1e^{\frac{w}{f}x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

and

$$y=t_2e^{\frac{w}{f}x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Corresponding points on the profiles, e.g., P and Q, have a *common subtangent* of the constant value $\frac{f}{w}$ for

$$NT=PN\tan NPT\left(=y\frac{dx}{dy}\right)=\frac{f}{w}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

$$\text{Area } PNOA=\int_0^x ydx=t_1\left(\frac{f}{w}e^{\frac{w}{f}x}-\frac{f}{w}\right)=\frac{f}{w}(Y_1-t_1). \quad . \quad . \qquad (7)$$

where $PN=Y_1$.

$$\text{Area } QNOB=\int_0^x ydx=\frac{f}{w}(Y_2-t_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

where $QN=Y_2$.

$$\text{Therefore the area } QPAB=\frac{f}{w}(Y_1+Y_2-\overline{t_1+t_2})=\frac{f}{w}(T'-T), \quad . \qquad (9)$$

where $PQ=Y_1+Y_2=T'$.

Thus the area of the portion under consideration is equal to the product of the subtangent and the difference of thickness at top and bottom.

Lines of Resistance with Reservoir Empty.—Let g_1 be the point in which the vertical through the C. of G. of the portion $OAPN$ intersects PN. Then

$$Ng_1\times\text{area } OAPN=\int_0^x ydx\frac{y}{2}, \quad \text{and therefore}$$

$$Ng_1(Y_1-t_1)\frac{f}{w}=\frac{1}{2}\frac{f}{w}\int_{t_1}^{Y_1}ydy=\frac{1}{4}\frac{f}{w}(Y_1^2-t_1^2);$$

or

$$Ng_1=\frac{Y_1+t_1}{4}.$$

So if g_2 be the point in which the vertical through the C. of G. of the portion $OBQN$ intersects QN,

$$Ng_2 = \frac{Y_2 + t_2}{4}.$$

Let G be the point in which the vertical through the C. of G. of the whole mass $ABQP$ intersects PQ. Then

$$NG \times \text{area } ABQP = Ng_1 \times \text{area } AONP - Ng_2 \times \text{area } BONQ,$$

or

$$NG\frac{f}{w}(Y_1 - t_1 + Y_2 - t_2) = \frac{1}{4}\frac{f}{w}(Y_1{}^2 - t_1{}^2) - \frac{1}{4}\frac{f}{w}(Y_2{}^2 - t_2{}^2).$$

Therefore $$NG = \frac{(Y_1{}^2 - t_1{}^2) - (Y_2{}^2 - t_2{}^2)}{4(Y_1 - t_1 + Y_2 - t_2)}.$$

The horizontal distance between G and a vertical through the middle point of AB

$$= NG - \frac{1}{2}(t_1 - t_2) = \frac{(Y_1 - t_1)^2 - (Y_2 - t_2)^2}{4(Y_1 - t_1 + Y_2 - t_2)} = \frac{(Y_1 - t_1) - (Y_2 - t_2)}{4}$$

= one half of the horizontal distance between the verticals through the middle points of AB and QP.

The locus of G can therefore be easily plotted.

Lines of Resistance with Reservoir Full.—Let R be the centre of resistance in PQ (Fig. 365).

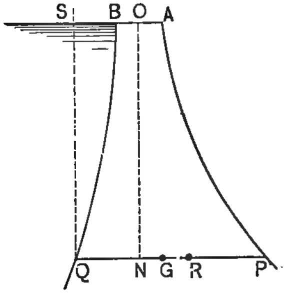


FIG. 365.

Draw the vertical QS, and consider the equilibrium of the mass $QSAPQ$.

Let w' = weight of a cubic foot of water.

$\frac{w'x^2}{2}\frac{x}{3}$ = moment of water-pressure against QS about R
= moment of weight of QBS about R + moment of weight of $QPAB$ about R,

or

$$\frac{w'x^3}{6} = \text{moment of } QBS \text{ about } R + \frac{f}{w}(T' - T)wGR.$$

The first term on the right-hand side of this equation is generally very small and may be disregarded, the error being on the safe side.

In such case

$$GR=\frac{1}{6}\frac{w'}{f}\frac{x^3}{T'-T}.$$

Also the *mean* intensity of the vertical pressure

$$=p_0=\frac{w\times\text{area } APQB}{PQ}=f\left(1-\frac{T}{T'}\right),$$

and the *maximum* intensity of the vertical pressure

$$=p_1=\frac{2R}{(\frac{3}{2}-3q)T'}=\frac{4}{3}f\frac{\left(1-\frac{T}{T'}\right)}{1-2q}$$

or

$$=\frac{R}{T'}(1+6q)=f(1+6q)\left(1-\frac{T}{T'}\right).$$

General Case.—Let the profile be of any form, and consider any portion *ABQP*, Fig. 366.

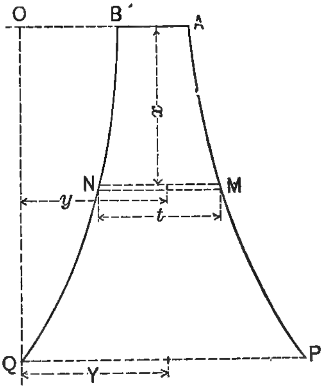


FIG. 366.

Take the vertical through *Q* as the axis of *x*, and the horizontal line coincident with top of wall as the axis of *y*.

The horizontal distance ($\bar{y}$) between the axis of *x* and the vertical through the C. of G. of the portion under consideration is given by the equation

$$\bar{y}\int_0^x tdx=\int_0^x tydx,$$

t being the width, *dx* the thickness, and *y* the horizontal distance from *OQ* of the C. of G. of any layer *MN* at a depth *x* from the top.

When the reservoir is *empty*, the deviation of the centre of resistance from the centre of base

$$=qT=Y-\bar{y}<\frac{T}{6}.$$

When the reservoir is *full*, let $q'T$ be the deviation of the centre of resistance from the centre of the base, and disregard the moment of the weight of the water between OQ and the profile BQ. Then

$$q'T=\frac{\text{moment of water-pr.}\pm\text{moment of wt. of } OBQ}{\text{weight of } ABQP}\mp Y$$

$$=\pm\frac{1}{6}\frac{w'x^2}{w\int_0^x tdx}\pm\bar{y}\mp Y.$$

Hence

$$(q\pm q')T=\frac{1}{6}\frac{w'x^2}{w\int_0^x tdx}.$$

15. Relations between the Elastic Constants.—In an isotropic body, i.e., in a body whose properties are the same in all directions if three simple stresses, p, of the same magnitude and sign are applied in three different directions, there is no distortion of form and the volume only is changed. If δV is the change in the volume V, the strain is measured by the ratio $\frac{\delta V}{V}$, and if K is a coefficient such that $\frac{\delta V}{V}=\frac{p}{K}$, then K or the ratio of the stress to the strain is called the *bulk modulus* or the *modulus of cubic compressibility*.

The *linear* dimensions of the body change equally. Let a be one of these dimensions. Then $V\propto a^3$,

and
$$\frac{\delta V}{V}=\frac{3a^2\cdot\delta a}{a^3}=3\frac{\delta a}{a},$$

if the cubic strain is small.

Hence the *linear strain* $=\frac{\delta a}{a}=\frac{1}{3}\frac{\delta V}{V}=\frac{1}{3}\frac{p}{k}.$

Relation between G and E.—A shear stress q along the face AC, Fig. 367, of the square $ABDC$ distorts the square into the rhombus $BA'C'D$.

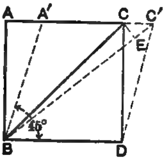


FIG. 367.

Draw CE perpendicular to BC'. Then the strain along the diagonal BC

$$=\frac{C'B-CB}{CB}=\frac{C'E}{CB}=\frac{1}{2}\frac{CC'}{CD}$$

$$=\tfrac{1}{2}\text{ (shearing strain),}$$

or
$$e=\frac{\phi}{2}.$$

Again, it has already been shown that a pair of equilibrating stresses q, Fig. 368, develope an equal tensile stress q along one diagonal and an equal compressive stress q along the other. Hence the total longitudinal strain

$$=\frac{q}{E}+\frac{q}{\sigma E}=\frac{\phi}{2}=\frac{1}{2}\,\frac{q}{G},$$

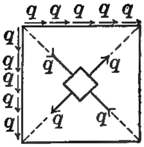


FIG. 368.

σ being Poisson's ratio, Chapter IV.

Therefore
$$G=\frac{E}{2}\left(\frac{\sigma}{\sigma+1}\right).$$

Relation between G, E, and K.—Let a pull of intensity p act upon the upper and lower faces BD and FH of an elementary cube, Fig. 369, and let this stress be divided into three portions each equal to $\frac{p}{3}$. Without changing the condition of stress it may be assumed that a pull and a push, each of intensity $\frac{p}{3}$, act upon the remaining faces.

Combining the pull $\frac{p}{3}$ on the faces BD and FH, with the push $\frac{p}{3}$ on the faces BE and CH, the result is a simple shear along a plane making an angle of 45° with the axis of the pull.

Combining again a second pull $\frac{p}{3}$ on the faces BD and FH with the push on the faces *AH* and *BG*, the result is also a simple shear in a plane at right angles to the first plane and making an angle of 45° with the axis of the pull. There is now left a pull $\frac{p}{3}$ on each face of the cube, producing a cubic dilatation. Then

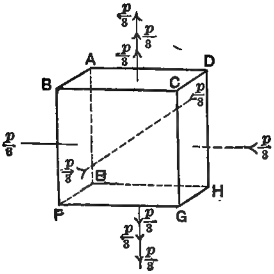


FIG. 369.

the lineal strain due to the bulk strain

$$=\frac{1}{3}\left(\frac{\frac{1}{3}p}{k}\right)=\frac{1}{9}\frac{p}{K};$$

the lineal strain due to the first shearing strain

$$=\frac{1}{2}\left(\frac{\frac{1}{3}p}{G}\right)=\frac{1}{6}\frac{p}{G};$$

the lineal strain due to the second shearing strain

$$=\frac{1}{2}\left(\frac{\frac{1}{3}p}{G}\right)=\frac{1}{6}\frac{p}{G}.$$

Thus the total longitudinal strain $=\frac{1}{9}\frac{p}{K}+\frac{1}{6}\frac{p}{G}+\frac{1}{6}\frac{p}{G}$

$$=\frac{p}{3}\left(\frac{1}{3K}+\frac{1}{G}\right)=\frac{p}{E},$$

and therefore

$$K=\frac{GE}{9G-3E} \quad \text{and} \quad E=\frac{9KG}{3K+G}.$$

Again, the lateral strain $=\frac{1}{6}\frac{p}{G}-\frac{1}{9}\frac{p}{K}=\frac{p}{3}\left(\frac{1}{2G}-\frac{1}{3K}\right)$, and therefore Poisson's ratio is given by

$$\sigma=\frac{\frac{p}{3}\left(\frac{1}{3K}+\frac{1}{G}\right)}{\frac{P}{3}\left(\frac{1}{2G}-\frac{1}{3K}\right)}=\frac{6K+2G}{3K-2G}=\frac{2G}{E-2G},$$

from which

$$G=\frac{\sigma E}{2(\sigma+1)},$$

and therefore

$$K=\frac{\sigma E}{3(\sigma-2)}.$$

The preceding results may also be obtained in the following manner:

Let a solid body be strained uniformly, i.e., in such a manner that lines of particles which are parallel in the free state remain parallel in the strained state, their lengths being altered in a given ratio which is practically very small. Lines of particles which are oblique to each other in the free state are generally inclined at different angles in the strained state, and their lengths are altered in different ratios.

Let the straining of the body convert a rectangular portion $ABCD$ (Fig. 370) into the rectangle $AB'C'D'$, where $AB' = (1+\alpha)AB$ and $AD'=(1+\beta)AD$.

Now α and β are very small, so that their joint effect may be considered to be equal to the sum of their *separate* effects. Hence:

First. Let a simple longitudinal strain in a direction parallel to AB convert the rectangle $ABCD$ into the rectangle $AB'ED$, where $BB'=\alpha\cdot AB$.

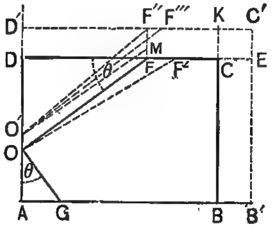


FIG. 370.

A line OF will move into the position OF', when $FF'=\alpha\cdot DF$, and

$$\textit{the strain along } OF=\frac{OF'-OF}{OF}$$

$$=\frac{FF'\cos\theta}{OF}=\frac{\alpha\cdot DF\cos\theta}{OF}=\alpha\cos^2\theta,$$

θ being the angle OFD.

Also, the "*distortion* or *deviation from rectangularity*"

$$=\text{angle } FOF'=\frac{FF'\sin\theta}{OF}=\frac{\alpha\cdot DF\sin\theta}{OF}=\alpha\cos\theta\sin\theta.$$

Second. Let a simple longitudinal strain in a direction parallel to AD convert the rectangle $ABCD$ into the rectangle $ABKD'$, where $DD'=\beta\cdot AD$.

The line OF will move into the position $O'F''$, where $OO'=\beta\cdot AO$

and $F''F = DD' = \beta \cdot AD$; therefore

$$\text{the strain along } OF = \frac{O'F'' - OF}{OF}.$$

Draw $O'M$ parallel to OF. Then

$$O'F'' - OF = O'F'' - O'M = F''M \sin\theta = (F''F - FM) \sin\theta$$

$$= (DD' - OO') \sin\theta = \beta(AD - AO) \sin\theta$$

$$= \beta \cdot OD \sin\theta; \text{ therefore}$$

$$\textit{the strain along } OF = \frac{\beta \cdot OD \sin\theta}{OF} = \beta \sin^2\theta.$$

The *distortion* = the angle $F''O'M$

$$= \frac{F''M \cos\theta}{OF} = \frac{\beta \cdot OD \cos\theta}{OF} = \beta \sin\theta \cos\theta.$$

Hence when the strains are simultaneous, the line OF will take the position $O'F'''$ between $O'F''$ and OF', and

$$\text{the } \textit{total strain} \text{ along } OF = \alpha \cos^2\theta + \beta \sin^2\theta;$$

$$\text{the } \textit{total distortion} = (\alpha - \beta) \sin\theta \cos\theta.$$

Again, draw a line OG perpendicular to OF.
The angle $OGA = 90° - \theta$, and hence, from the above,

$$\text{the } \textit{total strain} \text{ along } OG = \alpha \sin^2\theta + \beta \cos^2\theta,$$

and $$\text{the corresponding distortion} = (\alpha - \beta) \sin\theta \cos\theta.$$

Denote the strain along OF by e_1, that along OG by e_2, and each of the equal distortions by $\frac{\phi}{2}$. Then

$$e_1 + e_2 = \alpha + \beta.$$

Again, if OF, OG, Fig. 371, are the sides of a rectangle enclosed in the rectangle $ABCD$, the straining will convert the rectangle into an oblique figure with its opposite sides parallel. The lengths of adjacent sides are altered by the amounts e_1 and e_2, and the angle

θ by ϕ. The above results may also be considered to hold true if the straining, instead of being uniform, varies continuously from point to point.

Consider a unit cube $ABCD$ subject to stresses of intensity p_1 and p_2 upon the parallel faces AD, BC and AB, DC. By Art. 1, Chap. IV,

$$\alpha=\frac{p_1}{E}-\frac{p_2}{\sigma E},$$

$$\beta=\frac{p_1}{\sigma E}+\frac{p_2}{E},$$

and the strain perpendicular to the face $ABCD=-\frac{p_1}{\sigma E}-\frac{p_2}{\sigma E}$.

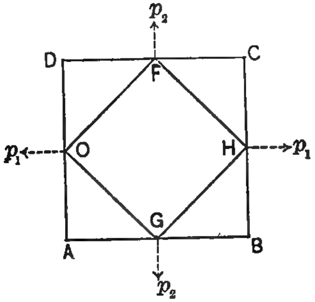


FIG. 371.

If the stresses are of equal intensity but of opposite kind, i.e., if the one is a tension and the other a compression,

$p_1=-p_2=p$, suppose; then

$$\alpha=-\beta=\frac{p}{E}\left(1+\frac{1}{\sigma}\right), \text{ and the third strain is } nil.$$

Thus the volume of the *strained* solid

$$=(1+\alpha)(1-\alpha)(1)=1-\alpha^2=1, \text{ approximately,}$$

so that the volume is not sensibly changed.

Also, if $OGHF$ is an enclosed square, O being the middle point of AD, $\theta=45°$, and

$$e_1=e_2=\frac{\alpha+\beta}{2}=0=\text{strain along } OF \text{ or } OG,$$

and the distortion = change in angle O

$$=\phi=2\frac{\alpha-\beta}{2}\doteq 2\alpha=\frac{2p}{E}\left(1+\frac{1}{\sigma}\right).$$

This result may be at once deduced from the figure. For

$$\tan\frac{FOG}{2}=\frac{OD}{FD}=\frac{1+\beta}{1+\alpha}=\frac{1-\alpha}{1+\alpha}=\tan\left(\frac{90°-\phi}{2}\right),$$

or

$$\frac{1-\alpha}{1+\alpha}=\frac{1-\tan\frac{\phi}{2}}{1+\tan\frac{\phi}{2}}=\frac{1-\frac{\phi}{2}}{1+\frac{\phi}{2}},$$

since ϕ is very small. Hence

$$\frac{\phi}{2}=\alpha.$$

As already shown in Art. 2, shearing cannot take place along one plane only, and at any point of a strained solid the shears along planes at right angles are of equal intensity. The effect of such stresses is merely to produce a *distortion of figure,* and generally without sensible change of volume.

Thus shears of intensity q along the parallel faces of the unit square $ABCD$ will merely distort the square into a rhombus $ABC'D'$ (Fig. 372). Denoting the change of angle by ϕ, and assuming that the "stress is proportional to the strain,"

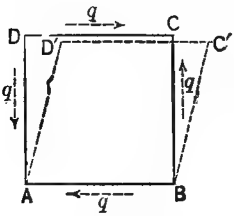


FIG. 372.

$$q=G\phi,$$

where G is a coefficient called the *modulus of rigidity,* and depends upon a *change of form.* It is stated in the same units as are employed to specify the stress.

Consider a section along the diagonal BD.

The stresses on the faces AB, AD, and on CB, CD, resolved parallel and perpendicular to BD, are evidently equivalent to *nil* and a normal force $q\sqrt{2}$ respectively. Thus there is no sliding tendency along BD, but the two portions ABD and CBD exert upon each other a pull, or tension, of intensity $\frac{q\sqrt{2}}{BD}=\frac{q\sqrt{2}}{\sqrt{2}}=q$.

Similarly it may be shown that there is no tendency to slide along AC, but that the two portions ABC and ADC exert upon each other a pressure of intensity q. The straining due to the shearing stresses is, therefore, identical with that produced by a thrust and tension of equal intensity upon planes at 45°. Hence, as proved above,

$$q = G\phi = G\frac{2q}{E}\left(1+\frac{1}{\sigma}\right),$$

and therefore

$$G = \frac{E\sigma}{2(\sigma+1)} = \frac{q}{\phi}.$$

Now σ rarely exceeds 4, and hence G is generally $< \frac{2}{5}E$.

Again,

$$K = \frac{\sigma E}{3(\sigma-2)} = \frac{2}{3}\frac{\sigma+1}{\sigma-2}G.$$

Therefore

$$\sigma = \frac{6K+2G}{3K-2G}.$$

16. General Equations of Stress.—Let x, y, z be the co-ordinates with respect to three rectangular axes of any point O in a strained body.

Consider the equilibrium of an element of the body in the form of an indefinitely small parallelopiped with its edges $OA(=dx)$, $OB(=dy)$, $OC(=dz)$ parallel to the axes of x, y, z. It is assumed that the faces of the element are sufficiently small to allow of the distribution of stress over them being regarded as uniform. The resultant force on each face will therefore be a single force acting at its middle point.

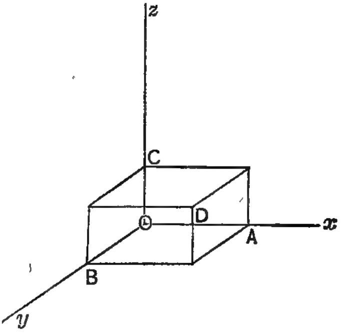


FIG. 373.

Let X_1, Y_1, Z_1 be the components parallel to the axes, x, y, z of the resultant force per unit area on the face BC.

" X_2, Y_2, Z_2 be the corresponding components for the face AC.

" X_3, Y_3, Z_3 be the corresponding components for the face AB.

These components are functions of x, y, z, and therefore become

$$-\left(X_1+\frac{dX_1}{dx}dx\right), \quad -\left(Y_1+\frac{dY_1}{dx}dx\right), \quad -\left(Z_1+\frac{dZ_1}{dx}dx\right)$$

for the adjacent face AD;

$$-\left(X_2+\frac{dX_2}{dy}dy\right),\quad -\left(Y_2+\frac{dY_2}{dy}dy\right),\quad -\left(Z_2+\frac{dZ_2}{dy}dy\right)$$

for the adjacent face BD;

$$-\left(X_3+\frac{dX_3}{dz}dz\right),\quad -\left(Y_3+\frac{dY_3}{dz}dz\right),\quad -\left(Z_3+\frac{dZ_3}{dz}dz\right)$$

for the adjacent face DC.

Hence the total stress parallel to the axis of x

$$=X_1dydz-\left(X_1+\frac{dX_1}{dx}dx\right)dydz+X_2dzdx-\left(X_2+\frac{dX_2}{dy}dy\right)dzdx$$
$$+X_3dxdy-\left(X_3+\frac{dX_3}{dz}dz\right)dxdy$$
$$=-\left(\frac{dX_1}{dx}+\frac{dX_2}{dy}+\frac{dX_3}{dz}\right)dxdydz.$$

Similarly, the total stress parallel to the axis of y

$$=-\left(\frac{dY_1}{dx}+\frac{dY_2}{dy}+\frac{dY_3}{dz}\right)dxdydz,$$

and the total stress parallel to the axis of z

$$=-\left(\frac{dZ_1}{dx}+\frac{dZ_2}{dy}+\frac{dZ_3}{dz}\right)dxdydz.$$

Let ρ be the density of the mass at O, and let P_x, P_y, P_z be the components parallel to the axes of x, y, z of the *external* force, per unit mass, at O.

$\rho dxdydzP_x$ is the component parallel to the axis of x of the external force on the element;

$\rho dxdydzP_y$ is the component parallel to the axis of y of the external force on the element;

$\rho dxdydzP_z$ is the component parallel to the axis of z of the external force on the element.

The element is in equilibrium.

Therefore
$$\left.\begin{aligned}\frac{dX_1}{dx}+\frac{dX_2}{dy}+\frac{dX_3}{dz}&=\rho P_x;\\ \frac{dY_1}{dx}+\frac{dY_2}{dy}+\frac{dY_3}{dz}&=\rho P_y;\\ \frac{dZ_1}{dx}+\frac{dZ_2}{dy}+\frac{dZ_3}{dz}&=\rho P_z.\end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

These are the general equations of stress.

Again, take moments about axes through the centre of the element parallel to the axes of co-ordinates, and neglect terms involving $(dx)^2dydz$, $dx(dy)^2dz$, $dxdy(dz)^2$. Then

$$Y_3=Z_2, \quad Z_1=X_3, \quad \text{and} \quad X_2=Y_1. \quad . \quad . \quad . \quad (2)$$

Adopting Lame's notation, i.e., taking

N_1, N_2, N_3 as the normal intensities of stress at O on planes perpendicular to the axes of x, y, z;

T_1 as the tangential intensity of stress at O on a plane perpendicular to the axis of x if due to a stress parallel to the axis of y, or on a plane perpendicular to the axis of y if due to a stress parallel to the axis of x; and T_2, T_3 similarly, equations (1) become

$$\left.\begin{aligned}\frac{dN_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}&=\rho P_x;\\ \frac{dT_3}{dx}+\frac{dN_2}{dy}+\frac{dT_1}{dz}&=\rho P_y;\\ \frac{dT_2}{dx}+\frac{dT_1}{dy}+\frac{dN_3}{dz}&=\rho P_z.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Next consider the equilibrium of a tetrahedral element having three of its faces parallel to the co-ordinate planes. Let l, m, n be the direction-cosines of the normal to the fourth face.

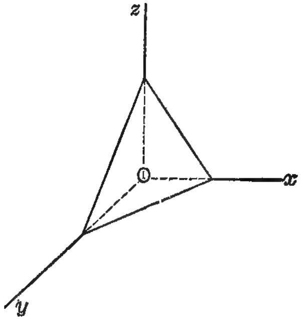


FIG. 374.

Also, let X, Y, Z be the components parallel to the axes of x, y, z of the intensity of stress R on the fourth face.

$$X=lN_1+mT_3+nT_2+\tfrac{1}{3}\rho P_x ldx.$$

But the last term disappears in the limit when the tetrahedron is indefinitely small, and hence

$$\left.\begin{aligned}X&=lN_1+mT_3+nT_2;\\ Y&=lT_3+mN_2+nT_1;\\ Z&=lT_2+mT_1+nN_3.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

These three equations define R in direction and magnitude when the stresses on the three rectangular planes are known.

Let it be required to determine the planes upon which the stress is wholly normal. We have

$$X=lR, \quad Y=mR, \quad Z=nR. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting these values of X, Y, Z in eqs. (4) and eliminating l, m, n, we obtain

$$R^3-R^2(N_1+N_2+N_3)+R(N_1N_2+N_2N_3+N_3N_1)-T_1^2-T_2^2-T_3^2$$
$$-(N_1N_2N_3-N_1T_1^2-N_2T_2^2-N_3T_3^2+2T_1T_2T_3)=0, \quad . \quad . \quad . \quad (6)$$

a cubic equation giving three real values for R, and therefore three sets of values for l, m, and n, showing that there are three planes at O on each of which the intensity of stress is wholly normal. These planes are at right angles to each other and are called *principal planes*, the corresponding stresses being *principal stresses*. They are the principal planes of the quadric

$$N_1x^2+N_2y^2+N_3z^2+2T_1yz+2T_2zx+2T_3xy=c. \quad . \quad . \quad . \quad (7)$$

For the equation to the tangent plane at the extremity of a radius r whose direction-cosines are l, m, n is

$$Xrx+Yry+Zrz=c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and the equation of the parallel diametral plane is

$$Xx+Yy+Zz=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The direction-cosines of the perpendicular to this plane are

$$\frac{X}{R}=l, \quad \frac{Y}{R}=m, \quad \frac{Z}{R}=n,$$

so that the resultant stress R must act in the direction of this perpendicular.

Hence the intensities of stress on the planes perpendicular to the axes of the quadric (7) are wholly normal.

Refer the quadric to its principal planes as planes of reference. All the T's vanish and its equation becomes

$$N_1x^2+N_2y^2+N_3z^2=c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Also, the general equations (3) become

$$\left.\begin{aligned}\frac{dN_1}{dx}&=\rho P_x;\\ \frac{dN_2}{dy}&=\rho P_y;\\ \frac{dN_3}{dz}&=\rho P_z.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Again,

$$\left(\frac{X}{N_1}\right)^2+\left(\frac{Y}{N_2}\right)^2+\left(\frac{Z}{N_3}\right)^2=l^2+m^2+n^2=1. \quad . \quad . \qquad (12)$$

Consider X, Y, Z as the co-ordinates of the extremity of the straight line representing R in direction and magnitude. Equation (12) is then the equation to an ellipsoid whose semi-axes are N_1, N_2, N_3. As a plane at O turns around O as a fixed centre, the extremity of a line representing the intensity of stress R on the plane will trace out an ellipsoid. This ellipsoid is called the *ellipsoid of stress.*

Note 1. The coefficients in the cubic equation (6) are invariants. Thus, $N_1+N_2+N_3$ is constant, or the sum of three normal intensities of stress on three planes placed at right angles at any point of a strained body is the same for all positions of the three planes.

Note 2. The perpendicular p from O on the tangent plane, equation (8),

$$=\frac{c}{Rr}=p, \quad \text{and therefore} \quad R=\frac{c}{pr}. \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Note 3. Let the stress be the same for *all* positions of the plane at O. Then $N_1=N_2=N_3$, and the ellipsoid (12) becomes a sphere. The stress is therefore everywhere normal, and the body must be a perfect fluid. Conversely, if the stress is everywhere normal, the body must be a perfect fluid, the ellipsoid becomes a sphere, and therefore $N_1=N_2=N_3$.

17. Relation between Stress and Strain.—In Art. 15 it was shown that when the size and figure of a body are altered in two dimensions, there is an *ellipse of strain* analogous to the ellipse of stress. If the alteration takes place in three dimensions, it may be similarly shown that every state of strain may be represented by an *ellipsoid of strain* analogous to the ellipsoid of stress. The axes of the ellipsoid are the principal axes of strain, and every strain may be resolved into three simple strains parallel to these axes.

It is assumed that the strains remain very small, that the stresses developed are proportional to the corresponding strains, and that their effects may be superposed.

Consider an element of the unstrained body in the form of a rectangular parallelopiped having its edges $PQ(=h)$, $PR(=k)$, $PS(=l)$ parallel to the axes of co-ordinates.

When the body is strained, the element becomes distorted, the new edges being $P'Q'$, $P'R'$, $P'S'$.

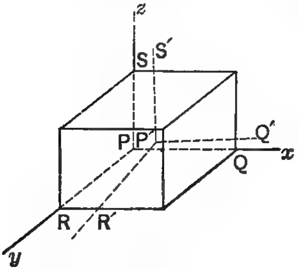


FIG. 375.

Let x, y, z be the co-ordinates of P.

Let $x+u$, $y+v$, $z+w$ be the co-ordinates of P'.

By Taylor's Theorem the co-ordinates with respect to P' of

$$Q' \text{ are } h\left(1+\frac{du}{dx}\right),\quad h\frac{dv}{dx},\quad h\frac{dw}{dx};$$

$$R' \text{ are } k\frac{du}{dy},\quad k\left(1+\frac{dv}{dy}\right),\quad k\frac{dw}{dy};$$

$$S' \text{ are } l\frac{du}{dz},\quad l\frac{dv}{dz},\quad l\left(1+\frac{dw}{dz}\right).$$

Therefore

$$\left.\begin{aligned} P'Q' &= h\left(1+\frac{du}{dx}\right);\\ P'R' &= k\left(1+\frac{dv}{dy}\right);\\ P'S' &= l\left(1+\frac{dw}{dz}\right).\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (14)$$

$$\left.\begin{array}{l}\text{Hence strain parallel to axis of } x=\dfrac{P'Q'-PQ}{PQ}=\dfrac{du}{dx}=Ee_x;\\ \text{`` \quad `` \quad `` \quad `` `` ``} \quad y=\dfrac{P'R'-PR}{PR}=\dfrac{dv}{dy}=Ee_y;\\ \text{`` \quad `` \quad `` \quad `` `` ``} \quad z=\dfrac{P'S'-PS}{PS}=\dfrac{dw}{dz}=Ee_z.\end{array}\right\} \quad . \quad (15)$$

Again, $\cos Q'P'R'$

$$=\frac{\left(1+\dfrac{du}{dx}\right)\dfrac{du}{dy}+\left(1+\dfrac{dv}{dy}\right)\dfrac{dv}{dx}+\dfrac{dw}{dy}\dfrac{dw}{dx}}{\left[\left\{1+\left(\dfrac{du}{dx}\right)^2+\left(\dfrac{dv}{dx}\right)^2+\left(\dfrac{dw}{dx}\right)^2\right\}\left\{\left(\dfrac{du}{dy}\right)^2+\left(1+\dfrac{dv}{dy}\right)+\left(\dfrac{dw}{dy}\right)^2\right\}\right]^{\frac{1}{2}}}.$$

In the limit this reduces to

$$\left.\begin{array}{ll} & \cos Q'P'R'=\dfrac{du}{dy}+\dfrac{dv}{dx}.\\ \text{Similarly,} & \cos Q'P'S'=\dfrac{du}{dz}+\dfrac{dw}{dx};\\ & \cos R'P'S'=\dfrac{dw}{dy}+\dfrac{dv}{dz}.\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Volume of unstrained element $=hkl$;

Volume of distorted element $=hkl\left(1+\dfrac{du}{dx}\right)\left(1+\dfrac{dv}{dy}\right)\left(1+\dfrac{dw}{dz}\right)$

multiplied by the cosines of small angles

$$=hkl\left(1+\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right) \quad \text{in the limit.}$$

Therefore

$$\frac{\text{Difference of volume}}{\text{Vol. of unstrained element}}=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} \quad . \quad . \quad . \quad (17)$$

$=E(e_x+e_y+e_z)$

$=$ the *volume* or *cubic* strain.

18. Isotropic Bodies, i.e., bodies possessing the same elastic properties in all directions.

A normal stress of intensity N_1 parallel to the axis of x produces a simple longitudinal strain $\frac{N_1}{E}$, and two simple lateral strains, each $=-\frac{N_1}{\sigma E}$, parallel to the axes of y and z, E being the ordinary modulus of elasticity and σ Poisson's ratio (Art. 1, Chap. IV).

Normal stresses N_2, N_3 parallel to the axes of y and z may be similarly treated.

Let the three normal stresses act simultaneously and superpose the results. Then

$$\left.\begin{array}{ll}\text{total strain parallel to axis of } & x=\frac{N_1}{E}-\frac{N_2+N_3}{\sigma E}=\frac{du}{dx}=Ee_x;\\ \text{`` `` `` `` `` ``} & y=\frac{N_2}{E}-\frac{N_3+N_1}{\sigma E}=\frac{dv}{dy}=Ee_y;\\ \text{`` `` `` `` `` ``} & z=\frac{N_3}{E}-\frac{N_1+N_2}{\sigma E}=\frac{dw}{dz}=Ee_z.\end{array}\right\} \quad . \quad (18)$$

The form in which these equations are given is due to Grashof. Solving for N_1, N_2, N_3,

$$\left.\begin{array}{l}N_1=\frac{\sigma(\sigma-1)E^2}{(\sigma+1)(\sigma-2)}e_x+\frac{\sigma E^2}{(\sigma+1)(\sigma-2)}(e_y+e_z),\\ N_2=\frac{\sigma(\sigma-1)E^2}{(\sigma+1)(\sigma-2)}e_y+\frac{\sigma E^2}{(\sigma+1)(\sigma-2)}(e_z+e_x),\\ N_3=\frac{\sigma(\sigma-1)E^2}{(\sigma+1)(\sigma-2)}e_z+\frac{\sigma E^2}{(\sigma+1)(\sigma-2)}(e_x+e_y),\end{array}\right\} \quad . \quad . \quad . \quad (19)$$

which may be written in the form

$$\left.\begin{array}{l}N_1=AEe_x+\lambda E(e_y+e_z),\\ N_2=AEe_y+\lambda E(e_z+e_x),\\ N_3=AEe_z+\lambda E(e_x+e_y),\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where

$$A=\frac{\sigma(\sigma-1)E}{(\sigma+1)(\sigma-2)},$$

$$\lambda=\frac{\sigma E}{(\sigma+1)(\sigma-2)}.$$

and λ is the *coefficient of dilatation.*

Again, the straining changes the angle RPS by an amount $\frac{dw}{dy}+\frac{dv}{dz}$, producing two tangential stresses, each equal to $G\left(\frac{dw}{dy}+\frac{dv}{dz}\right)$, parallel to the axes of y and z. Therefore

Similarly,

$$\left.\begin{aligned} T_1&=G\left(\frac{dw}{dy}+\frac{dv}{dz}\right). \\ T_2&=G\left(\frac{du}{dz}+\frac{dw}{dx}\right), \\ T_3&=G\left(\frac{du}{dy}+\frac{dv}{dx}\right). \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (21)$$

G is called the coefficient of *rigidity* or *transverse elasticity.* It is designated n in Thomson and Tait's notation, and μ in Lamé's notation.

Relation between A, λ, and G.—Equations (20) and (21) preserve the same forms whatever rectangular axes may be chosen.

Keep the axis of z fixed and turn the axes of x and y through an angle α.

Let N_1' be the normal stress parallel to the new axis of x. Then

$$N_1'=N_1\cos^2\alpha+N_2\sin^2\alpha+2T_3\sin\alpha\cos\alpha. \; . \; . \quad (22)$$

Let x', y' and u', v' be the new co-ordinates and displacements.

Therefore $$N_1'=A\frac{du'}{dx'}+\lambda\left(\frac{dv'}{dy'}+\frac{dw'}{dz'}\right)=(A-\lambda)\frac{du'}{dx'}+\lambda\theta. \; . \; . \quad (23)$$

For $\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}$, $=\theta=\frac{du'}{dx'}+\frac{dv'}{dy'}+\frac{dw'}{dz'}$, is an invariant.

The values of N_1' given by eqs. (22) and (23) must be identical. Now

$$\left.\begin{aligned} x&=x'\cos\alpha-y'\sin\alpha, \quad y=x'\sin\alpha+y'\cos\alpha, \\ u'&=u\cos\alpha+v\sin\alpha, \quad v'=-u\sin\alpha+v\cos\alpha. \end{aligned}\right\} \; . \; . \quad (24)$$

Hence $$\frac{du'}{dx'}=\frac{du}{dx'}\cos\alpha+\frac{dv}{dx'}\sin\alpha$$

$$=\frac{du}{dx}\cos^2\alpha+\frac{dv}{dy}\sin^2\alpha+\left(\frac{du}{dy}+\frac{dv}{dx}\right)\sin\alpha\cos\alpha$$

$$=\frac{du}{dx}\cos^2\alpha+\frac{dv}{dy}\sin^2\alpha+\frac{T_3}{G}\sin\alpha\cos\alpha;$$

and by eq. (23),

$$N_1' = (A-\lambda)\left(\frac{du}{dx}\cos^2\alpha + \frac{dv}{dy}\sin^2\alpha + \frac{T_3}{G}\sin\alpha\cos\alpha\right) + \lambda\theta. \qquad (25)$$

Also, by eqs. (20) and (22),

$$N_1' = (A-\lambda)\left(\frac{du}{dx}\cos^2\alpha + \frac{dv}{dy}\sin\alpha + \frac{2T_3}{A-\lambda}\sin\alpha\cos\alpha\right) + \lambda\theta. \qquad (26)$$

Eqs. (25) and (26) must be identical.

Therefore
$$G = \frac{A-\lambda}{2} = \frac{\sigma E}{2(\sigma+1)} = \mu = n. \quad . \quad . \quad . \quad . \qquad (27)$$

Adding together eqs. (20),

$$N_1 + N_2 + N_3 = (A+2\lambda)E(e_x+e_y+e_z)$$
$$= \frac{\sigma E^2}{\sigma-2}(e_x+e_y+e_z).$$

It may be easily shown that the normal stresses can each be separated into a fluid pressure p and a distorting stress.

Hence, putting

$$N_1 = N_2 = N_3 = p = \frac{E_2}{3(\sigma-2)}(e_x+e_y+e_z),$$

$$\text{the cubic elasticity} = \frac{p}{\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}} = \frac{\sigma E}{3(\sigma-2)} = K. \quad . \quad . \quad . \qquad (28)$$

19. Applications.—1. *Traction.*—One end of a cylindrical bar of isotropic material is fixed and the bar is stretched in the direction of its length. The axis of the bar is the only line not moved laterally by contraction.

Take this line as the axis of x.

The displacements u, v, w of any point x, y, z may be expressed in the form

$$u = \alpha x, \quad v = -\beta y, \quad w = -\beta z. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

By eqs. (20) and (29),

$$N_1 = \alpha A - 2\beta\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

$$N_2 = -\beta A + \lambda(\beta+\alpha) = N_3. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

By eqs. (21) and (29), all the tangential stresses vanish.

Hence, since N_1, N_2, N_3 are constant, and since the equations of internal equilibrium contain only differential coefficients of the stresses, the hypothesis, eq. (29), satisfies these equations.

First. Let $N_2=0=N_3$; i.e., let no external force act upon the curved surface. Then

$$-\beta A+\lambda(\beta+\alpha)=0,$$

or

$$\frac{\beta}{\alpha}=\frac{\lambda}{A+\lambda}=\frac{1}{\sigma}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Thus, the coefficient of contraction is less than the coefficient of expansion.

Again, by eqs. (30) and (32),

$$\frac{N_1}{\alpha}=A-2\lambda\frac{\beta}{\alpha}=A-\frac{2\lambda}{\sigma}=E. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Second. If the bar, instead of being free to move laterally, has its surface acted upon by a uniform pressure P, then

$$N_2=N_3=P.$$

By eqs. (31) and (32),

$$\frac{\beta}{\alpha}=\frac{AP-\lambda N_1}{\lambda(N_1+2P)-AN_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

For example, let P be sufficient to prevent lateral contraction. Then $\beta=0$ and, by eqs. (31) and (35),

$$\alpha A=N_1=\frac{AP}{\lambda}=(\sigma-1)P.$$

2. *Torsion.*—(*a*) Let a circular cylinder (hollow or solid) of length l undergo torsion around its axis (the axis of x), and let t be the angle through which one end is twisted relatively to the other. A point in a transverse section distant x from the latter will be twisted through an angle $x\frac{t}{l}$.

The displacements u, v, w of the point x, y, z in this section may be expressed in the form

$$u=0, \quad v=-zx\frac{t}{l}, \quad w=+yx\frac{t}{l}.$$

By eqs. (20) and (21),

$$N_1=0=N_2=N_3,$$

and

$$T_1=0, \quad T_2=+Gy\frac{t}{l}, \quad T_3=-Gz\frac{t}{l}.$$

The algebraic sum of the moments of T_2, T_3 with respect to the axis

$$=G\frac{t}{l}(y^2+z^2)=G\frac{t}{l}r^2,$$

r being the distance of the point (x, y, z) from the axis.

Hence the moment $M=Pp$ (Chap. IX), of the couple producing torsion

$$=G\frac{t}{l}\int r^2dS=G\frac{t}{l}J=G\theta J,$$

dS being an element of the area at (x, y, z), J the polar moment of inertia, and θ the torsion per unit of length of the cylinder, or the *rate of twist.*

The torsional rigidity of a solid cylinder

$$=\frac{M}{\theta}=GJ=\frac{G}{2}\pi R^4,$$

R being the radius of the cylinder.

(*b*) Torsion of a bar of *elliptic* section.

The displacements u, v, w may now be expressed in the form

$$u=F(y, z), \quad v=-\theta xz, \quad w=\theta xy.$$

Therefore
$$\frac{du}{dx}=0=\frac{dv}{dy}=\frac{dw}{dz},$$

$$N_1=0=N_2=N_3,$$

$$T_1=0,\quad T_2=G\left(\frac{du}{dz}+\theta y\right),\quad T_3=G\left(\frac{du}{dy}-\theta z\right). \quad . \quad . \quad (36)$$

Hence, by the general eqs. (3),

$$\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Also, the surface stresses are zero, and therefore

$$T_3\frac{dz}{ds}-T_2\frac{dy}{ds}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Hence, by eqs. (35),

$$\frac{du}{dy}dz-\frac{du}{dz}dy=\theta(zdz+ydy). \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

This equation must hold true at the surface.
Let the equation to the elliptic section be

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Then

$$\frac{dz}{dy}=-\frac{c^2y}{b^2z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

and, by eq. (38),

$$c^2y\frac{du}{dy}+b^2z\frac{du}{dz}=-\theta yz(b^2-c^2). \quad . \quad . \quad . \quad . \quad (42)$$

$u=dyz$ satisfies this last equation and also eq. (36), if

$$d=-\theta\frac{b^2-c^2}{b^2+c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Again, the algebraic sum of the moments T_2, T_3 with respect to the axis of x

$$=G\left(\frac{du}{dz}+\theta y\right)y-G\left(\frac{du}{dy}-\theta z\right)z$$

$$=G\{(d+\theta)y^2-(d-\theta)z^2\}$$

$$=\frac{2G\theta}{b^2+c^2}(c^2y^2+b^2z^2). \quad . \quad . \quad . \quad (44)$$

The total moment (M) of the couple producing torsion

$$=\frac{2G\theta}{b^2+c^2}\int(c^2y^2+b^2z^2)dS$$

$$=G\theta\frac{\pi b^3c^3}{b^2+c^2},$$

and the torsional rigidity

$$=\frac{M}{\theta}=G\frac{\pi b^3c^3}{b^2+c^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

(c) Torsion of a bar of rectangular section.

As in case (b), u must satisfy the equation

$$\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Also, the equations of condition corresponding to eq. (38) are

$$\frac{du}{dy}-\theta z=0 \quad \text{when} \quad y=\pm b, \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

and

$$\frac{du}{dz}+\theta y=0 \quad \text{when} \quad z=\pm c; \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

$2b$ and $2c$ ($b<c$) being the sides of the rectangle. The total moment of torsion, viz., $\int(T_2y-T_3z)dS$, is then found to be

$$M=16b^3cG\theta\left\{\frac{1}{3}-\frac{64}{\pi^5}\frac{b}{c}\sum_0^\infty\frac{\tan h(2n+1)\frac{\pi c}{2b}}{(2n+1)^5}\right\}. \quad . \quad (49)$$

If $b=c$, i.e., if the section is a square, eq. (48) becomes

$$M=.843462JG\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

J ($=\frac{8}{3}b^4$) being the polar moment of inertia with respect to the axes. (Chap. IX.)

If $\frac{b}{c}$ is very small, eq. (48) becomes

$$M=16b^3cG\theta\left(\frac{1}{3}-.21\frac{b}{c}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

The torsional rigidity of a rectangular section is sometimes expressed by the formula

$$\frac{M}{\theta}=\frac{5}{18}\frac{b^3c^3}{b^2+c^2}G. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52)$$

For the further treatment of this subject the student is referred to St. Venant's edition of Clebsch, and to Thomson and Tait's Natural Philosophy.

3. *Work done in the small strain of a body* (Clapeyron's Theorem).—Multiply eqs. (3) by $u\,dx\,dy\,dz$, $v\,dx\,dy\,dz$, $w\,dx\,dy\,dz$, and find the triple integral of their sum throughout the whole of the solid.

The terms involving the components P_x, P_y, P_z may be disregarded, as the deformations due to their action are generally inappreciable.

Also,

$$\int\int\int\frac{dN_1}{dx}u\,dx\,dy\,dz$$

$$=\int\int(N_x'u_x'-N_x''u_x'')dy\,dz-\int\int\int N_1\frac{du}{dx}dx\,dy\,dz;$$

N_x', N_x'' being the values of N_1 at the two points in which the line parallel to the axis of x cuts the surface of the body, and u_x' u_x'' the corresponding values of u.

Let dS, dS' be the elementary areas of the surface at these points and l', l'' the cosines of the angles between the normals to these elements and the axis of x.

The double integral on the right-hand side of the last equation then becomes

$$\int\int(N_x'l'u_x'dS-N_x''l''u_x''dS')=\Sigma(N_1ludS).$$

Treating the other terms similarly,

$$0=\Sigma\{(N_1l+T_3m+T_2n)u+(T_3l+N_2m+T_1n)v$$
$$+(T_2l+T_1m+N_3n)w\}dS$$
$$-\int\int\int dx\,dy\,dz\left\{N_1\frac{du}{dx}+N_2\frac{dv}{dy}+N_3\frac{dw}{dz}\right.$$
$$\left.+T_1\left(\frac{dv}{dz}+\frac{dw}{dy}\right)+T_2\left(\frac{dw}{dx}+\frac{du}{dz}\right)+T_3\left(\frac{du}{dy}+\frac{dv}{dx}\right)\right\}.$$

Hence *the work done* $=\frac{1}{2}\Sigma(Xu+Yv+Zw)dS$

$$=\tfrac{1}{2}\int\int\int dx\,dy\,dz\left\{\frac{\lambda+G}{G(3\lambda+2G)}(N_1+N_2+N_3)^2\right.$$
$$\left.-\frac{1}{G}(N_1N_2+N_2N_3+N_3N_1-T_1^2-T_2^2-T_3^2)\right\}$$

$$=\tfrac{1}{2}\int\int\int dx\,dy\,dz\left\{\frac{(N_1+N_2+N_3)^2}{E}\right.$$
$$\left.-\frac{N_1N_2+N_2N_3+N_3N_1-T_1^2-T_2^2-T_3^2}{G}\right\},$$

E being the ordinary modulus of elasticity.

20. Transmission of Energy.—Consider a small parallelopiped of the material, moving in any direction OR with a velocity v, the components of v parallel to the axes OX, OY, OZ being v_x, v_y, and v_z, respectively.

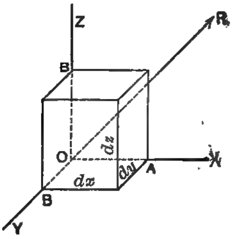


FIG. 376.

Let f_{xx}, f_{yy}, f_{zz}, . . . be the mean intensities of stress over the faces which are at right angles to the axis defined by the first suffix, and in the direction of the axis defined by the second suffix.

Let T_x, T_y, T_z be the energies transmitted across the three faces perpendicular to OX, OY, and OZ, respectively. Then

$$T_x=dy\,dz(f_{xx}v_x+f_{xy}v_y+f_{xz}v_z)+\text{energy of resilience}+\text{energy of motion}$$
$$=dy\,dz(f_{xx}v_x+f_{xy}v_y+f_{xz}v_z)+dx\,dy\,dz\left\{\left(\frac{p^2_{xx}}{2E}+\frac{p^2_{xy}}{2G}+\frac{p^2_{xz}}{2G}\right)+\frac{\rho v^2}{2g}\right\}.$$

Disregarding the energies of resilience and motion,

$$T_x=dy\,dz(f_{xx}v_x+f_{xy}v_y+f_{xz}v_z).$$

So also $$T_y=dz\,dx(f_{yx}v_x+f_{yy}v_y+f_{yz}v_z)$$

and $$T_z=dx\,dy(f_{zx}v_x+f_{zy}v_y+f_{zz}v_z).$$

Ex. 15. *Find the least diameter of a shaft which will transmit* 1000 *H.P. at a surface velocity of* 10 *f/s. and a maximum working stress of* 10,000 *lbs.*

Let f_r, v_r f/s. be the stress and velocity at any radius r, and let θ be the inclination of the radius to the axis Oy. Then

$$f_{xx}=0, \quad f_{xy}=-f_r \sin\theta, \quad f_{xz}=f_r\cos\theta;$$

$$v_x=0, \quad v_y=-v_r\sin\theta, \quad v_z=v_r\cos\theta.$$

Therefore $\quad T_x=dy\,dz\,(f_rv_r\sin^2\theta+f_rv_r\cos^2\theta)=dy\,dz\,f_rv_r.$

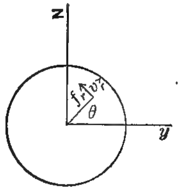


FIG. 377.

If R is the radius of the shaft, f the surface stress, and V the surface velocity,

$$f_r=\frac{r}{R}f \quad \text{and} \quad v_r=\frac{r}{R}V.$$

Therefore $$T_x=dy\,dz\frac{fV}{R^2}r^2,$$

and the total work transmitted across the section

$$=\Sigma(T_x)=\frac{fV}{R^2}\Sigma(r^2\,dy\,dz)=\frac{fV}{R^2}\frac{\pi R^4}{2}.$$

Hence, $$\frac{10000\times10}{7}11R^2=550\times1000,$$

$$\text{or} \quad R^2=3.5$$

$$\text{and} \quad R=1.87 \text{ ins.}$$

21. Thick Cylinders.—If the thickness of the cylinder-shell is large as compared with the radius, it cannot be assumed that the stress is uniformly distributed over this thickness. As in the case of thin cylinders, however, it will be assumed—

(*a*) that the metal is homogeneous and free from initial strain;

(*b*) that the pressures are uniformly distributed over the internal and external surfaces.

It will also be assumed—

(*c*) that the cylinder-ends are free;

(*d*) that the annulus forming the section of the cylinder is composed of an infinite number of concentric rings.

Under these conditions the straining of the cylinder cannot affect its cylindrical form. Hence right sections of the cylinder in the unstrained state remain planes after the straining, so that the longitudinal strain, i.e., the strain in the direction of the cylinder's length, is the same at every point.

Consider an elementary ring bounded by the radii r and $r+dr$. At any point of this ring,

Let q be the normal (i.e., radial) stress;

" f be the hoop stress, i.e., the stress tangential to the ring;

" s be the stress at right angles to the plane of the ring;

" α, β, γ be the corresponding strains;

" E, σE be coefficients of direct and lateral elasticity;

" q_0, f_0 and q_1, f_1 be the values of q and f at the outer and inner surfaces of the cylinder respectively.

Then, since q, f, and s are principal stresses,

$$\alpha=\frac{q}{E}-\frac{f+s}{\sigma E},\quad \beta=\frac{f}{E}-\frac{s+q}{\sigma E},\quad \text{and}\quad \gamma=\frac{s}{E}-\frac{f+q}{\sigma E}. \quad . \quad . \quad (1)$$

But γ is constant. Also, since the ends are free, the total longitudinal pressure on a transverse section is *nil*, and hence it may be inferred that s is zero at every point. Adopting this value of s,

$$f+q=\text{a constant}=c.$$

The radial stress q diminishes from the inner to the outer surface and may be represented by the ordinates of some curve q_0qq_1, in which r_0q_0, rq, and r_1q_1 define the radial stresses at the outer surface of radius r_0, at the radius r, and at the inner surface of radius r_1. Then

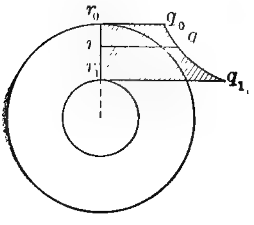


FIG. 378.

$2\times$area $r_0q_0q_1r_1=$ the total resistance to separation between the two halves of the cylinder

$$=2(q_1r_1-q_0r_0),$$

or area $r_0q_0q_1r_1=q_1r_1-q_0r_0$.

So, area $rqq_1r_1=q_1r_1-qr=-\int_{r_1}^{r}fdr$,

the sign being negative as f and q are opposite kinds of stresses, i.e., one is a tension and the other a pressure.

Therefore $$\frac{d}{dr}(qr)=f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Take $$f=\frac{f+q}{2}+\frac{f-q}{2}=B+C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and $$q=\frac{f+q}{2}-\frac{f-q}{2}=B-C. \qquad (4)$$

The two principal stresses f and q may now be regarded as consisting of a pair of equal stresses of the same kind and of intensity $B\left(=\frac{f+q}{2}\right)$, and a pair of equal stresses of opposite kind and of intensity $C\left(=\frac{f-q}{2}\right)$.

First. Assume that only the stresses $\pm C$ are acting and that therefore

$$B=0=\frac{f+q}{2}, \quad \text{or} \quad f=-q.$$

Then $$f=-q=\frac{d}{dr}(qr)=q+r\frac{dq}{dr},$$

or $$2q+r\frac{dq}{dr}=0,$$

and therefore $$qr^2=\text{a constant}=A.$$

Thus $$C=\frac{f+q}{2}=-q=-\frac{A}{r^2}. \qquad (5)$$

Second. Assume that only the stresses B are acting and that therefore

$$C=0=\frac{f-q}{2}, \quad \text{or} \quad f=q.$$

Then $$f=q=\frac{d}{dr}(qr)=q+r\frac{dq}{dr},$$

or $$\frac{dq}{dr}=0.$$

Therefore $$q=\text{a constant }=f=B. \qquad (6)$$

Third. Assuming now that the two pairs of stresses act together and superposing the results just obtained,

$$f=B-\frac{A}{r^2}. \qquad (7)$$

and
$$q=B+\frac{A}{r^2}. \qquad (8)$$

But $q=q_0$ when $r=r_0$, and $q=q_1$ when $r=r_1$.

Therefore
$$q_0=B+\frac{A}{r_0^2} \quad \text{and} \quad q_1=B+\frac{A}{r_1^2},$$

from which it follows that

$$A=\frac{(q_1-q_0)r_1^2r_0^2}{r_0^2-r_1^2} \quad \text{and} \quad B=\frac{q_0r_0^2-q_1r_1^2}{r_0^2-r_1^2}.$$

Hence
$$q=\frac{q_0r_0^2-q_1r_1^2}{r_0^2-r_1^2}+\frac{q_1-q_0}{r^2}\,\frac{r_1^2r_0^2}{r_0^2-r_1^2} \qquad (9)$$

and
$$f=\frac{q_0r_0^2-q_1r_1^2}{r_0^2-r_1^2}-\frac{q_1-q_0}{r^2}\,\frac{r_1^2r_0^2}{r_0^2-r_1^2}. \qquad (10)$$

When $r=r_1$, $f=f_1$; and when $r=r_0$, $f=f_0$.

Therefore
$$\left.\begin{aligned} f_1&=\frac{2q_0r_0^2-q_1(r_0^2+r_1^2)}{r_0^2-r_1^2} \\ \text{and} \quad f_0&=\frac{q_0(r_0^2+r_1^2)-2q_1r_1^2}{r_0^2-r_1^2}. \end{aligned}\right\} \qquad (11)$$

Eqs. (11) are employed in the design of gun-barrels.

Again, take $q_0=0$. Then

$$\left.\begin{aligned} f_1&=-q_1\frac{r_0^2+r_1^2}{r_0^2-r_1^2} \\ \text{and} \quad f_0&=-q_1\frac{2r_1^2}{r_0^2-r_1^2}. \end{aligned}\right\} \qquad (12)$$

Therefore
$$\frac{f_1}{f_0}=\frac{r_0^2+r_1^2}{2r_1^2}>1.$$

Eqs. (12) are used in the design of the cylinders for hydraulic presses, accumulators, etc. Also, since $q_1=-f_1\dfrac{r_0^2-r_1^2}{r_0^2+r_1^2}$, q_1 must always be less than f_1 whatever the thickness of the cylinder may be, and if f_1 is the safe working stress of the material, q_1 is the maximum intensity of pressure to which the cylinder can be subjected. This result is, of course, based on the assumption that $q_0=0$.

By eq. (10), since $f=f_1$ when $r=r_1$, then

$$f_1=\frac{q_0r_0^2-q_1r_1^2}{r_0^2-r_1^2}-\frac{(q_1-q_0)r_0^2}{r_0^2-r_1^2}=\frac{2q_0r_0^2-q_1(r_1^2+r_0^2)}{r_0^2-r_1^2}.$$

Therefore

$$\frac{r_1^2}{r_0^2}=\frac{f_1+q-2q_0}{f_1-q}=\frac{f_1+q}{f_1-q}, \text{ approximately, if } q_0 \text{ is small.}$$

Hence

$$\frac{r_1}{r_0}=\left(\frac{f_1+q}{f_1-q}\right)^{\frac{1}{2}}=\left(1+\frac{q}{f}\right)^{\frac{1}{2}}\left(1-\frac{q}{f}\right)^{-\frac{1}{2}}$$

$$=\left(1+\frac{1}{2}\frac{q}{f}-\frac{1}{8}\frac{q^2}{f^2}\right)\left(1+\frac{1}{2}\frac{q}{f}+\frac{3}{8}\frac{q^2}{f^2}\right)$$

$$=1+\frac{q}{f}+\frac{1}{2}\frac{q^2}{f^2}, \text{ approximately.}$$

Therefore

$$\frac{r_1}{r_0}-1=\frac{\text{thickness of cylinder}}{r_0}=\frac{q}{f}\left(1+\frac{1}{2}\frac{q}{f}\right). \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

The result given by eq. (13) may be used as a second approximation for the thickness, a first approximation being that given by eq. (1) in Chap. IV, Art. 8.

Ex. 16. *The barrel of a gun is made up of three rings A, B, C, the bore of the gun and the diameters of the rings being in the ratio of 2:4:5:6. The ring B is shrunk upon A, and the ring C upon B, the resulting pressures at the surfaces of contact being 4 tons/sq. in. If the firing of the gun produces an internal pressure of 20 tons/sq. in., find the stresses induced in the gun.*

Let r_1, r_2, r_3, r_4 be the radii of the four surfaces;

q_1, q_2, q_3, q_4 " " radial stresses at "
f_1, f_2, f_3, f_4 " " hoop stresses " " before firing;
F_1, F_2, F_3, F_4 " " " " " " after "

Before firing. For ring A, $q_1=0$, $q_2=4$ t./sq. in. Therefore

$$f_1=\frac{8r_2^2}{r_2^2-r_1^2}=\frac{32}{3} \text{ t/sq. in.} \quad \text{and} \quad f_2=4\,\frac{r_2^2+r_1^2}{r_2^2-r_1^2}=\frac{20}{3} \text{ t/sq. in.}$$

For ring B, $q_2=4$ t/sq. in. $=q_3$. Therefore

$$f_2=\frac{8r_3^2-4(r_3^2+r_2^2)}{r_3^2-r_2^2}=4 \text{ t/sq. in.} \quad \text{and} \quad f_3=\frac{4(r_3^2+r_2^2)-8r_2^2}{r_3^2-r_2^2}=4 \text{ t/sq. in.}$$

For ring C, $q_3=$ 4 t./sq. in., $q_4=0$. Therefore

$$f_3=-4\frac{r_4^2+r_3^2}{r_4^2-r_3^2}=-\frac{244}{11}\text{t/sq. in.} \quad \text{and} \quad f_4=-\frac{8r_3^2}{r_4^2-r_3^2}=-\frac{200}{11} \text{ t/sq. in.}$$

After firing. Let P be the pressure per square inch at the surface between A and B.

Let Q be the pressure per square inch at the surface between B and C.

For ring A, $q_1=20$, $q_2=P$. Therefore

$$F_1=\frac{2Pr_2^2-20(r_2^2+r_1^2)}{r_2^2-r_1^2}=\frac{8}{3}P-\frac{100}{3} \quad \text{and} \quad F_2=\frac{5}{3}P-\frac{40}{3}.$$

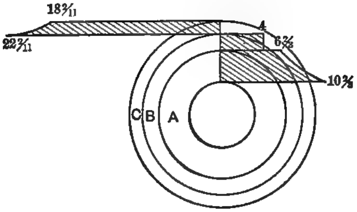


FIG. 379.—Initial Stresses in Metal before Firing.

For ring B, $q_2=P$, $q_3=Q$. Therefore

$$F_2=\frac{2Qr_3^2-P(r_3^2+r_2^2)}{r_3^2-r_2^2}=\frac{50}{9}Q-\frac{41}{9}P$$

and

$$F_3=\frac{Q(r_3^2+r_2^2)-2Pr_2^2}{r_3^2-r_2^2}=\frac{41}{9}Q-\frac{32}{9}P.$$

Assuming that the F_2 for ring A is equal to the F_2 for ring B, then

$$\frac{5}{3}P-\frac{40}{3}=\frac{50}{9}Q-\frac{41}{9}P,$$

or

$$28P-25Q=60.$$

For ring C, $q_3=Q$ and $q_4=0$. Therefore

$$F_3=-Q\,\frac{r_4^2+r_3^2}{r_4^2-r_3^2}=-\frac{61}{11}Q$$

$$F_4=-2Q\,\frac{r_3^2}{r_4^2-r_3^2}=-\frac{50}{11}Q.$$

Assuming that the F_3 for ring B is equal to the F_3 for ring C, then

$$\tfrac{41}{9}Q-\tfrac{32}{9}P=-\tfrac{61}{11}Q,$$

or
$$1000Q=352\,P,$$

or
$$Q=.352\,P.$$

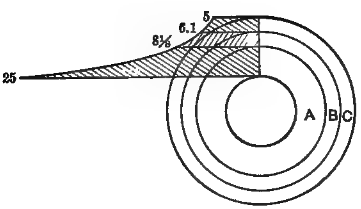


FIG. 380.—Stresses Induced by Firing.

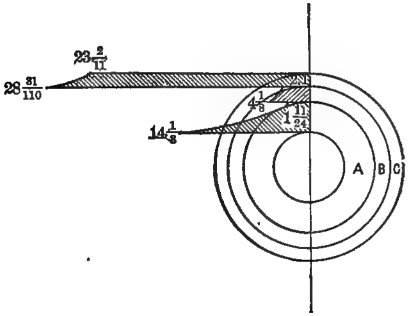


FIG. 381.—Combined Stresses.

But $28P-25Q=60$.

Therefore $P=3\tfrac{1}{8}$ t/sq. in. and $Q=1.1$ t/sq. in.

Hence the stresses *after* firing are

$$F_1=-25\text{ t/sq. in.},\quad F_2=-8\tfrac{1}{8}\text{ t/sq. in.},$$

$$F_3=-6.1\text{ t/sq. in.},\quad F_4=-5\text{ t/sq. in.},$$

and the total resultant stresses are

$$f_1+F_1=\tfrac{32}{3}-25=-14\tfrac{1}{3}\text{ t/sq. in.},$$

$$f_2+F_2=\tfrac{20}{3}-8\tfrac{1}{8}=-\tfrac{35}{24}\text{ t/sq. in.},$$

$$f_3+F_3=-\tfrac{244}{11}-6.1=-28\tfrac{31}{110}\text{ t/sq. in.},$$

$$f_4+F_4=-\tfrac{200}{11}-5=-\tfrac{255}{11}\text{ t/sq. in.}$$

Figs. 379, 380, and 381 represent respectively the stresses in the material of the gun developed *before* firing, *by* firing, and these stresses combined.

EXAMPLES.

1. The principal stresses on two planes *aa'* and *bb'*, at a point *O*, are thrusts of 94 and 26 lbs. per square inch. Find the kind, intensity, and obliquity of the stresses on a third plane inclined at 65° to *aa'* and *bb'*.

Ans. 46.2 lbs. (thrust); 34° 19'.

2. At a point within a strained solid the stress on one plane is a tension of 120 lbs. per square inch with an obliquity of 30°, and on a second plane a compression of 60 lbs. with an obliquity of 60°.

Find (1) the principal stresses; (2) the planes of principal stress; (3) the plane on which the stress is wholly tangential.

Ans. (1) A tension of 127.75 lbs. and a thrust of 47.11 lbs./sq. in.;
(2) $\gamma = 21° 40'$; (3) $\gamma = 31° 16'$ and stress $= 77.576$ lbs./sq. in.

3. At a point within a solid there is on some one plane a thrust of 200 lbs. per square inch of obliquity 15°, and on another a thrust of 80 lbs. per square inch of obliquity 30°. Find the principal stresses at that point, the position of the axis of greatest principal stress relative to the first plane, and the inclination of the two planes to each other.

Ans. 213.028 and 58.152 lbs./sq. in.

4. At a point within a strained solid the stresses on two planes at right angles to each other are a thrust of $30\sqrt{2}$ lbs. and a tension of 60 lbs. per square inch, the obliquities being 45° and 30° respectively. Determine (*a*) the principal stresses; (*b*) the ellipse of stress; (*c*) the intensity of stress upon a plane inclined at 60° to the major axis.

Ans. (*a*) A tension of 61.76 lbs. and a thrust of 39.80 lbs.;
(*c*) A thrust of 57.06 lbs.

5. At a point in a plane the stress on one plane is a thrust of 150 lbs. per square inch with an obliquity of 15°, and on another plane the stress is a thrust of 90 lbs. with an obliquity of 30°. Find the principal stresses, the position of the axis of greater principal stress relatively to the first plane, and the inclinations of the two planes to each other.

6. At a point within a strained solid there are two conjugate stresses, viz., a tension of 200 lbs. and a thrust of 150 lbs. per square inch, the common obliquity being 30°. Find (*a*) the principal stresses; (*b*) the maximum shear and the direction and magnitude of the corresponding resultant stress; (*c*) the resultant stress upon a plane inclined at 30° to the axis of greatest principal stress.

Ans. (*a*) A tension of 204.65 lbs. and a thrust of 146.95 lbs. per square inch;
(*b*) 175.8 lbs. per square inch.; 177.95 lbs. in a direction making an angle of 9° 20' with the axis of greatest principal stress;
(*c*) 163.3 lbs. per square inch.

7. From external conditions the stresses on two planes at a point in a solid are thrusts of 54 and 30 lbs. per square inch and are inclined at 10° and 26° respectively to the normal of these planes. Find the principal stresses at the point, the position of the axis of greatest principal stress relatively to the first plane, and the inclination of the two planes to each other.

Ans. 55.93 and 20.97 lbs./sq. in.; 16° 17½'; 65° 33'.

8. At a point within a strained solid the stress on one plane is a tension of 50 lbs. per square inch with an obliquity of 30°, and upon a second plane is a compression of 150 lbs. per square inch with an obliquity of 45°. Find (*a*) the principal stresses; (*b*) the angle between the two planes; (*c*) the plane upon which the resultant stress is a shear, and the amount of the shear.

Ans. (*a*) $p_1 = 179.98$ lbs. (comp.), $p_2 = -46.12$ lbs. (tens.);
(*b*) 61° 31′;
(*c*) 91.11 lbs.; $\gamma = 26° \, 51'$.

9. The principal stresses at a point in a strained solid are a tension of 300 lbs. and a thrust of 160 lbs. Find the resultant stresses on a plane making an angle of 30° with the plane of principal stresses. Also find the plane upon which the stress is wholly a shear, and determine the intensity of this shear.

Ans. 272 lbs./sq. in.; 36° 9′; 219 lbs./sq. in.

10. The total stress at a point *O* upon a plane *AB* is 60 lbs. per square inch, and its obliquity is 30°; the normal component upon a plane *CD* at the point *O* is 40 lbs. per square inch; *CD* is perpendicular to *AB*. Find (*a*) the total stress upon *CD*, and also its obliquity; (*b*) the principal stresses at *O*; (*c*) the equal conjugate stresses at *O*.

Ans. (*a*) $\tan^{-1}(\frac{3}{4})$; 50 lbs.;
(*b*) 76.57 lbs. and 15.39 lbs.;
(*c*) 34.32 lbs.; obliquity = 41° 42′.

11. At a point in a strained body the stresses are a tension of 255 lbs. and 171 lbs. per square inch. Find the stress on a plane inclined at 27° to the plane of principal stress, and its obliquity. *Ans.* 240 lbs./sq. in.; 8° 8′.

12. The principal stresses at a point in a strained mass are a tension of 300 lbs. and a thrust of 160 lbs. per square inch. Find (*a*) the obliquity and intensity of stress on the plane at 30° to the plane of greatest stress; (*b*) the intensity of the tangential stress on the plane upon which that stress is greatest; (*c*) the angle to the plane of greatest stress of the plane upon which the stress is entirely a shear. Also find the intensity of the shear.

Ans. (*a*) 47° 7′, 272 lbs.; (*b*) 230 lbs.; (*c*) 53° 52′, 219 lbs.

13. If the principles of the ellipse of stress are applicable within a mass of earth, and if at any point of the mass the stress upon a plane is double its conjugate stress, the angle between the two stresses being 20° 28′, show that the angle of repose of the earth is 27° 58′.

14. At a point in a strained solid the intensity of shear on two planes at right angles is 24 lbs.; the obliquity of the resultant stress on one plane is $\sin^{-1} \cdot 8$, and of the resultant stress on the other plane is $\sin^{-1} \cdot 6$. Find (*a*) the magnitudes of the two forces. Also find (*b*) the stress upon a plane bisecting the two planes in question; (*c*) the principal stresses at the point.

Ans. (*a*) 40 lbs., 30 lbs.; (*b*) 49.5 lbs.

15. At a point within a solid there is a pair of conjugate stresses of 30 and 40 lbs. per square inch. Their common obliquity is 10°. Find the principal stresses and the angle which the normal to the plane of conjugate stresses makes with the plane of principal stresses.

Ans. $43\frac{1}{2}$ and $27\frac{1}{2}$ lbs./sq. in.; 30° 20′.

16. At a point within a strained solid the stress on one plane is a tension of 100 lbs. per square inch with an obliquity of 30°, and on a second plane a compression of 50 lbs. with an obliquity of 60°. Find (*a*) the angle between the planes; (*b*) the plane upon which the stress is wholly a shear; (*c*) the planes of principal stress; also (*d*) find the conjugate stresses which have a common obliquity of 45°.

Ans. (*a*) 50° 5′; (*b*) 64.6 lbs.; $\gamma = 31° 16'$;
(*c*) $p_1 = 106.46$ (tens.), $p_2 = -39.26$ (compr.);
(*d*) 92.63 lbs. (tens.), 45.11 lbs. (compr.).

17. At a point within a strained mass the principal stresses at a given point are in the ratio of 3 to 1. Find the ratio of the conjugate stresses at the same point having the common obliquity 30°. Also find the inclination of the axis of greatest principal stress to the horizontal.

Ans. Equal; 60°.

18. A bar of iron is at the same time under a direct tensile stress of 5000 lbs. per square inch and a shearing stress of 3500 lbs. per square inch. What would be the resultant equivalent tensile stress on the material?

Ans. 6801 lbs.

19. Taking the safe tensile stress of wrought iron to be 5 tons per square inch, determine whether it would be safe to subject a piece of wrought iron to a tensile stress of 3.92 tons per square inch together with a shear stress of 3.36 tons per square inch. *Ans.* Unsafe, since max. stress = 5.85 tons.

20. At a point in a solid there is a tensile stress of 6 tons/sq. in. and a shear stress of 4 tons/sq. in. Find the principal stresses.

Ans. 8 and −2 tons/sq. in.

21. The principal stress at a point in a plane section of a strained solid is f tons. If the shear and normal (tensile or compressive) stresses at the same point are equal, find their values. *Ans.* $.618f$.

22. The shearing stress at a point in a section of a strained mass is 4 tons. Find the normal stress (tensile or compressive) at the same point if the max. principal stress is not to exceed 6 tons. *Ans.* $3\frac{1}{3}$ tons.

23. At a point in a plane section of a strained mass there is a shear stress q and a normal stress p equal to one half of the max. principal stress at the same point. Show that $q = p\sqrt{2}$.

24. A round shaft is in torsion, and the shear stress produced across the section near the circumference is 8000 lbs. per square inch. At the same section the shaft is subjected to bending, and a compressive stress of 6000 lbs. per square inch is produced. What is the greatest compressive stress in the material there? *Ans.* 11,544 lbs.

25. Taking 6500 tons/sq. in. as the modulus of rigidity (G) for steel, find the deflection *due to shearing* of a 2″×2″ steel bar, fixed at one end and loaded at the other with a weight of 2 tons, the length of the bar being 10 ins.

Ans. 1.

26. The crank-shaft of an engine is 5 in. in diameter; the distance from the centre of the bearing to the point opposite the centre of the crank-pin, *AB* in Fig. 341, is 12 ins.; the half-stroke *AC* in figure is 16 ins., and the

pressure applied to the crank-pin in the direction of BC is 5000 lbs. Find the greatest intensity of thrust; tension and shearing stress; and the angle θ made by the line of principal stress with the axis of the shaft.

Ans. 6530 lbs.; 4080 lbs; 27°.

27. In an overhanging crank the crank radius is 16 ins. and the distance between the centre of the crank-pin and the centre of the near crank-shaft bearing is 12 ins. When the connecting-rod is at right angles to the crank the thrust along the rod is 5000 lbs. Estimate the greatest tensile and shearing stresses which may occur in the crank-shaft, the diameter of the crank-shaft being 5 ins.

28. A propeller-shaft 8 in. in diameter is subjected to a thrust of 100 tons uniformly distributed over its two ends, and a twisting moment of 30 foot-tons. Find the greatest intensity of thrust and shearing stress.

29. A model for illustrating shear consists of a frame of four equal bars, each 16 in. in length, pivoted at the ends and connected at diagonally opposite corners by springs. With one side fixed and the opposite side subjected to stress in the direction of its length the following observations were obtained:

Load in Pounds.	Movement of Side.	Extension of 1st Diagonal and Compression of 2d in Inches.
0	0	0
2	.44	.30
4	.88	.61
6	1.31	.89
8	1.77	1.19

Deduce the modulus (G) for this frame, and show that the longitudinal strain is one half the shear strain.

30. A wall 12 ft. high and 3 ft. thick weighs 120 lbs. per cubic foot. It is built in earth weighing 100 lbs. per cubic foot and having an angle of repose of 30°. Find the depth of the foundation consistent with the equilibrium of the earth.

31. A wall is built up in layers, the water face being plumb and the rear stepped. If t be the thickness of the nth layer and y the depth of water above its lower face, show that width of layer × thickness of layer $= \sqrt{4A^2 + 6Atz + mty^3} - 2A$; A being the sectional area of the wall above the layer in question, z the horizontal distance between the water face and the line of action of the resultant weight above the layer, t the layer's thickness, and m the ratio of the specific weights of the water and masonry.

32. Assuming Rankine's theory, find the pressure on the vertical face of a retaining-wall 30 ft. high which retains earth sloping up from the top at the angle of repose, viz., 30°.

(Weight of masonry = 128 lbs. per cubic foot; weight of earth = 120 lbs. per cubic foot.) *Ans.* 46,764 lbs.

33. A wall 8 ft. high supports an embankment having a slope of 20°, the weight of the material being 120 lbs. per cubic foot, and the angle of repose 25°. Find the average intensity of the thrust on the wall, the total thrust, the horizontal component of thrust, and the overturning moment.

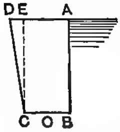


FIG. 382.

34. A wall retaining water on the vertical face AB overhangs by an amount DE. If $BC=\frac{1}{2}AB$, and if the deviation of the centre of resistance in BC from the middle point O is $<\frac{3}{8}BC$, show that the safe maximum value of DE is $.17325AB$, the specific weight of the wall being twice that of the water.

35. A vertical retaining-wall is strengthened by means of vertical rectangular anchor-plates having their upper and lower edges 18 and 22 ft. respectively below the surface. Find the holding power per foot of width, the earth weighing 130 lbs. per cubic foot and having an angle of repose of 30°.

Ans. 27,733⅓ lbs.

36. A vertical rectangular anchor-plate of depth h ft. has its horizontal centre line z ft. below the ground surface, which is horizontal. Find the maximum holding power of the plate per foot of width.

Ans. $\frac{4wzh \sin\phi}{\cos^2\phi}$, ϕ being the angle of repose of the earth and w its specific weight.

37. A ditch is 6 ft. deep, is cut in clay, and has vertical faces shored up with boards. A horizontal strut is placed at intervals of 5 ft., 2 ft. from the bottom. Taking $\sin\theta=.276$, find the thrust on the strut and also the greatest thrust which may be put on the strut before the earth without will heave up.

Ans. 6120 lbs.; 18,825 lbs.

38. The weight of a building is 5000 tons. The area of the concrete bottom of the foundations is 2000 square feet. At what depth ought it to be below the level of the soil, if the soil is such that $\phi=42°$, and if the soil weighs 100 lbs./cub. ft.?

39. A wall of rectangular section, 18 ft. high and 8 ft. thick, weighs 125 lbs. per cubic foot. Find the maximum intensity of pressure at the base when the wall retains (*a*) water level with the top, (*b*) earth level with the top; the angle of repose in the latter case being 30° and the earth weighing 100 lbs. per cubic foot. *Ans.* (*a*) 5189 lbs.; (*b*) 3491 lbs.

40. A brickwork pier 18 ins. square supports a load of 4 tons; the resultant presssure acts at a distance of 4 ins. from the centre of the pier. Calculate the maximum and minimum stresses in the brickwork.

Ans. 4.15 tons/sq. ft. in comp.; .59 ton/sq. ft. in tension.

41. The total vertical pressure on a horizontal section of a wall of masonry is 100 tons per foot length of wall. The thickness of the wall is 4 ft., and the centre of stress is 6 ins. from the centre of thickness of the wall. Determine the intensity of stress at the opposite edges of the horizontal joint.

Ans. 43.75 and 6.25 ton/sq. ft.

42. The slope of a cutting is 1 in $1\frac{1}{2}$. The earth weighs 120 lbs. per cubic foot and has an angle of repose of 36°. Find the average intensity of pressure on a vertical plane extending 3 ft. below the ground surface. Also find the total pressure on the vertical plane and the overturning moment at the foot of the plane. *Ans.* 93.191 lbs.; 279.573 lbs.; 223.633 ft.-lbs.

43. A wall 20 ft. high and 6 ft. thick retains earth on one side level with the top, and on the other the earth rises up the wall at its natural slope, viz., 45°, to the height of 5 ft. Will the wall stand or fall?

(Weight of masonry per cubic foot = 130 lbs.; of earth = 120 lbs.)

Find the locus of the centres of pressure of successive layers.

Ans. Overturning moment = 4128 ft.-lbs.; moment of stability $= 93600q + 750(\frac{14}{3} - 6q) = 36912\frac{1}{2}$ ft.-lbs. if $q = \frac{3}{8}$. The wall is stable.

44. A wall 12 ft. high, 2 ft. wide at the top and 3 ft. wide at the bottom, is constructed of masonry weighing 120 lbs. per cubic foot. The overturning force on the rear face of the wall, which is plumb, is a horizontal force P acting at 4 ft. from the base. Find P so that the deviation of the centre of pressure in the base may not exceed $\frac{1}{6}$ ft. The centre of pressure being fixed at 2 in. from the middle of the base, show that $\frac{2}{5}$ of the section may be removed without altering its stability, and find the increase in the inclination of the resultant pressure on the base to the vertical consequent on the removal.
Ans. 360 lbs.; tangents of angles are in ratio of 5 to 3.

45. A reservoir wall is 4 ft. wide at top, has a front batter of 1 in 12, a rear batter of 2 in 12, and is constructed of masonry weighing 125 lbs. per cubic foot; the maximum compression is not to exceed $10,526\frac{6}{19}$ lbs. per square foot. Find the limiting height of the wall. *Ans.* 16 ft.

46. A masonry pier 12 ft. high and 9′×9′ in section weighs 125 lbs. per cubic foot and carries a platform hinged above its centre. The end of the platform 3 ft. from the hinge is secured to the pier by a bolt extending to the bottom of the pier. At the other end of the platform, $33\frac{3}{4}$ ft. from the hinge, a weight W is placed. Find W so that the deviation of the centre of resistance at the base of the pier from the middle point of the base may not exceed $3\frac{3}{8}$ ft. *Ans.* 20,250 lbs.

47. A wall 4 ft. wide at the top, with a front batter of 1 in 8 and a rear batter of 1 in 12, is 30 ft. high. Will the wall be stable or unstable (1) when it retains water level with the top; (2) when it retains earth? (Weight of masonry per cubic foot = 125 lbs.; of earth = 112 lbs.; angle of repose = 30°; and $q = \frac{3}{8}$.)

Ans. (1) Moment of weight = 128,863 ft.-lbs.; overturning moment = 281,250 ft.-lbs., and the wall is therefore unstable.
(2) Moment of weight = 148,251 ft.-lbs.; overturning moment = 168,000 ft.-lbs., and wall is therefore unstable.

48. A dock wall, plumb at the rear and having a face with a batter of 1 in 24, is 20 ft. high and 9 ft. wide at the base. Counterforts are built at intervals of 12 ft., projecting 3 ft. from the rear and 6 ft. wide. Determine the thickness of an equally strong wall without counterforts, with the same face-batter and also plumb in the rear.

If the walls are founded in earth weighing 112 lbs. per square foot and having an angle of repose of 32°, find the least depth of foundation in each case, the masonry weighing 125 lbs. per cubic foot.

Ans. 10.95 ft.; 2.72 ft.; 2.71 ft.

49. Determine the limiting depths of foundation for (*a*) a wall of rectangular section 20 ft. high; (*b*) for a wall of trapezoidal section having plumb rear and front faces 4 and 20 ft high respectively. (Angle of repose of earth $=30°$; weight of earth $=112$ lbs. per cubic foot; of masonry $=140$ lbs.)

Ans. (*a*) 3.22 ft.; (*b*) 1.93 ft.

50. A wall with one face vertical is built up as shown by the figure. If w is the specific weight of the wall, n the number of sections, and f the maximum stress in the material, show that

$$\Sigma\left(\frac{1}{y}\right)=\frac{w}{f}\,\frac{n(2a+nx)}{a+x}.$$

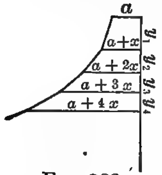


FIG. 383.

51. A wall of rectangular section 20 ft. high and 8 ft. wide weighs 125 lbs./cu. ft. and retains earth level with the top of the wall and weighing 100 lbs./cu. ft. The angle of repose of the earth is 30°. Find the distance of the centre of resistance from the middle of the base and the maximum stress in the material of the wall. How will these results be modified if water is substituted for the earth?

Ans. $2\frac{2}{9}$ ft., 7500 lbs./sq. ft.; $1\frac{9}{16}$ ft., $5470\frac{10}{39}$ lbs./sq. ft.

52. A wall has a front and a rear batter of 1 in 12, is 24 ft. high, and weighs 125 lbs./cu. ft. The water on one face rises to the top of the wall. Determine the width of the base (*a*) if $q=\frac{1}{4}$, (*b*) if the stress in the material is nowhere to exceed 12,000 lbs./sq. ft. *Ans.* 14.3 ft.; 20.4 ft.

53. A wall of height h has a horizontal base and a top width of a ft. The rear and front faces have batters of nb and b respectively. Show that

$$q=\frac{hb(3n-1)+a}{hb(n+1)+a}.$$

54. Find the total pressure per foot of breadth against a retaining-wall 12 ft. deep, when loose earth, weighing 120 lbs. per cubic foot, presses against it, the top layer being horizontal and of the same height as the wall. Also calculate the overturning moment.

55. The front face of a wall is plumb, and the rear face, which retains water level with the top of the wall, has a batter. The density of the wall is twice that of the water. If the width of the base is N times the width of the top, find the deviation of the centre of pressure in the base from the middle of the base, and if this deviation is $\frac{1}{6}$ of the thickness of the base, show that the height of the wall is $(N^2+1)^{\frac{1}{2}}$ times the width of the top, and find the maximum intensity of pressure in the base.

56. Earth slopes up from the top of the vertical faces of a wall at an angle

of 20°. The earth weighs 130 lbs. per cubic foot, and has an angle of repose of 30°. Find the average intensity of pressure and the total pressure on the wall. Also find the overturning moment.

57. A wall with a plumb rear face is to be 30 ft. high and 4 ft. wide at the top; the earth slopes up from the inner edge at the angle of 20°, 30° being the angle of repose. Assuming Rankine's theory, determine the proper width of the base, the masonry weighing 144 lbs. per cubic foot and the earth 110 lbs. *Ans.* If $q=\frac{1}{4}$, $t=10.94$ ft.

58. A wall 6 ft. wide at the bottom, plumb at the rear, and with a front batter of 1 in 12, retains water level with the top. Find (*a*) the limiting position of the centre of pressure at the base so that the stress may be nowhere negative.

How (*b*) high may the wall be built when subjected to this condition? (A cubic foot of masonry $=125$ lbs.)

Ans. (*a*) 12 in. from middle point of base; (*b*) height $=8.9$ ft.

59. A wall of rectangular section, weighing 125 lbs. per cubic foot, retains earth on one side level with the top of the wall. The earth is laid in horizontal layers and has an angle of repose of 25°. The height of the wall is 6 ft. Find the average intensity of pressure on the wall, the total pressure, and the overturning moment. Also find the thickness of the wall if the centre of resistance deviates from the middle point of the base $\frac{1}{8}$ of the thickness.

60. The section of a retaining-wall is a parallelogram, the upper and lower faces being horizontal and one of the diagonals being vertical. The wall retains water level with the top. If the thickness of the wall is 6 feet, find the height (1) so that the stress may nowhere exceed 12,000 lbs. per square foot; (2) so that $q=\frac{1}{6}$. (Weight of masonry $=125$ lbs. per cubic foot.)

Ans. (1) 23.02 ft.; (2) 20.78 ft.

61. A wall of rectangular section supports water at one side level with the top. How much of the wall can be cut away by a plane through the toe if the distance of the centre of pressure at the base from the toe is $\frac{1}{3}$ of the width of the base?

With the section of the wall thus modified, show that its height must be double the width of the base, and also show that the tangents of the angles which the resultant pressures at the base make with the vertical are in the ratio of 4 to 3. The weight of the masonry per cubic foot is twice that of the water. *Ans.* One quarter of the wall.

62. Show how to design the section of a brick wall, 25 ft. in height and weighing 110 lbs. per cubic foot, which is to retain earth weighing 100 lbs. per cubic foot level with the top of the wall. The angle of repose of the earth behind the wall $=30°$; of the earth below the wall $=25°$. Resistance of brickwork to crushing $=40$ tons/sq. ft. Factor of safety $=8$. Pressure of earth below wall is not to exceed $1\frac{1}{3}$ tons/sq. ft. The coefficient of friction for the brickwork $=.74$, and factor of safety against sliding $=1.2$. Front batter $=1$ in 12. The rear face is vertical.

63. The front and rear faces of a wall retaining water level with the top have a batter of 1 in 12. The height of the wall is 12 feet. Find the thickness of the wall (*a*) if q is .25; (*b*) if the stress at the base is nowhere to exceed 12,000 lbs./sq. ft. (Weight of masonry = 125 lbs. per cubic foot.)

Ans. (*a*) 7.078 ft.; (*b*) 5.547 ft.

64. A vertical rectangular retaining-wall of height h has a row of rectangular counterforts. The width of a counterfort is equal to the distance between two consecutive counterforts. Find the thickness of a rectangular wall giving the same moment of stability. If V_1 is the volume of the counterforted wall, V_2 the volume of the equivalent uniform wall, and V_3 the difference between the volumes of the portions of the wall with and without counterforts, show that $V_2^2 - V_1^2 = V_3^2$.

65. *ABCD* is the section of a masonry wall retaining water level with the top and weighing 125 lbs. per cubic foot. The water face *BC* is vertical and 40 ft. in height, the base *CD* horizontal and twice the width of the horizontal upper face *AB*, and the rear of the wall slopes uniformly from *A* to *B*. Find the width of the base so that the stress in no part can exceed 10,000 lbs. per square foot.

What must the value of q so that the triangular portion of masonry *ABD* may be removed without altering the stability of the wall?

66. The upper half of the section of a masonry wall is a rectangle 4 ft. wide, and the lower half a rectangle 6 ft. wide, one face being plumb. Find the height of the wall so that the stress on the base may nowhere exceed 10,000 lbs. per square foot when the wall retains water (*a*) on the plumb face; (*b*) on the stepped face. (Masonry weighs 125 lbs. per cubic foot.)

Ans. (*a*) 13.08 ft.; (*b*) 12.2 ft.

67. A masonry dam h ft. high is a right-angled triangle *ABC* in section, and retains water on the vertical face *AB*. Show that the thickness t of the base *BC* is given by $t^2 = \frac{4h^2}{5(6q+1)}$, qt being the deviation of the centre of pressure in the base from the middle point.

Also show that the thickness will be given by $t^2 = \frac{4h^2}{3(6q+1)}$ if the rock upon which the wall is built is seamy, and if it is assumed that the communication between the water in the seams and that in the reservoir produces an upward pressure upon the base *BC*, varying uniformly from that equivalent to the head at *B* to *nil* at *C*. If $q = \frac{1}{6}$, show that, in order that the wall may slide, the coefficient of friction must be less than 67 per cent in the first and 81 per cent in the second case. (Weight of a cubic foot of masonry = $2\frac{1}{2} \times$ weight of cubic foot of water.)

68. A wall 30 ft. high is of triangular section *ABC*, the face *AB* being plumb, and water being retained on the side *AC* level with the top of the wall; the masonry weighs 125 lbs. per cubic foot. Find the thickness of the base *BC* (*a*) when $q = \frac{3}{8}$; (*b*) when stress in masonry is not to exceed 10,000 lbs. per square foot; (*c*) when $q = \frac{3}{8}$ and the wall also retains earth

on the side *AB* level with the top, the angle of repose being 30°. The weight of the earth per cubic foot $=144$ lbs.

Ans. (*a*) 17.69 ft.; (*b*) 19.72 ft.; (*c*) 8.52 ft.

69. The faces of a reservoir wall 4 ft. wide at top and 40 ft. high have the same batter, and water rises on one side to within 6 ft. of the top. Find the width of base, assuming (*a*) that the pressure on the horizontal base is to be nowhere negative; (*b*) that the pressure varies uniformly and at no point exceeds 5068 lbs. per square foot. (Weight of masonry $=125$ lbs. per cubic foot.)

Ans. (*a*) 24.32 ft.; (*b*) 30 ft.

70. A wall of an isosceles triangular section with a base 36 ft. wide has to retain water level with its top. How high may such a wall be built consistent with the condition that the stress in the masonry is nowhere to exceed $10{,}546\frac{7}{8}$ lbs. per square foot? (Weight of masonry per cubic foot $=125$ lbs.)

Ans. 54 ft.

71. The faces *AB*, *AC* of a wall are parabolas of equal parameters having their vertices at *B* and *C*; water rises on one side to the top of the wall. Determine the thickness of the horizontal base *BC*, (*a*) for a wall 50 ft. high; (*b*) for a wall 100 ft. high, so that the pressure on the base may at no point exceed 10,000 lbs. per square foot. Also (*c*) compare the volume of such wall with the volume of an equally strong wall of the same height, but with a section in the form of an isosceles triangle with its vertex at *A*. (Weight of masonry $=125$ lbs. per cubic foot.)

Ans. (*a*) 32.44 ft.; (*b*) 90.35 ft.; (*c*) In case (*a*) ratio $=7:\sqrt{114}$;
" (*b*) " $=\sqrt{136}:21$.

72. The section of a wall $48\frac{2}{3}$ ft. high and weighing 125 lbs./cu. ft. is a parabola with the vertex at the highest point. The wall retains water on one side level with the top, and the stress is nowhere to exceed 10,000 lbs./sq. ft. Find the width of the base, which is horizontal. Find the width of a wall of rectangular section and of the same height which might be substituted for the above, and compare the volumes of the two walls.

Ans. $31\frac{1}{9}$ ft.; 39.04 ft.; .53.

73. How will the results in the preceding example be modified if the walls are each 64 ft. in height?

Ans. 54.58 ft.; 92.1 ft.; .4.

74. The section *ABCD* of a retaining-wall for a reservoir has a vertical face *BC* and a parabolic water-face *AD*, with the vertex at *D*. The width of the base $DC=4\times$ width of the top *AB*. If $AB=6$ ft., find the height of the wall, and trace the curves of resistance (*a*) when the reservoir is full; (*b*) when empty. (Cubic foot of masonry $=2\times$ cubic foot of water.)

Ans. 32 ft. if $q=\frac{1}{4}$, and then max. comp. $=8000$ lbs. per sq. ft.

75. The figure represents the section of the upper portion of a masonry dam which has to retain water level with the top of the dam. The face *AC* is plumb for a depth of 73 ft. The width of the section is constant and $=22\frac{1}{2}$ ft. for a depth $AB=40$ ft.

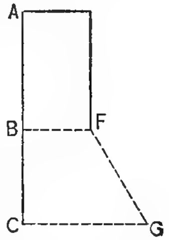


FIG. 334.

Find the maximum stress in the masonry at the horizontal bed *BF*.

With the same maximum stress show that the width of the horizontal bed CG, FG being straight, is 75.386 ft. (Masonry weighs 130 lbs. per cubic foot.)

Ans. $14{,}0491\frac{9}{1559}$ lbs. per sq. ft.

76. When a cylindrical bar is twisted show that it is subjected to shears along transverse and radial longitudinal sections, or to tensions and compressions on helices at 45° to the axis.

77. Find the work done in gradually and uniformly compressing a body of volume V_1 to the volume V_2, p being the final intensity of pressure and K the modulus of compression. Also show that the intensity of stress is constant throughout the body. *Ans.* $\frac{p^2V_1}{2K}$.

78. A bar is stretched under a force of intensity p. If the bar is prevented from contracting, find the lateral stress; also find the extension.

Ans. $\frac{p}{\sigma-1}$; $\frac{p}{E}\frac{\sigma^2-\sigma-2}{\sigma(\sigma-1)}$.

79. For mild steel, $E=15{,}000$ and $G=6000$ tons/sq. in. Find K and Poisson's ratio. *Ans.* 10,000; 4.

80. For wrought iron, $K=10{,}000$ and $G=5750$ tons/sq. in. Find E and Poisson's ratio. *Ans.* 14,500 tons/sq. in.; $\frac{148}{37}$.

81. For copper, $K=12{,}000$ and $E=7800$ tons/sq. in. Find G and Poisson's ratio. *Ans.* 2802 tons/sq. in.; $\frac{120}{47}$.

82. Taking the value of the coefficient of elasticity (E) and the coefficient of rigidity (G) to be 15,000 and 5750 tons for steel, 13,950 and 5450 tons for wrought iron, and 9500 and 3750 tons for cast iron, find the coefficient of elasticity of volume (K), and also the values of A and the coefficient of dilatation (λ), assuming the metals to be isotropic.

	σ	K	A	λ
Ans. Steel.	$3\frac{2}{7}$	$12777\frac{7}{9}$	$\frac{32}{9}G$	$\frac{14}{9}G$
Wrought iron.	$3\frac{35}{61}$	$10559\frac{3}{8}$	$\frac{157}{48}G$	$\frac{61}{48}G$
Cast iron.	$3\frac{3}{4}$	$6785\frac{5}{7}$	$\frac{22}{7}G$	$\frac{8}{7}G$

83. A copper cube $3''\times3''\times3''$ is subjected to a hydrostatic pressure of 4.48 tons/sq. in.; find the volume of the strained solid, K being 12,000 tons/sq. in. *Ans.* 26.99 cubic inches.

84. Ten cubic inches of wrought iron and 10 cubic inches of water are subjected to fluid pressure of 3 tons per square inch; find the new volumes. If the iron is spherical, what are the old and new diameters?

85. A body is distorted without compression or expansion; find the work done. *Ans.* $\frac{1}{4\mu}\int\{N_1^2+N_2^2+N_3^2+2(T_1^2+T_2^2+T_3^2)\}dS$.

86. Find the work required to twist a hollow cylinder of external radius R_1, internal radius R_2, and length l through an angle α.

Ans. $\mu\frac{\pi\alpha^2}{4l}(R^4-R_2^4)$.

87. Prove that torsion is equivalent to a shear at each point.

88. Show that a simple elongation is equivalent to a cubical dilation and a pair of shearing or distorting stresses.

89. Find the resultant shearing stress at any point in the surface of the transverse section of an elliptic cylinder. (Art. 19, Case *b*.)

Ans. $2\theta\frac{G}{p}\frac{b^2c^2}{b^2+c^2}$, p being the perpendicular from the centre upon the tangent to the ellipse at the given point, and $2b$, $2c$ the major and minor axes.

90. A cylinder undergoes torsion round its axis. Show that the *curves of no traction* are concentric circles.

91. Find the least sectional area of a rope which will transmit 500 H.P. at 100 f/s, the max. working stress being 10,000 lbs./sq. in.

Ans. .55 ins.

92. Find the min. diam. of a shaft which will transmit 500 H.P. at a surface velocity of 5 f/s, the safe working stress being 10,000 lbs./sq. in.

Ans. 3.75 in.

93. Find the efficiency of a shaft of length l, and also the greatest length of shaft which will turn itself without fracture, μ being the coefficient of bearing friction, f the max. working stress, d the diam., and v the surface velocity.

Ans. $1-2\mu\frac{wl}{f}$; $\frac{f}{2\mu w}$.

94. Find the law of variation in the radius of a shaft so that the max. stress may be the same for each section of the shaft. Also find the efficiency of the shaft.

Ans. $\log_e\frac{r_0}{r}=\frac{2}{3}\mu d\frac{l}{f}$; $1-2e^{-2\mu\frac{dl}{f}}$.

95. Show that the hoop (f) or radial (q) stresses at any radius r in a thick hollow cylinder or sphere are connected by the relation

$$\frac{d}{dr}(qr)=f.$$

96. A tube of wrought iron, of 2 ins. internal and 4 ins. external radius, is subjected to an internal pressure of 50,000 lbs./sq. in., the pressure on the outside being *nil*. Find the tensile stress in the material at any radius r.

Ans. $\frac{266667}{r^2}+16667$.

97. Find the max. and min. stress in the walls of a thick cylinder of 8 ins. internal and 14 in. external diam., the inside fluid pressure being 2000 lbs./sq. in.

Ans. 4670 and 1520 lbs./sq. in.

98. The cast-iron cylinder of an hydraulic press has an external diameter twice the internal, and is subjected to an internal pressure of p tons per square inch. Find the principal stresses at the outer and inner circumferences. Also, if the pressure is 3 tons per square inch, and if the internal diameter is

10 in., find the work done in stretching the cylinder circumferentially, E being 8000 tons.

Ans. At inner circumference, $q=p$, a thrust, and $f=-\frac{5}{3}p$, a tension.
At outer circumference, $q=0$, and $f=-\frac{2}{3}p$, a tension.
Work = 126 ft.-lbs. per square foot of surface.

99. The chamber of a 27-ton breech-loader has an external diameter of 40 ins. and an internal diameter of 14 ins. Under a powder pressure of 18 tons per square inch, find the principal stresses at the outer and inner circumferences, and also the work done; E being 13,000 tons.

Ans. At inner, $q=18$ tons, compression; at outer, $q=0$.
At inner, $f=-23\frac{4}{39}$ tons, tension; at outer, $f=-5\frac{1}{39}$ tons, tension.
Work $=1\frac{1}{3}$ ft.-tons per sq. ft. of surface.

100. What should be the thickness of a 9-in. cylinder (*a*) which has to withstand a pressure of 8000 lbs. per square inch, the maximum allowable tensile stress being 24,000 lbs. per square inch; (*b*) which has to withstand a pressure of 6000 lbs. per square inch; the maximum allowable tensile stress being 10,000 lbs. per square inch? *Ans.* (*a*) 1.86 in.; (*b*) $4\frac{1}{2}$ in.

101. A tube of wrought iron, inside radius 3 ins., outside 4 ins., outside pressure 0. What is the inside pressure required to produce a maximum tensile stress of 15,000 lbs. per square inch? Find the fractional increase in size of the inside radius. *Ans.* 4200 lbs./sq. in.; .00161 in.

102. A tube of wrought iron, inside radius 2 ins., outside 3 ins., no pressure inside; outside pressure = 4200 lbs. per square inch outside. Find the circular compressive stress at any point, and also the diminution of the outer radius.

Ans. $7560+\dfrac{30240}{r^2}$; .00099 in.

103. A metal cylinder of internal radius r and external radius nr is subjected to an internal pressure of p tons per square inch. Show that the total work done in stretching the cylinder circumferentially is $\dfrac{6p^2r}{E}\dfrac{n^2+1}{n^2-1}$ ft.-tons per square foot of surface, E being the metal's coefficient of elasticity.

104. The metal of a cast-iron hydraulic press of 508 mm. diameter is 222 mm. thick, and has to bear an internal pressure of 402 atmospheres. Find the maximum hoop stress. *Ans.* 684k/cm.

105. What must be the thickness of the metal in a cast-iron hydraulic press of 500 mm. diameter which has to bear an internal pressure of 280 atmospheres together with a maximum tension of 5k/mm.? *Ans.* 20.3 cm.

106. An hydraulic press has to raise a weight of 1,162,400k. The internal pressure is 573 atmospheres and the diameter of the cylinder is 560 mm. The tension in the metal is not to exceed 700k/cm^2. Find the thickness of the metal. *Ans.* 41 cm.

107. The actual thickness of the metal in the preceding example was 25.4 cm. What was the maximum tensile stress induced in lifting the given weight? *Ans.* 948.5k/cm.

108. The thickness of the metal in a cast-iron cylinder of 108 mm. diam-

eter is 35 mm. Find the greatest hoop stress in the metal due to a maximum internal pressure of 250*k*/cm. *Ans.* 510k/cm^2.

109. In a thick cylinder the initial tensile stress on the outer skin and the initial compressive stress on the inner skin are each 3 tons/sq. in. Find the resultant stresses on the outside and inside when the cylinder is subjected to an internal fluid pressure of 4½ tons/sq. in. *Ans.* 4.2 and 4.5 tons/sq. in.

110. A gun of 12 ins. internal and 24 ins. external diameter is subjected to a maximum internal pressure of 40,000 lbs. per square inch. Find the stress produced at $r=6$, 9, and 12 ins. *Ans.* 66,666; 37,037; 4444 lbs./sq. in.

111. Pipes of a water-pressure supply company are to withstand a possible pressure of 1000 lbs. per square inch; they are of 6 in. internal diameter. What is the outside diameter, the safe tensile stress of the metal being 3000 lbs. per square inch? *Ans.* 8.485 ins.

112. Assuming that the annulus forming the section of a cylindrical boiler is composed of a number of infinitely thin rings, show that the pressure at the circumference of a ring of radius r is $\frac{A}{r^{1+m}}$ per unit of surface, and that the circumferential stress is $\frac{B}{r}+\frac{A}{mr^{m+1}}$, A and B denoting arbitrary constants, and m being the coefficient of lateral contraction. Find the values of A and B, p_0 and p_1 being respectively the internal and external pressures.

113. Show that in the case of a spherical boiler the pressure and circumferential stress are respectively $\frac{A}{1+m}$ and $\frac{B}{r^2}+\frac{2A}{(m-1)r^{m-3}}$. Find A and B.

114. An hydraulic cylinder having an internal diameter of 6 ins. and a thickness of 3½ ins. is subjected to an internal pressure of 2 tons per square inch. Draw a curve of hoop stress for this cylinder to scale.

115. A cast-iron water-main is 30 ins. internal diameter and 1 in. thick. What is the greatest head of water which it can safely stand? If the pipes are wrought iron, what ought its thickness to be if the working stress is 10,000 lbs. per square inch, and if the longitudinal seams are 60 per cent of the strength of the unhurt plate?

116. The 3-in. plunger for the cast-steel cylinder of an intensifier is connected with a piston which works in a 48-in. cylinder under a pressure equivalent to a 120-ft. head of water. Find the proper thickness of the metal of the intensifier, allowing a maximum stress of 20,000 lbs. per square inch. *Ans.* 2.46 in.

117. The barrel of a gun consists of two rings A and B, the bore of the gun and the diameters of the inner and outer rings being in the ratio of 1:2:4. The ring A is shrunk upon B, producing a pressure of 5 tons per square inch at the surface of contact. If the firing of the gun produces an internal pressure of 20 tons/sq. in., find the stresses induced in the gun. *Ans.* $-19\frac{1}{3}, +1\frac{2}{3}$ tons; $-15, -6$ tons.

CHAPTER VI.

FRICTION.

1. Sliding Friction.—Friction is the resistance to motion which is always developed when two substances, whether solid, liquid, or gaseous, are pressed together and are compelled to move the one over the other. If P is the mutual pressure, and if F is the force which must act tangentially at the point of contact to produce motion, the ratio of F to P is called the coefficient of friction and may be denoted by f. The value of f does not depend upon the nature of any *single* substance, but upon the nature and condition of the surfaces of contact of a *pair* of substances. It is not the same, e.g., for iron upon iron as for iron upon bronze or upon wood; neither is it the same when the surfaces are dry as when lubricated.

The laws of friction as enunciated by Coulomb are:

(1) That f is independent of the velocity of rubbing; (2) that f is independent of the extent of surfaces in contact; (3) that f depends only on the nature of the surfaces in contact.

The friction between two surfaces at rest is greater than when they are in motion, but a slight vibration is often sufficient to change the friction of rest to that of motion.

Morin's elaborate friction experiments completely verified these laws within certain limits of pressure (from $\frac{3}{4}$ lb. to 128 lbs. per square inch) and velocity (the maximum velocity being 10 ft. per second), and under the conditions in which they were made.

A few of his more important results are given in the following table.

The apparatus employed in carrying out these experiments consisted of a box which could be loaded at pleasure, and which was made to slide along a horizontal bed by means of a cord

Material.	State of Surfaces.	Coefficient of Friction.
Wood on wood.	Dry.	.25 to .5
Metal on wood.	Dry.	.2 " .6
" " "	Wet.	.22 " .26
Metal on metal.	Dry.	.15 " .2
" " "	Wet.	.3
Metal and wood on each other or each on itself	Slightly oily.	.15
	Occasionally lubricated as usual.	.07 to .08
	Constantly lubricated.	.05

passing over a pulley and carrying a weight at the end. The contact-surfaces of the bed and box were formed of the materials to be experimented upon. The pull was measured and recorded by a spring dynamometer.

More recent experiments, however, have shown that Coulomb's laws cannot be regarded as universally applicable, but that f depends upon the velocity, the pressure, and the temperature. At very low velocities Morin's results have been verified (Fleeming Jenkin). At high velocities f rapidly diminishes as the velocity increases. Franke, having carefully examined the results of various series of experiments, especially those of Poirée, Bochet, and Galton, has suggested the formula

$$f = f_0{}^{-\alpha v},$$

v being the velocity and f_0, α, coefficients depending upon the nature and condition of the rubbing surfaces.

For example,

$f_0 = .29$ and $\alpha = .04$ for cast iron on steel with dry surfaces.

$f_0 = .29$ and $\alpha = .02$ for wrought iron on wrought iron with dry surfaces.

$f_0 = .24$ and $\alpha = .0285$ for wrought iron on wrought iron with slightly damp surfaces.

Ball has shown that at very low pressures f increases as the pressure diminishes, while Rennie's experiments indicate that at very high pressures f rapidly increases with the pressure, and this is perhaps partly due to a depression, or to an abrasion of the rubbing surfaces.

2. Inclined Plane.—Let a body of weight P slide uniformly up an inclined plane under a force Q inclined at an angle β to the plane.

Let F be the friction resisting the motion, R the pressure on the plane, and α the plane's inclination.

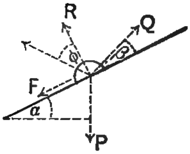


Fig. 385.

The two equations of equilibrium are

$$F = Q\cos\beta - P\sin\alpha$$

and

$$R = -Q\sin\beta + P\cos\alpha.$$

Therefore $$\frac{F}{R} = \frac{Q\cos\beta - P\sin\alpha}{-Q\sin\beta + P\cos\alpha} = \text{coefficient of friction} = f.$$

Let the resultant of F and R make an angle ϕ with the normal to the plane. Then

$$\tan\phi = \frac{F}{R} = \frac{Q\cos\beta - P\sin\alpha}{-Q\sin\beta + P\cos\alpha}, \quad \text{or} \quad \frac{Q}{P} = \frac{\sin(\alpha+\phi)}{\cos(\beta-\phi)}.$$

ϕ is called the *angle of friction.* It has also been called the *angle of repose,* since a body will remain at rest on an inclined plane so long as its inclination does not exceed the angle of friction.

If $\alpha = 0 = \beta$, then $\frac{Q}{P} = \tan\phi = f$.

The work done in traversing a distance $x = Q\cos\beta x$. If Q is variable, the work done $= \int_0^x Q\cos\beta dx$.

3. Wedge.—The wedge, or key, is often employed to connect members of a structure, and is generally driven into position by the blow of a hammer. It is also employed to force out moisture from materials by inducing a pressure thereon.

The figure represents a wedge descending vertically under a continuous pressure P, thus producing a lateral motion in the horizontal member C, which must therefore exert a pressure Q upon the vertical face AB.

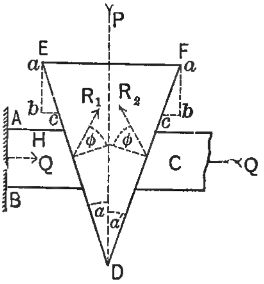


Fig. 386.

The member H is fixed, and it is assumed that the motion of the machine is uniform, so that the wedge and C are in a state of relative equilibrium.

Let R_1, R_2 be the reactions at the faces DE, DF, respectively, their directions making an angle ϕ, equal to the angle of friction with the normals to the corresponding faces.

Let α be the angle between DE and the vertical, α' the angle between DF and the vertical.

Consider the wedge, and neglect its weight, which is usually inappreciable as compared with P.

Resolving vertically,

$$R_1 \cos(90^\circ - \overline{\alpha+\phi}) + R_2 \cos(90^\circ - \overline{\alpha'+\phi})$$

$$= P = R_1 \sin(\alpha+\phi) + R_2 \sin(\alpha'+\phi). \quad (1)$$

Resolving horizontally,

$$R_1 \sin(90^\circ - \overline{\alpha+\phi}) - R_2 \sin(90^\circ - \overline{\alpha'+\phi}) = 0,$$

or $\quad R_1 \cos(\alpha+\phi) = R_2 \cos(\alpha'+\phi). \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$

Consider the member C, and neglect its weight.

Resolving horizontally,

$$R_1 \cos(\alpha+\phi) = Q = R_2 \cos(\alpha'+\phi). \quad . \ . \ . \ . \ . \quad (3)$$

Assuming the wedge is isosceles, as is usually the case, $\alpha = \alpha'$, and therefore,

by eq. (2), $R_1 = R_2$, and by eq. (1), $2R_1 \sin(\alpha+\phi) = P$. (4)

Hence, by eqs. (3) and (4),

$$\frac{Q}{P} = \frac{\cot(\alpha+\phi)}{2} = \frac{\text{external resistance overcome}}{\text{effort exerted}}. \quad . \ . \ . \quad (5)$$

(*N.B.*—This ratio of resistance to effort is termed the *mechanical advantage*, or *purchase*, of a machine.)

Suppose the motion of the machine reversed, so that Q becomes the effort and P the resistance.

The reactions R_1, R_2 now fall *below* the normals, and the equations of relative equilibrium are the same as the above, with $-\phi$ substituted for ϕ. Thus

$$\frac{Q}{P} = \tfrac{1}{2} \cot(\alpha-\phi). \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (6)$$

The two cases may be included in the expression

$$\frac{Q}{P}=\tfrac{1}{2}\cot(\alpha\pm\phi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

For a given value of P, Q increases with α.

If there were no friction, ϕ would be zero, and eq. (7) would become

$$\frac{Q}{P}=\frac{\cot\alpha}{2}.$$

Thus the effect of friction may be allowed for by assuming the wedge frictionless, but with an angle *increased* by 2ϕ in the *first* case, and *diminished* by 2ϕ in the *second* case.

Again, when P is the effort and Q the resistance, eq. (5) shows that if $\alpha+\phi>90°$, the ratio $\frac{Q}{P}$ is negative, which is impossible, while if $\alpha+\phi=90°$, $\frac{Q}{P}$ is zero, and in order to overcome Q, however small it might be, P would require to be infinitely great. Hence

$$\alpha+\phi \text{ must be } <90°,$$

and below this limit $\frac{Q}{P}$ diminishes as ϕ increases.

Similarly, it may be shown from eq. (7) that when Q is the effort and P the resistance,

$$\phi \text{ must be } <\alpha,$$

and that below this limit $\frac{Q}{P}$ increases with ϕ.

Efficiency.— During the uniform motion of the machine let any point a descend vertically to the point b. The corresponding horizontal displacement is evidently $2bc$.

$$\text{The motive work}=P\cdot ab;$$

$$\text{" useful work}\quad=Q\cdot 2bc.$$

Hence the efficiency $=\frac{Q \cdot 2bc}{P \cdot ab}=\frac{Q}{P} 2 \tan \alpha = \tan \alpha \cot (\alpha+\phi)$, by eq. (5).

This is a maximum for a given value of ϕ when

$$\alpha = 45^\circ - \frac{\phi}{2},$$

and the max. efficiency $= \tan \left(45^\circ - \frac{\phi}{2}\right) \cot \left(45^\circ + \frac{\phi}{2}\right)$

$$= \left(\frac{1 - \tan \frac{\phi}{2}}{1 + \tan \frac{\phi}{2}} \right)^2 = \frac{1 - \sin \phi}{1 + \sin \phi}.$$

For the *reverse* motion the efficiency

$$= \frac{P \cdot ab}{Q \cdot 2bc} = \cot \alpha \tan (\alpha - \phi).$$

This is a maximum when $\alpha = 45^\circ + \frac{\phi}{2}$. Thus the

$$\text{max. efficiency} = \cot \left(45^\circ + \frac{\phi}{2}\right) \tan \left(45^\circ - \frac{\phi}{2}\right) = \frac{1 - \sin \phi}{1 + \sin \phi}.$$

4. Screws.—A screw is usually designed to produce a linear motion or to overcome a resistance in the direction of its length. It is set in motion by means of a couple acting in a plane perpendicular to its axis. A reaction is produced between the screw and nut which must necessarily be equivalent to the couple and resistance, *the motion being steady.*

Take the case of a *square*-threaded * screw. It may be assumed that the reaction is concentrated along a *helical* line whose diameter, *d*, is a mean between the external and internal diameters of the thread, and that its distribution along this line is uniform. It will also be supposed that the axes of the couple and screw are coincident, so that there will be no lateral pressure on the nut.

---

* Square-threaded screws work more accurately than those with a V thread but the efficiency of the latter has been shown to be very little less than that of the former (Poncelet). On the other hand, the V thread is the stronger, much less metal being removed in cutting it than is the case with a square thread. Again with a V thread there is a tendency to burst the nut, which does not obtain in a screw with a square thread.

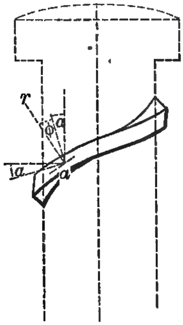


FIG. 387.

Let M be the driving couple;

Q " " axial resistance to be overcome;

r " " reaction at any point a of the helical line, and

ϕ " " angle between its direction and the normal at a; ϕ is the angle of friction.

Let α " " angle between the tangent at a and the horizontal; α is called the *pitch-angle.*

Since the reaction between the screw and nut must be equivalent to M and Q, then

Q = algebraic sum of vertical components of the reactions at all points of the line of contact,

$$=\Sigma[r\cos(\alpha+\phi)]=\cos(\alpha+\phi)\Sigma(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and M = algebraic sum of the moments with respect to the axis of the horizontal components of the reactions at all points of the line of contact,

$$=\Sigma\left[r\sin(\alpha+\phi)\frac{d}{2}\right]=\frac{d}{2}\sin(\alpha+\phi)\Sigma(r). \quad . \quad . \quad . \quad (2)$$

Let the couple consist of two equal and opposite forces P, acting at the ends of a lever of length p, so that $M=Pp$.

Then, by eqs. (1) and (2),

$$\frac{Q}{M}=\frac{Q}{Pp}=\frac{2}{d}\cot(\alpha+\phi),$$

and the *mechanical advantage*

$$\frac{Q}{P}=\frac{2p}{d}\cot(\alpha+\phi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If $\phi=0$, $\frac{Q}{P}=\frac{2p}{d}\cot\alpha$ and the effect of friction may be allowed for by assuming the screw frictionless, but with a pitch-angle equal to $\alpha+\phi$.

Again, let the figure represent one complete turn of the thread developed in the plane of the paper. CD is the corresponding length of the thread; DE the circumference πd; CE, parallel to the axis, the pitch h; and CDE the pitch-angle α.

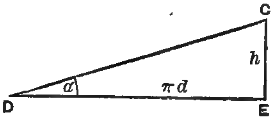


FIG. 388.

The motive work in one revolution $=M2\pi=Pp2\pi$.
The useful work done in one revolution $=Qh$.

$$\text{Hence the efficiency}=\frac{Qh}{Pp2\pi}=\frac{2p}{d}\cot(\alpha+\phi)\frac{h}{p2\pi}$$

$$=\frac{h}{\pi d}\cot(\alpha+\phi)=\tan\alpha\cot(\alpha+\phi). \quad . \quad . \quad (4)$$

This is a maximum when $\alpha=45^\circ-\frac{\phi}{2}$, its value then being

$$\frac{1-\sin\phi}{1+\sin\phi}.$$

In practice, however, α is generally much smaller, efficiency being sacrificed to secure a large mechanical advantage, which, according to eq. (3), increases as α diminishes.

If $\alpha+\phi=90^\circ$, $\frac{Q}{P}=0$, so that to overcome Q, however small it may be, would require an infinite effort P.

Therefore $\qquad \alpha+\phi<90^\circ.$

Suppose the pitch-angle sufficiently coarse to allow of the screw being *reversed*. Q now becomes the effort and P the resistance. The direction of r falls on the other side of the normal, and the relation between P and Q is the same as above, $-\phi$ being substituted for ϕ. Thus

$$\frac{Q}{P}=\frac{2p}{d}\cot(\alpha-\phi),$$

and therefore the mechanical advantage

$$=\frac{P}{Q}=\frac{d}{2p}\tan(\alpha-\phi).$$

If $\alpha=\phi$, $\frac{P}{Q}=0$, and to overcome P, however small it may be, Q would require to be infinite.

Therefore $$\alpha>\phi.$$

If $\alpha<\phi$, reversal of motion is impossible, and the screw then possesses the property, so important in practice, of serving to fasten securely together different structural parts, or of locking machines.

Again, it may be necessary to take into account the friction between the nut and its seat, as well as the friction at the end of the screw. The corresponding moments of friction with respect to the axis are (Art. 8)

$$f\frac{Q}{3}\frac{d_1^3-d_2^3}{d_1^2-d_2^2} \quad \text{and} \quad f\frac{Q}{3}d',$$

f being the coefficient of friction, d_1, d_2 the external and internal diameters of the seat, and d' the diameter of the end of the screw.

5. Endless Screws (Fig. 389).—A screw is often made to work with a toothed wheel, as, for example, in raising sluice-gates, when the screw is also made sufficiently fine to prevent, by friction alone, the gates from falling back under their own weight. The theory is very similar to the preceding. Let the screw drive. A tooth rises on the thread, and the wheel turns against a tangential resistance Q, which is approximately parallel to the axis of the screw.

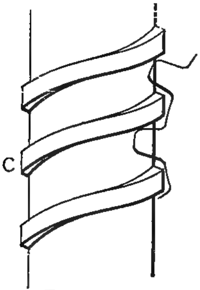


FIG. 389.

Let Fig. 390 represent one complete turn of the thread developed in the plane of the paper, α being the pitch-angle as before.

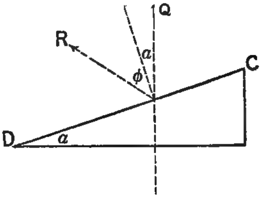


FIG. 390.

Consider a tooth. It is acted upon by Q in a direction parallel to the axis, and by the reaction R between the thread and tooth, making an angle ϕ (the angle of friction) with the normal to the thread CD.

Therefore $$Q=R\cos(\alpha+\phi).$$

Again, the horizontal component of R, viz., $R\sin(\alpha+\phi)$, has a moment $R \sin (\alpha+\phi)\frac{d}{2}$ with respect to the axis of the screw, and this must be equivalent to the moment of the driving couple, viz., Pp (Art. 4). Therefore

$$Pp=R\frac{d}{2}\sin (\alpha+\phi).$$

Thus the relation between P and Q is the same as in the preceding article.

Similarly, if the wheel acts as the driver,

$$\frac{P}{Q}=\frac{d}{2p}\tan (\alpha-\phi).$$

6. Rolling Friction.—The friction between a rolling body and the surface over which it rolls is called rolling friction. Prof. Osborne Reynolds has given the true explanation of the resistance to rolling in the case of elastic bodies. The roller produces a deformation of the surfaces in contact, so that the distance rolled over is greater than the actual distance between the terminal points. This he verified by experiment, and concluded that the resistance to rolling was due to the sliding of one surface over the other, and that it would naturally increase or diminish with the deformation. In proof of this he found, for example, that the resistance to an iron roller on India-rubber is *ten* times as great as the resistance when the roller is on an iron surface. Hence the harder and smoother the surfaces, the less is the rolling friction. The resistance is not sensibly affected by the use of lubricants, as the advantage of a smaller coefficient of friction is largely counteracted by the increased tendency to slip. Other experiments are yet required to show how far the resistance is modified by the speed.

Generally, as in the case of ordinary roadways, the resistance is chiefly governed by the amount of the deformation of the surface and by the extent to which its material is crushed. Let a roller of weight W (Fig. 391) be on the point of motion under the action of a horizontal pull R.

The resultant reaction between the surfaces in contact must pass through the point of intersection of R and W. Let it also cut the surface in the point B.

Let d be the horizontal distance between B and W;
p " " vertical " " B " R.

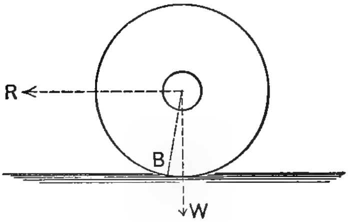


FIG. 391.

Taking moments about B,

$$Rp=Wd,$$

or $$R=\text{the resistance}=W\frac{d}{p}.$$

Coulomb and Morin inferred, as the result of a series of experiments, that d is independent of the load upon the roller as well as of its diameter,* but is dependent upon the nature of the surfaces in contact.

7. Journal Friction. — Experiments indicate that f is not the same for curved as for plane surfaces, and in the ordinary cases of journals turning in well-lubricated bearings the value of f is probably governed by a combination of the laws of fluid friction and of the sliding friction of solids.

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* Dupuit's experiments led him to the conclusion that d is proportional to the square root of the diameter, but this requires further verification.

Let μ be the coefficient of sliding friction.

The resistance of the roller to sliding is μW, and "rolling" will be insured if $R<\mu W$, i.e., if $\frac{d}{p}<\tan\phi$, which is generally the case so long as the direction of R does not fall below the centre of the roller.

Assume that R is applied at the centre. The radius r may be substituted for p, since d is very small, and hence

$$R=W\frac{d}{r}.$$

An equation of the same form applies to a wheel rolling on a hard roadway over obstacles of small height, and also when rolling on soft ground. In the latter case the resistance is proportional to the product of the weight upon the wheel into the depth of the rut, and the depth for a small arc is inversely proportional to the radius.

Experiments on the tractional resistance to vehicles on ordinary roads are few in number and incomplete, so that it is impossible to draw therefrom any general conclusion.

From the experiments carried out by Easton and Anderson it would appear that the value of d in inches varies from 1.6 to 2.6 for wagons on soft ground, and that the resistance is not sensibly affected by the use of springs. Upon a hard road in fair condition the resistance was found to be from $\frac{1}{4}$ to $\frac{1}{6}$ of that on soft ground, the average value of d being $\frac{1}{2}$ inch, and was very sensibly diminished by the use of springs.

The bearing part of the journal is generally truly cylindrical, and is terminated by shoulders resting against the ends of the step in which the journal turns.

Consider a journal in a semicircular bearing with the cap removed. When the cap is screwed on, the load upon the journal will be increased by an amount approximately equal to the tension of the bolts. Let P be the load.

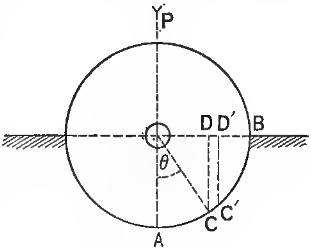


FIG. 392.

Assume that the line of action of the load is vertical and that it intersects the axis of the shaft. This load is balanced by the reaction at the surface of contact, but much uncertainty exists as to the manner in which this reaction is distributed. There are two extremes, the one corresponding to a normal pressure of constant intensity at every point of contact, the other to a normal pressure of an intensity varying from a maximum at the lowest point A to a minimum at the edge of the bearing B.

Let l be the length of the bearing, and consider a small element ΔS at any point C, the radius OC $(=r)$ making an angle θ with the vertical OA.

First. Let p be the constant normal intensity of pressure.

$$P=\Sigma(p\Delta S\cos\theta l)=pl\Sigma(DD')=2plr.$$

$$\textit{Frictional resistance}=\Sigma(fp\Delta Sl)=fpl\Sigma(\Delta S)=fpl\pi r=fP\frac{\pi}{2}.$$

The frictional resistance probably approximates to this limit when the journal is new.

Second. Let $p=p_0\cos\theta$, so that the intensity is now proportional to the depth CD and varies from a maximum p_0 at A to *nil* at B. This, perhaps, represents more accurately the pressure at different points when the journal is worn. Therefore

$$P=\Sigma(p\Delta S\cos\theta l)=\Sigma(p_0\Delta S\cos^2\theta l)$$

$$=2p_0lr\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta=p_0lr\frac{\pi}{2},$$

and the *frictional resistance* $=\Sigma(fp\Delta Sl)=2fp_0lr=fP\frac{4}{\pi}$.

Hence the frictional resistance lies between $fP\frac{\pi}{2}$ and $fP\frac{4}{\pi}$.

It may be represented by μP, μ being a coefficient of friction to be determined in each case by experiment.

The total *moment* of frictional resistance must necessarily be equal and opposite to the moment M of the couple twisting the shaft; i.e.,

$$M=\mu Pr.$$

Thus the total reaction at the surface of contact is equivalent to a single force P tangential to a circle of radius μr having its centre at O and called the *friction circle.*

The work absorbed by axle friction per revolution

$$=M\cdot 2\pi=2\mu\pi Pr.$$

The work absorbed by axle friction per minute

$$=2\mu\pi PrN=\mu Pv,$$

N being the number of revolutions and v the velocity per minute.

The work absorbed by frictional resistance produces an equivalent amount of heat, which should be dissipated at once in order to prevent the journal from becoming too hot. This may be done by giving the journal sufficient *bearing surface* (an area equal to the product of the diameter and the length of the bearing), and by the employment of a suitable unguent.

Suppose that h units of heat per square inch of bearing surface (ld) are dissipated per minute.

Let l inches be the length and d inches the diameter of the journal. Then hdl = heat-units dissipated = heat-units equivalent to frictional resistance

$$=\frac{\mu\pi PdN}{12J}=\frac{\mu Pv}{12J},$$

J being Joule's equivalent, or 778 ft.-lbs.,

or $$\frac{12Jh}{\mu\pi}=\frac{PN}{l} \quad \text{and} \quad \frac{12Jh}{\mu}=\frac{Pv}{ld}.$$

Let $\frac{P}{ld}=p=$ pressure per square inch of bearing surface. Then

$$pv=\frac{12Jh}{\mu}=\text{a constant.}$$

In Morin's experiments d varied from 2 to 4 in., P from 330 lbs. to 2 tons, and v did not exceed 30 ft. per minute; so that pv was <5000, and the coefficient of friction for the given limits was found to be the same for sliding friction.

Much greater values of pv occur in modern practice.

Rankine gives $p(v+20)=44{,}800$ as applicable to locomotives.

Thurston gives $pv=60{,}000$ as applicable to marine engines and to stationary steam-engines.

Frictional wear prevents the diminution of l below a certain limit at which the pressure per unit of *bearing surface* exceeds a value p given by the formula

$$P=pld=pkd^2,$$

where

$$\kappa=\frac{l}{d}.$$

In practice $k=\frac{1}{2}$ for slow-moving journals (e.g., joint-pins), and varies from $1\frac{1}{2}$ to 3 for journals in continuous motion. The best practice makes the length of the journal equal to four diameters (i.e., $k=4$) for mill-shafting.

Again, if the journal is considered a beam supported at the ends,

$$CPl=\frac{qd^3}{32}\pi,$$

q being the maximum permissible stress per square inch, and C a coefficient depending upon the method of support and upon the manner of the loading. Therefore

$$d^2\propto\frac{k}{q}.$$

For a given value of P, d diminishes as q increases. Also, it has been shown that the work absorbed by friction is directly proportional to d.

Hence, for both reasons, d should be a minimum and the shaft should be made of the strongest and most durable material. In practice the pressure per square inch of bearing surface may be taken at about 2 tons per square inch for cast iron, 3¾ tons per square inch for wrought iron, and 6¼ tons per square inch for cast steel.

It would appear, however, from the recent experiments of Tower and others, that the nature of the material *might* become of minor importance, while that of a suitable lubricant would be of paramount importance. They show that the friction of properly lubricated journals follows the laws of fluid friction much more closely than those of solid friction, and that the lubrication might be made so perfect as to prevent any absolute contact between the journal and its bearing. The journal would therefore *float* in the lubricant, so that there would be no metallic friction. The loss of power due to frictional resistance, as well as the consequent wear and tear, would be very considerably diminished, while the load upon the journal might be increased to almost any extent.

Again, Tower's experiments indicate that the friction diminishes as the temperature rises, a result which had already been experimentally determined by Hirn. It was also inferred by Hirn that if the temperature were kept uniform, the friction would be approximately proportional to $\sqrt{v}$, and Thurston has enunciated the law that, with a cool bearing, the friction is approximately proportional to $\sqrt[5]{v}$ for all speeds exceeding 100 ft. per minute.

With a speed of 150 ft. per minute and with pressures varying from 100 to 750 lbs. per square inch, Thurston found experimentally that f varied inversely as the square root of the intensity of the pressure. The same law, but without any limitations as to speed or pressure, had been previously stated by Hirn.

8. Pivots.—Pivots are usually cylindrical, with the circular edge of the base removed and sometimes with the whole of the base rounded. Conical pivots are employed in special machines in which, e.g., it is important to keep the axis of the shaft in an invariable position. Spherical pivots are often used for shafts subject to sudden shocks or to a lateral movement.

(*a*) *Cylindrical Pivots.*—If the shafts are to be run slowly, the intensity of pressure (p) on the step should not be so great as to squeeze out the lubricant. Reuleaux gives the following rules:

The maximum value of p in pounds per square inch should be 700 for wrought iron on gun-metal, 470 for cast iron on gun-metal, and 1400 for wrought iron on lignum-vitæ.

For rapidly moving shafts

$$d = c\sqrt{Pn},$$

n being the number of revolutions per minute, c a coefficient to be determined by experiment $(= .0045)$, and P the load upon the pivot.

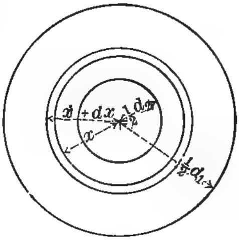


FIG. 393.

Suppose the surface of the step to be divided into rings, and let one of these rings be bounded by the radii x, $x+dx$.

In one revolution the work absorbed by the friction of this ring

$$= \mu p \cdot 2\pi x \cdot dx \cdot 2\pi x.$$

Hence the *total* work absorbed in one revolution

$$= \int_{\frac{d_2}{2}}^{\frac{d_1}{2}} 4\mu p \pi^2 x^2 dx = \frac{\mu p \pi^2}{6}(d_1^3 - d_2^3) = \frac{2}{3}\mu\pi P\frac{d_1^3 - d_2^3}{d_1^2 - d_2^2},$$

where $$P = \frac{p\pi}{4}(d_1^2 - d_2^2),$$

and d_1, d_2 are the external and internal diameters of the surface in contact.

If the *whole* of the surface is in contact, $d_2 = 0$, and the work absorbed $= \frac{2}{3}\mu\pi P d_1$.

Again, the *moment* of friction for the ring

$$= \mu p \cdot 2\pi x \cdot dx \cdot x = 2\mu\pi p x^2 dx,$$

and the total moment

$$= \int_{\frac{d_2}{2}}^{\frac{d_1}{2}} 2\mu\pi p x^2 dx = \frac{2}{3}\mu\pi p\frac{d_1^3 - d_2^3}{8}$$

$$= \frac{\mu p \pi}{12}(d_1^3 - d_2^3) = \frac{\mu P}{3}\frac{d_1^3 - d_2^3}{d_1^2 - d_2^2}.$$

If $d_2=0$, the moment $=\frac{\mu P}{3}d_1$.

Thus, in both cases, the work absorbed by friction $=2\pi$ times the moment of friction.

Let D be the *mean* diameter of the surface in contact $=\frac{d_1+d_2}{2}$.

Let $2y$ be the width of the surface in contact $=d_1-d_2$.

Then
$$\text{work absorbed}=\mu\pi P\left(D+\frac{y^2}{3D}\right).$$

Sometimes shafts have to run at high speeds and to bear heavy pressures, as, e.g., in screw-propellers and turbines. In order that there may be as little vibration as possible, p must be as small as practicable, and this is to some extent insured by using a collar-journal.

Let N be the number of collars, and let d_1, d_2 be the external and nternal diameters of a collar.

Then work absorbed by friction per revolution per collar

$$=\frac{\mu p\pi^2}{6}(d_1{}^3-d_2{}^3)=\tfrac{2}{3}\mu\pi\frac{P}{N}\frac{d_1{}^3-d_2{}^3}{d_1{}^2-d_2{}^2}=2\pi\times\text{moment of friction}.$$

According to Reuleaux, the mean diameter of a collar

$$=D=\sqrt[3]{\frac{Pn^2}{N^2}},$$

n being the number of revolutions per minute.

Also, the *width* of surface in contact $=d_1-d_2=.48\sqrt{D}$, and the maximum allowable pressure per square inch

$$=p=\frac{46940}{n}.$$

(*b*) *Wear.*—The wear at any point of the elementary ring must necessarily be proportional to the friction μp, and also to the amount of rubbing surface which passes over the point in a unit of time, i.e., to the velocity ωx, ω being the angular velocity of the shaft.

Hence the wear at any point is proportional to $\mu p\omega x$.

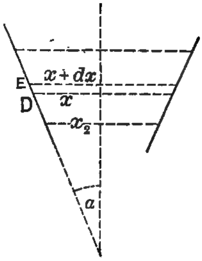


FIG. 394.

(c) *Conical Pivots.*—As before, suppose the surface of the step to be divided into a number of elementary rings. Two cases will be discussed:

First. Assume that the normal intensity of pressure p at the surface of contact is constant.

Let x, $x+dx$ be the distances of D and E, respectively, from the axis.

The total moment of friction

$$=\int_{x_2}^{x_1}\mu pDE\cdot 2\pi x\cdot x=\frac{2\mu p\pi}{\sin\alpha}\int_{x_2}^{x_1}x^2dx$$

$$=\frac{2}{3}\frac{\mu p\pi}{\sin\alpha}(x_1{}^3-x_2{}^3),$$

x_1, x_2 being the radii of the top and bottom sections of the step.

Also, P, the total load on the pivot,

$$=\int_{x_2}^{x_1}pDE\sin\alpha\cdot 2\pi x=2\pi p\int_{x_2}^{x_1}xdx$$

$$=\pi p(x_1{}^2-x_2{}^2).$$

Hence total moment of friction $=\dfrac{2}{3}\dfrac{\mu P}{\sin\alpha}\dfrac{x_1{}^3-x_2{}^3}{x_1{}^2-x_2{}^2}$.

Second. Assume that the wear is of such a nature that every point, e.g. D, descends vertically through the same distance.

Thus the normal wear $\propto\sin\alpha$,

$$\text{or}\quad \mu q\omega x\propto\sin\alpha,$$

$$\text{or}\quad px\propto\sin\alpha.$$

In the present case α is constant, and hence $px=$ a constant.

Thus total moment of friction

$$=\int_{x_2}^{x_1}\mu pDE\cdot 2\pi x\cdot x=\frac{2\mu px\pi}{\sin\alpha}\int_{x_2}^{x_1}xdx$$

$$=\frac{\mu px\pi}{\sin\alpha}\sin\alpha(x_1{}^2-x_2{}^2).$$

Also, $$P=\int_{x_2}^{x_1} pDE \sin\alpha\cdot 2\pi x$$

$$=2\pi px\int_{x_2}^{x_1} dx=2\pi px(x_1-x_2).$$

Hence total moment of friction $=\dfrac{\mu P}{2\sin\alpha}(x_1+x_2)$.

(*d*) *Schiele's Pivots.*—The object aimed at in these pivots is to give the step such a form that the wear and the pressure are the same at all points.

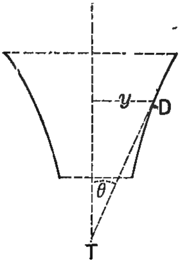


FIG. 395.

Let θ be the angle made by the tangent at any point of the step with the axis.

Let y be the distance of the point from the axis. Then

$$py\propto\sin\theta;$$

and hence, if p is constant,

$$y\propto\sin\theta\quad\text{or}\quad y\operatorname{cosec}\theta=\text{a const.}$$

is the equation of the generating line of the step. This line is known as the *tractrix*, and also as the *anti-friction* curve. If the tangent at D intersects the axis in T,

$$DT=y\operatorname{cosec}\theta=\text{a const.}$$

The curve may be traced by passing from one point to another and keeping the tangent DT of constant length.

The above equation may be written

$$y\frac{ds}{dy}=\text{a const.}=a,$$

or $$\frac{ds}{dx}=\frac{a}{y}\frac{dy}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2},$$

which may be easily integrated, the result being the analytical equation to the curve, viz.,

$$x=a\log_e\left(\frac{a-\sqrt{a^2-y^2}}{y}\right)+\sqrt{a^2-y^2}+\text{a const.}$$

Schiele or anti-friction pivots are suitable for high speeds, but have not been very generally adopted.

9. **Belts and Ropes.**—Let the figure represent a pulley movable about a journal at O, and let a belt (or rope), acted upon by forces T_1, T_2 at the ends, embrace a portion ABC of the circumference subtending an angle α at the centre.

In order that there may be motion in the direction of the arrow, T_1 must exceed T_2 by an amount sufficient to overcome the *frictional resistance* along the arc of contact and the *resistance to bending* due to the stiffness of the belt.

Consider first the frictional resistance, and suppose the belt to be *on the point of slipping.*

Any small element BB' $(=ds)$ of the belt is acted upon by a pull T tangential to the pulley at B, a pull $T-dT$ tangential to the pulley at B', by a reaction equivalent to a normal force Rds at the middle point of BB', and by a tangential force, or frictional resistance, μRds.

Let the angle $COB=\theta$, and the angle $BOB'=d\theta$.

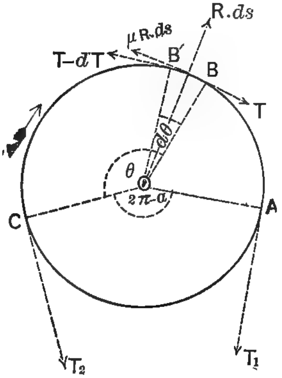


Fig. 396.

Resolving normally,

$$(T+T-dT)\sin\frac{d\theta}{2}-Rds=0. \quad \ldots\ldots \quad (1)$$

Resolving tangentially,

$$(T-\overline{T-dT})\cos\frac{d\theta}{2}-\mu Rds=0, \quad \ldots\ldots \quad (2)$$

μ being the coefficient of friction.

Now $d\theta$ being very small, $\sin\frac{d\theta}{2}$ is approximately $\frac{d\theta}{2}$, $\cos\frac{d\theta}{2}$ is approximately *unity,* and small quantities of the second order may be disregarded.

Hence eqs. (1) and (2) may be written

$$Td\theta-Rds=0, \quad \ldots\ldots\ldots \quad (3)$$

and

$$dT-\mu Rds=0. \quad \ldots\ldots\ldots \quad (4)$$

Therefore
$$dT = \mu T d\theta, \quad \text{or} \quad \frac{dT}{T} = \mu d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Integrating,
$$\log_e T = \mu\theta + C,$$

C being a constant of integration.

When $\theta = 0$, $T = T_2$, and hence $\log_e T_2 = C$.

Therefore
$$\log_e \frac{T}{T_2} = \mu\theta,$$

or
$$\frac{\mathbf{T}}{\mathbf{T}_2} = \mathrm{e}^{\mu\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

When $\theta = \alpha$, $T = T_1$, and hence
$$\frac{\mathbf{T}_1}{\mathbf{T}_2} = \mathrm{e}^{\mu\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

e being the number 2.71828, i.e., the base of the Naperian system of logarithms. The angle α may be called the angle of *lap*.

If α is increased by β, the new ratio of tensions will be $e^{\mu\beta}$ times the old ratio; so that if α increases in arithmetical progression, the ratio of tensions will increase in geometrical progression. This rapid increase in the ratio of the tensions, corresponding to a comparatively small increase in the arc of contact, is utilized in "brakes" for the purpose of absorbing surplus energy. For example:

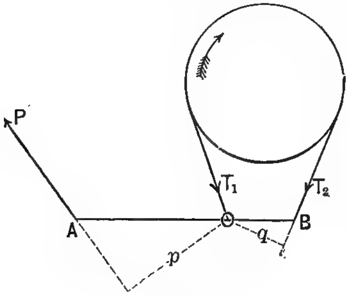


FIG. 397.

A flexible brake consisting of an iron or steel strap, or, again, of a chain, or of a series of iron bars faced with wood and jointed together, embraces about three fourths of the circumference of an iron or wooden drum. One end of the brake is secured to a fixed point O, and the other to the end B of a lever AOB turning about a fulcrum at O. A force applied at A will cause the brake to clasp the drum and so produce friction which will gradually bring the drum to rest.

Let ω be the angular velocity of the drum before the brake is applied.

Let I be the moment of inertia of the drum with respect to its axis.

The kinetic energy of the drum $=\dfrac{I\omega^2}{2}$.

When the brake is applied, the motion being in the direction of the arrow, let the greater and lesser tensions at its ends be T_1 and T_2 respectively.

Let n be the number of revolutions in which the drum is brought to rest. Then

$$\tfrac{1}{2}I\omega^2=(T_1-T_2)\pi dn, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

d being the diameter of the drum.

Also, if P is the force applied at A, and if p and q are the perpendicular distances of O from the directions of P and T_2 respectively,

$$Pp=T_2q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Again, $$T_1=T_2e^{\mu a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

a being the angle subtended at the centre by the arc of contact.

Hence, by eqs. (8), (9), (10),

$$n=\frac{qI\omega^2}{2Pp(e^{\mu a}-1)\pi d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

If the motion of the drum were in the opposite direction, q would be the perpendicular distance of O from the direction of T_1, and then $Pp=T_1q$.

Proceeding as before,

$$n'=\frac{qI\omega^2e^{\mu a}}{2Pp(e^{\mu a}-1)\pi d},$$

and therefore the number of turns in the second case, before the drum comes to rest, is $e^{\mu a}$ times the number in the first, which is consequently the preferable arrangement.

The coefficient of friction μ varies from .12 for greasy shop-belts on iron pulleys to .5 for new belts and hempen ropes on wooden drums. In ordinary practice an average value of μ for dry belts on iron pulleys is .28, and for wire ropes .24; if the belts are wet, μ is about .38.

Formulæ (6) and (7) are also true for non-circular pulleys.

10. Effective Tension.—The pull available for the transmission of power $=T_1-T_2=S$. Let H.P. be the horse-power transmitted, v the speed of transmission in feet per second, a the sectional area of the rope or belt, and s the stress per square inch in the *advancing* portion of the belt.

Then if T_1 and T_2 are in pounds,

$$\text{H.P.}=\frac{(T_1-T_2)v}{550}=\frac{Sv}{550}, \quad \text{and} \quad T_1=as.$$

The working tensile stress per square inch usually adopted for leather belts varies from 285 lbs. (Morin) to 355 lbs. (Claudel), an average value being 300 lbs. In wire ropes 8500 lbs. per square inch may be considered an average working tension.

Hempen ropes for the transmission of power generally vary from $4\frac{1}{2}$ to $6\frac{1}{2}$ ins. in circumference.

11. Effect of High Speed.—When the speed of transmission is great the effect of centrifugal force must be taken into account.

The centrifugal force of the elements $ds=\frac{wads}{g}\frac{v^2}{r}$, w being the specific weight of the belt or rope, and r the radius of the pulley.

Eq. (3) above now becomes

$$Td\theta-Rds-\frac{wads}{g}\frac{v^2}{r}=0,$$

or

$$Td\theta-\frac{wad\theta}{g}v^2-Rds=0;$$

and hence, by eq. (4),

$$\frac{dT}{T-\frac{wa}{g}v^2}=\mu d\theta.$$

Integrating,

$$\log_e\frac{T-\frac{wa}{g}v^2}{T_2-\frac{wa}{g}v^2}=\mu\theta,$$

since $T=T_2$ when $\theta=0$.

Also, $T = T_1$ when $\theta = \alpha$, and therefore

$$\frac{T_1 - \frac{wa}{g}v^2}{T_2 - \frac{wa}{g}v^2} = e^{\mu\alpha},$$

or

$$T_1 = T_2 e^{\mu\alpha} - \frac{wa}{g}v^2(e^{\mu\alpha} - 1).$$

the work transmitted per second

$$= (T_1 - T_2)v = \left(T_2 v - \frac{wa}{g}v^3\right)(e^{\mu\alpha} - 1),$$

which is a maximum and equal to $\frac{2}{3}T_2(e^{\mu\alpha} - 1)v$ when $v = \sqrt{\frac{T_2 g}{3wa}}$, and the two tensions are then in the ratio of $2e^{\mu\alpha} + 1$ to 3.

The speed for which no work is transmitted, i.e., the limiting speed, is given by

$$T_2 v - \frac{wa}{g}v^3 = 0, \quad \text{or} \quad v = \sqrt{\frac{T_2 g}{wa}}.$$

12. **Slip of Belts.**—A length l of the belt (or rope) becomes $l\left(1 + \frac{p_1}{E}\right)$ on the *advancing side* and $l\left(1 + \frac{p_2}{E}\right)$ on the *slack* side where $p_1 = \frac{T_1}{a}$ and $p_2 = \frac{T_2}{a}$, E being the coefficient of elasticity. Thus the advancing pulley draws on a greater length than is given off to the driven pulley, and its speed must therefore exceed that of the latter by an amount given by the equation

$$\frac{\text{reduction of speed, or slip}}{\text{speed of driving-pulley}} = \frac{l\left(1 + \frac{p_1}{E}\right) - l\left(1 + \frac{p_2}{E}\right)}{l\left(1 + \frac{p_1}{E}\right)} = \frac{p_1 - p_2}{E + p_1}.$$

The slip or creep of the belt measures the loss of work. In ordinary practice the loss with leather belting does not exceed 2 per cent, while with wire ropes it is so small that it may be disregarded.

13. Prony's Dynamometer.—This dynamometer is one of the commonest forms of friction-brake. The motor whose power is to be measured turns a wheel E which revolves between the wood block B and a band of wood blocks A. To the lower block is attached a lever of radius p carrying a weight P at the free end. By means of the screws C, D the blocks may be tightened around the circumference until the unknown moment of frictional resistance FR is equal to the known moment Pp, R being the radius of the wheel.

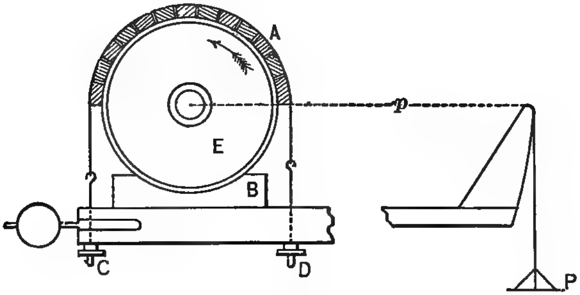


FIG. 398.

The weight P, which rests upon the ground when the screws are slack, is now just balanced.

The work absorbed by friction per minute $=2\pi RFn=2\pi Ppn$, n being the number of revolutions per minute.

14. Stiffness of Belts and Ropes.—The belt on reaching the pulley is bent to the curvature of the periphery, and is straightened again when it leaves the pulley. Thus an amount of work, increasing with the stiffness of the belt, must be expended to overcome the resistance to bending. As the result of experiment, this resistance has been expressed in the form $\frac{aT}{bR}$, T being the tension of the belt, a its sectional area, R the radius of the pulley, and b a coefficient to be determined.

According to Redtenbacher, $b=2.36$ ins. for hempen ropes.
" " " $b=1.67$ " " " "
" " Reuleaux, $b=3.4$ " " leather belts.

Let the figure represent a sheave in a pulley-block turning in the direction of the arrow about a journal of radius r.

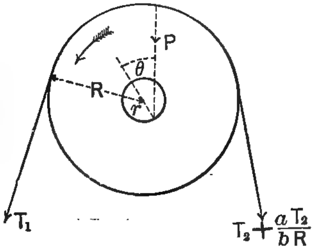


FIG. 399.

Let T_1 be the effort, T_2 the resistance.

The resistance due to the stiffness of the belt may be allowed for by adding $\frac{aT_2}{bR}$ to the force T_2. The frictional resistance at the journal surface is $P \sin \phi$ or fP, P being the resultant of T_1, T_2.

The motion being steady, and taking moments about the centre,

$$T_1R=\left(T_2+\frac{aT_2}{bR}\right)R+fPr,$$

or

$$T_1=T_2+\frac{aT_2}{bR}+f\frac{r}{R}P.$$

If T_1 and T_2 are parallel, $P=T_1+T_2$, and the last equation becomes

$$T_1=T_2+\frac{aT_2}{bR}+f\frac{r}{R}(T_1+T_2).$$

Let the pulley turn through a small angle θ.

The *counter-efficiency* of the sheave

$$=\frac{\text{motive work}}{\text{useful work}}=\frac{T_1\theta}{T_2\theta}=\frac{T_1}{T_2}=1+\frac{2fr}{R-fr}+\frac{a}{b}\frac{1}{R-fr}.$$

In the case of an endless belt connecting a pair of pulleys of radii R_1, R_2, the resistance due to stiffness may be taken equal to $\frac{aT}{b}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$, T being the mean tension $\left(=\frac{T_1+T_2}{2}\right)$.

The resistance due to journal friction $=frP\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$.

The useful resistance $=T_1-T_2=S$.

Hence the counter-efficiency

$$=1+\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\left(\frac{aT}{bS}+2fr\frac{P}{S}\right).$$

In wire ropes the stress due to bending may be calculated as follows:

Let x be the radius of a wire. The radius of its axis is sensibly the same as the radius R of the pulley.

The outer layers of the wire will be stretched, and the inner shortened, while the axis will remain unchanged in length. Hence

$$\frac{x}{R}=\frac{\text{change of length of outer or inner strands}}{\text{length of axis}}=\frac{\text{unit stress}}{E},$$

$$\text{and the unit stress due to bending}=E\frac{x}{R}.$$

15. Wheel and Axle.—Let the figure represent a wheel of radius p turning on an axle of radius r, under the action of the two tangential forces P and Q, inclined to each other at an angle θ.

The resultant R of P and Q must equilibrate the resultant reaction between the wheel and axle at the surface of contact.

Let the directions of P and Q meet in T.

If there were no friction, the resultant reaction and the resultant R would necessarily pass through O and T.

Taking friction into account, the direction of R will be inclined to TO. Let its direction intersect the circumference of the axle in the point A. The angle between TA and the normal AO at A, the motion being steady, is equal to the angle of friction; call it ϕ.

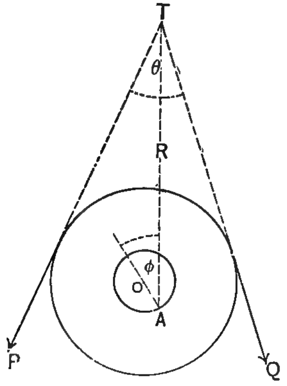


FIG. 400.

Taking moments about O,

$$Pp-Qp-Rr\sin\phi=0. \quad \ldots\ldots \quad (1)$$

Also,

$$R^2=P^2+Q^2+2PQ\cos\theta. \quad \ldots\ldots \quad (2)$$

Let $f=\sin\phi=\frac{\mu}{\sqrt{1+\mu^2}}$, μ being the coefficient of friction.

Eq. (1) may now be written

$$Pp-Qp-fRr=0. \quad \ldots \quad (3)$$

If P and Q are parallel in direction,

$$\theta=0 \quad \text{and} \quad R=P+Q.$$

Let the figure represent a wheel and axle.

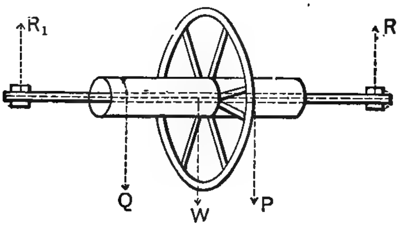


FIG. 401.

Let P be the effort and Q the weight lifted, the directions of P and Q being parallel.
" W be the weight of the "wheel and axle."
" R_1 and R_2 be the vertical reactions at the bearings.
" p be the radius of the wheel.
" q " " " axle.
" r " " " bearings.

Take moments about the axis. Then

$$Pp-Qq-R_1r\sin\phi-R_2r\sin\phi=0. \quad \ldots \quad (4)$$

But
$$R_1+R_2=W+P+Q. \quad \ldots \quad (5)$$

Hence
$$Pp-Qq=(W+P+Q)r\sin\phi=(W+P+Q)fr,$$

or
$$P(p-fr)=Q(q+fr)+fWr. \quad \ldots \quad (6)$$

Efficiency.—In turning through an angle θ,

$$\text{motive work} = Pp\theta,$$

$$\text{useful work} = Qq\theta;$$

therefore

$$\text{efficiency} = \frac{Qq\theta}{Pp\theta} = \frac{Qq}{Pp},$$

and the ratio $\frac{Q}{P}$ is given by eq. (6).

16. Toothed Gearing.—In toothed gearing the friction is partly rolling and partly sliding, but the former will be disregarded, as it is small as compared with the latter.

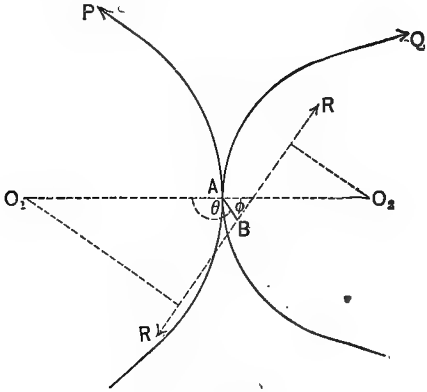


Fig. 402.

Let the pitch-circles of a pair of teeth in contact at the point B touch at the point A, and consider the action *before reaching* the line of centres O_1O_2, i.e., along the *arc of approach.*

The line AB is normal to the surfaces in contact at the point B.

Let R be the resultant reaction at B. Its direction, the motion being steady, makes an angle ϕ, equal to the angle of friction, with AB.

Let θ be the angle between O_1O_2 and AB.

Let the motive force and force of resistance be respectively equivalent to a force P tangential to the pitch-circle O_1, and to a force Q tangential to the pitch-circle O_2.

Let r_1, r_2 be the radii of the two wheels.

The work absorbed by friction in turning through the small arc ds

$$=(P-Q)ds. \qquad \qquad (1)$$

Consider the wheel O_1, and take moments about the centre.

$$Pr_1=R\{r_1 \sin (\theta-\phi)+x \sin \phi\}, \qquad \qquad (2)$$

where $AB=x$.

Similarly, from the wheel O_2,

$$Qr_2=R\{r_2 \sin (\theta-\phi)-x \sin \phi\}. \qquad \qquad (3)$$

Hence

$$\frac{Q}{P}=\frac{\sin (\theta-\phi)-\dfrac{x}{r_2}\sin \phi}{\sin (\theta-\phi)+\dfrac{x}{r_1}\sin \phi}, \qquad \qquad (4)$$

and therefore

$$P-Q=Q\frac{\left(\dfrac{1}{r_1}+\dfrac{1}{r_2}\right)x \sin \phi}{\sin (\theta-\phi)-\dfrac{x}{r_2}\sin \phi}. \qquad \qquad (5)$$

Hence the work absorbed by friction in the arc ds

$$=Q\frac{\left(\dfrac{1}{r_1}+\dfrac{1}{r_2}\right)x \sin \phi \, ds}{\sin (\theta-\phi)-\dfrac{x}{r_2}\sin \phi}. \qquad \qquad (6)$$

In precisely the same manner it can be shown that, *after leaving* the line of centres, i.e., in the *arc of recess*,

$$\frac{Q}{P}=\frac{\sin (\theta+\phi)-\dfrac{x}{r_2}\sin \phi}{\sin (\theta+\phi)+\dfrac{x}{r_1}\sin \phi}, \qquad \qquad (7)$$

and the work absorbed by friction in the arc ds

$$=Q\frac{\left(\frac{1}{r_1}+\frac{1}{r_2}\right)x\sin\phi ds}{\sin(\theta+\phi)-\frac{x}{r_2}\sin\phi}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The ratio $\frac{Q}{P}$ and the *loss of work* given by eqs. (4) and (6) are respectively greater than the ratio $\frac{Q}{P}$ and the *loss of work* given by eqs. (7) and (8), and therefore it is advisable to make the arc of approach as small as possible.

Again, by eq. (4), motion will be impossible if

$$\sin(\theta-\phi)+\frac{x}{r_1}\sin\phi=0;$$

i.e., if

$$\cot\phi=\cot\theta-\frac{x}{r_1\sin\theta},$$

and this can only be true if the direction of R passes through O_2.

Simple approximate expressions for the *lost work* and efficiency may be obtained as follows:

θ differs very little from 90°, and x is small as compared with r_2 and differs little from the corresponding arc s measured from A.

Hence the work absorbed by friction in the arc ds

$$=Q\tan\phi\left(\frac{1}{r_1}+\frac{1}{r_2}\right)sds=Q\mu\left(\frac{1}{r_1}+\frac{1}{r_2}\right)sds,$$

and the work lost in arc of approach s_1

$$=\int_0^{s_1}Q\mu\left(\frac{1}{r_1}+\frac{1}{r_2}\right)sds=Q\mu\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{s_1^2}{2}. \quad . \quad . \quad . \quad (9)$$

The useful work done in the same interval $=Qs_1$.

The *counter-efficiency* (reciprocal of efficiency)

$$=\frac{Qs_1+Q\mu\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{s_1^2}{2}}{Qs_1}=1+\mu\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{s_1}{2}. \quad . \quad . \quad (10)$$

Similarly for the arc of recess s_2,

$$\text{the lost work} = Q\mu\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{s_2^2}{2}, \quad . \quad . \quad . \quad (11)$$

$$\text{and the } \textit{counter-efficiency} = 1+\mu\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{s_2}{2}. \quad . \quad . \quad (12)$$

If $s_1 = s_2 = \text{pitch} = p = \frac{2\pi r_1}{n_1} = \frac{2\pi r_2}{n_2}$, n_1, n_2 being the number of teeth in the driver and the follower respectively, the expressions for the lost work given by eqs. (9) and (11) are identical, and those for the counter-efficiency given by eqs. (10) and (12) are also identical.

Thus the *whole* work lost during the action of a pair of teeth

$$= Q\mu\left(\frac{1}{r_1}+\frac{1}{r_2}\right)p^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and the counter-efficiency

$$= 1+\mu\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{p}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$= 1+\mu\pi\left(\frac{1}{n_1}+\frac{1}{n_2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This last equation shows that the efficiency increases with the number of teeth.

If the follower is an annular wheel $\frac{1}{r_1}-\frac{1}{r_2}$ must be substituted for $\frac{1}{r_1}+\frac{1}{r_2}$ in the above equations. Thus with an annular wheel the counter-efficiency is diminished and the efficiency, therefore, increased.

It has been assumed that R and Q are constant, as their variation from a constant value is probably small. It has also been assumed that only one pair of teeth are in contact. The theory, however, holds good when more than one pair are in contact, an effort and resistance corresponding to P and Q being supposed to act for each pair.

17. Bevel-wheels.—Let IA, IB represent the developments of the axes of the pitch-circles II_1, II_2 of a pair of bevel-wheels when the pitch-cones are spread out flat, O_1, O_2 being the corresponding centres.

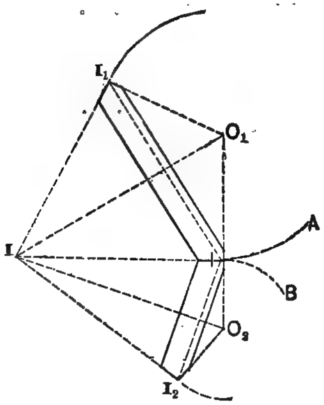


FIG. 403.

The preceding formulæ will apply to bevel-wheels, the radii being O_1I, O_2I, and the pitch being measured on the circumferences IA, IB.

18. Efficiency of Mechanisms.—Generally speaking, the ratio of the effort P to the resistance Q in a mechanism may be expressed as a function of the coefficient of friction μ. Thus

$$\frac{P}{Q}=F(\mu).$$

If, now, the mechanism is moved so that the points of application of P and Q traverse small distances $\varDelta x$, $\varDelta y$ in the directions of the forces,

$$\text{the efficiency}=\frac{Q\varDelta y}{P\varDelta x}=\frac{1}{F(\mu)}\frac{\varDelta y}{\varDelta x}.$$

TABLE OF COEFFICIENTS OF AXLE FRICTION.

	Dry.	Greasy and Wet.	Ordinary Lubrication.	Continuous Lubrication.	Pure Carriage-grease.	Lard and Plumbago.	Grease.
Bell-metal on bell-metal	....	......	.097				
Brass on brass	....	......	.079				
Brass on cast iron	....	......	.072	.049			
Cast iron on bell-metal	.194	.161	.075	.054	.065	......	.16
Cast iron on brass	.194	......	.075	.054			
Cast iron on cast iron	....	.137	.075	.054	......	......	.14
Cast iron on lignum-vitæ	.185	......	.1	.092	......	.11	.14
Lignum-vitæ on cast iron	....	......	.116	.17	......	......	.15
Lignum-vitæ on lignum-vitæ	....	......	......	.07			
Wrought iron on bell-metal	.251	.189	.075	.054	.09	.11	
Wrought iron on cast iron	....	......	.075	.054			
Wrought iron on lignum-vitæ	.187	......	.125				

But the ratio $\frac{\varDelta y}{\varDelta x}$ depends only upon the geometrical relations between the different parts of the mechanism, and will therefore remain the same if it is assumed that μ is zero. In such a case the efficiency would be perfect, or the motive work $(P\Delta x)$ would be equal to the useful work $(Q\Delta y)$, and therefore

$$1 = \frac{1}{F(0)} \frac{\Delta y}{\Delta x}.$$

Hence the efficiency $$= \frac{F(0)}{F(\mu)}.$$

EXAMPLES.

1. In a pair of four-sheaved blocks it is found that it requires a force P' to raise a weight $5P'$, and a force $5P'$ to raise a weight $15P'$. Show that the general relation between the force P and the weight W to be raised is given by

$$P = \frac{2}{5} W - P'.$$

Find the efficiency when raising the weights $5P'$ and $15P'$. *Ans.* $\frac{5}{8}$; $\frac{3}{8}$.

2. Find the mechanical advantage when an inch bolt is screwed up by a 15-in. spanner, the effective diameter of the nut being $1\frac{3}{8}$ ins., the diameter at the base of the thread .84 in., and .15 being the coefficient of friction. *Ans.* 80.

3. Find the turning moment necessary to raise a weight of 1000 lbs. by a vertical square-threaded screw having a pitch of 6 ins., the mean diameter of the thread being 4 ins. and the coefficient of friction $\frac{1}{3}$. *Ans.* 1930 in.-lbs.

4. The radii of the pulleys of a differential pulley-block are 6 ins. and $5\frac{3}{4}$ ins. Find the efficiency when a pull of 200 lbs. in the hauling-chain is required to raise a weight of 1 ton.

5. Find the mechanical efficiency of a screw-jack in which the load rotates with the head of the jack in order to eliminate collar friction. Threads per inch, 3; mean diameter of threads, $1\frac{3}{4}$ ins.; coefficient of friction, 0.14. Also find the efficiency when the load does not rotate. *Ans.* 30%; 17.1%.

6. In a lifting-machine, an effort of 26.6 lbs. just raises a load of 2260 lbs.; what is the mechanical advantage? If the efficiency is .755, what is the velocity ratio?

If on the same machine an effort of 11.8 lbs. raises a load of 580 lbs., what is now the efficiency?

7. The mean diameter of the threads of a $\frac{1}{2}$-in. bolt is 0.45 in., the slope of the thread 0.07, and the coefficient of friction 0.16. Find the tension of the bolt when pulled up by a force of 20 lbs. on the end of a spanner 12 ins. long. *Ans.* 1920 lbs.

8. If in a Weston pulley-block only 40 per cent of the energy expended is utilized in lifting the load, what should be the diameter of the smaller part of the compound pulley when the largest diameter is 8 ins. in order that a pull of 50 lbs. on the chain may raise a load of 550 lbs.? *Ans.* 7.42 ins.

9. The pitch of the screw of a lifting-jack is $\frac{3}{8}$ in. Disregarding friction, what force must be applied at the end of a 24-in. handle to raise a weight of 2664 lbs.? *Ans.* $6\frac{1}{8}$ lbs.

10. The law connecting a force P at the handle of a screw-jack with the weight W to be overcome is of the form

$$P = bW + c.$$

When $W = 2300$ lbs., $P = 30$ lbs., and when $W = 500$ lbs., $P = 10$ lbs.; find the values of the coefficients b and c.

Also, if the handle describes a circle of 19 ins., and if the pitch of the screw is $\frac{3}{8}$ in., find the velocity ratio. *Ans.* $b = \frac{1}{90}$, $c = 4\frac{4}{9}$; 318.47.

11. In the preceding example, find the efficiency in each case and also find the weight which must be lifted so that the efficiency may be 25 per cent.

12. A belt laps one half of a 14-in. pulley which makes 2000 revolutions per minute. The maximum tension is not to exceed 40 lbs. per inch of width. If $\mu = .28$, find the width of the belt and the power transmitted, the weight of the belt being 1.5 lbs. per square foot.

13. A belt embracing one half the circumference of a pulley transmits 10 H.P.; the pulley makes 30 revolutions per minute and is 7 ft. in diameter. Neglecting slip, find T_1 and T_2, μ being .125. *Ans.* $1541\frac{2}{3}$ lbs.; $1041\frac{2}{3}$ lbs.

14. How many ropes 4 ins. in circumference are required to transmit 200 H.P. from a pulley 16 ft. in diameter and making 90 revolutions per minute? *Ans.* 10.

15. A $\frac{1}{2}$-in. rope passes over a 6-in. pulley, the diameter of the axis being $\frac{1}{2}$ in.; the load upon the axis $= 2 \times$ the rope tension. Find the efficiency of the pulley, the coefficient of axle friction being .08 and the coefficient for stiffness .47. Hence also deduce the efficiency of a pair of three-sheaved blocks. *Ans.* .867; .427.

16. A belt laps 150° round a pulley of 3 ft. diameter making 130 revolutions per minute; the coefficient of friction is 0.35. What is the maximum pull on the belt when 20 H.P. is being transmitted and the belt is just on the point of slipping? *Ans.* .898 lb.

17. If the pulleys are 50 ft. centre to centre, and if the tight is three times the slack tension, find the length of the belt, the coefficient of friction being $\frac{1}{2}$ and the diameter of one of the pulleys 12 ins. *Ans.* 185.287 ft.

18. A 6-in. leather belt $\frac{1}{4}$ in. thick and weighing 0.4 lb. per lineal foot connects two pulleys, each 3 ft. in diameter, on parallel shafts, and is found to commence to slip when the moment of resistance is 400 ft.-lbs. and the revolutions are 500 per minute. Taking the coefficient of friction between the pulleys and belt to be .24, estimate the greatest and least tensions when on the point of slipping.

19. A strap is hung over a fixed pulley, and is in contact over an arc of length equal to two thirds of the total circumference. Under these circum-

stances a pull of 475 lbs. is found to be necessary in order to raise a load of 150 lbs. Determine the coefficient of friction between the strap and the pulley-rim. *Ans.* .275.

20. The tight tension on a 20-in. belt embracing one half the circumference of the pulley is 1200 lbs. Find the maximum work the belt will transmit, the thickness of the belt being .2 in. and its weight .0325 lb. per cubic inch. (Coefficient of friction =.28.) *Ans.* 68.75 H.P.

21. An endless belt weighing $\frac{1}{2}$ lb. per lineal foot connects two 35-in. pulleys and transmits 5 H.P., each pulley making 300 revolutions per minute. the tight and slack tensions, μ being .28.

22. Find the width of belt necessary to transmit 10 H.P. to a pulley 12 ins. in diameter, so that the greatest tension may not exceed 40 lbs. per inch of the width when the pulley makes 1500 revolutions per minute, the weight of the belt per square foot being 1.5 lbs., the angle of wrapping 180 degrees, and taking the coefficient of friction as 0.25. *Ans.* 8 ins.

23. A rope is run three times round a post, one end being held tight by a force of 10 lbs. Find the pull on the other end which will produce slip. (μ=.25.) *Ans.* 1000 lbs.

24. A rope-pulley carrying 20 ropes is 16 ft. in diameter and transmits 600 H.P. when running at 90 revolutions per minute. Taking μ=.7 and the angle of contact =180°, find the tension on the tight and slack sides.

25. In a rope-drive the rope weighs 0.8 lb. per foot. The tension being 290 lbs. on the tight side and 80 lbs. on the slack side, find approximately the sag in the two cases, supposing the shafts to be 50 ft. apart. Obtain the formula you employ.

26. A pulley 3 ft. 6 ins. in diameter and making 150 revolutions per minute, drives, by means of a belt, a machine which absorbs 7 H.P. What must be the width of belt so that its greatest tension shall be 70 lbs. per inch of width, it being assumed that the tension in the driving side is *twice* that on the slack side?

27. The efficiency of a single-rope pulley is found to be 94 per cent. Over how many of such pulleys must the rope pass in order to make it self-sustaining, i.e., to have an efficiency of under 50 per cent? *Ans.* 12.

28. A cable from a ship is wound three times round a post and a force equal to the weight of 100 lbs. is applied at the other end; how much energy is destroyed when the ship is brought to rest after dragging the cable 10 ft. the coefficient of friction being .2?

29. Power is transmitted from a pulley 5 ft. in diameter, running at 110 revolutions per minute, to a pulley 8 ins. in diameter. Thickness of belt =0.24 in.; modulus of elasticity of belt, 9000 lbs. per square inch; tension on tight side per inch of width =60 lbs.; ratio of tensions, 2.3 to 1. Find the revolutions per minute of the small pulley. *Ans.* 792.

30. A belt weighing $\frac{1}{2}$ lb. per lineal foot connects two 42-in. pulleys, on making 240 revolutions per minute. Find the limiting tension for which work will be transmitted. Also find the tight and slack tensions and the efficiency when the belt transmits 5 H.P. (Diameter of axle $=2$ in.; coefficient of friction $=.28$.) *Ans.* $30\frac{1}{4}$ lbs.; 106.82 lbs.; 44.32 lbs.

31. A circular saw makes 1000 revolutions per minute and is driven by a belt 3 ins. wide and $\frac{1}{5}$ in. thick, its weight per cubic inch being .0325 lb. The belt passes over a 10-in. pulley embracing one half the circumference and transmits 6 H.P. Find the tight and slack tensions, the coefficient of friction being .28. *Ans.* 130.16 lbs.; 54.56 lbs.

32. The most efficient speed of a 10-in.$\times\frac{1}{4}$-in. belt weighing .0325 lb. per cubic inch is 80 ft. per second, the corresponding tight and slack tensions being in the ratio of 7 to 3. The coefficient of friction is $\frac{1}{3}$. Find the angle subtended at the centre of the pulley by the arc of contact. Also find the tight and slack tensions and the work transmitted.

33. A cotton rope $1\frac{1}{2}$ ins. in diameter weighs 0.72 lb. per lineal foot and may be worked with 430 lbs. total tension. Find the horse-power transmitted at 60, 80, and 100 feet per second velocity of rope, the tight tension being $4\frac{1}{2}$ times the slack tension.

34. In a travelling-crane the driving-rope runs at 5000 ft. per minute. Find the tension due to centrifugal action, having given that a rope 1 in. in diameter weighs 0.28 lb. per foot of length. *Ans.* 60.4 lbs.

35. In an endless belt passing over two pulleys, the least tension is 150 lbs., the coefficient of friction .28, and the angle subtended by the arc of contact 148°. Find the greatest tension. The diameter of the larger wheel is 78 ins., of the smaller 10 ins., of the bearings 3 ins. Find the efficiency. A tightening-pulley is made to press on the slack side of the belt. Assuming that the working tension is to the coefficient of elasticity in the ratio of 1 to 80, find the increment of the arc of contact on the belt-pulley, the tension of the slack side, and the force of the tightening-pulley. *Ans.* 309 lbs.

36. Two pulleys 3 ft. 6 ins. in diameter, running at 150 revolutions per minute, are connected by a leather belt weighing 0.6 lb. per foot in length. Taking $\mu=.3$, find the greatest tension in the belt when transmitting $7\frac{1}{2}$ H.P. *Ans.* 360 lbs.

37. A flexible band embracing three fourths of the circumference of a brake-pulley keyed on a revolving shaft has one extremity attached to the end *A* of the lever *AOB*, and the other to the *fixed* point *O* (between *A* and *B*) about which the lever oscillates. The pressure between the band and pulley is effected by a force applied at right angles to the lever at the end *B*. Show that the time in which the axle is brought to rest is about $2\frac{1}{2}$ times as great when revolving in one direction as in the opposite. ($f=.2$.)

38. The power of an engine making *n* revolutions per minute is tested by a Prony brake having its arm of length *r* connected with a spring balance which registers a force *P*. The arm is vertical and the weight *W* of the brake

is supported by a stiff spring fixed vertically below the centre of the wheel. What error in B.H.P. would be introduced by placing the spring x ft. away from the central position? *Ans.* $\frac{BWx}{Pr}$, B being the B.H.P.

39. A string of wood blocks embraces the 24-in. pulley of an engine, one end of the string being attached to a load of 112 lbs. and the other to a spring balance which indicates 12 lbs. when the pulley is making 60 revolutions per minute. Find the work given out at the brake and the coefficient of friction between the blocks and pulley. *Ans.* 1.143 H.P.; .355.

40. In a Prony-brake test of a Westinghouse engine the blocks were fixed to a 24-in. fly-wheel with a 6-in. face, and the balance reading was 48 lbs.; the distance from centre of shaft to centre of balance, measured horizontally, was 30 ins., and the number of revolutions per minute was 624. Find the horse-power. *Ans.* 14.3.

41. An engine makes 150 revolutions per minute. If the diameter of the brake-pulley is 45 ins. and the pull on the brake is 50 lbs., find the B.H.P. *Ans.* 2.67.

42. A small water-motor is tested by a tail dynamometer. The pulley is 18 ins. in diameter; the weight is 60 lbs.; the spring registers a pull of 50 lbs.; the number of revolutions per minute $=500$. Find the B.H.P. *Ans.* $\frac{5}{7}$.

43. A Reynolds water-brake has recesses of 9 ins. internal and 18 ins. external diameter. Find the velocity in feet per second with which the water must circulate so as to absorb 15 H.P. at 200 revolutions per minute, the axis of the recesses being at 45° to the plane of the disk. *Ans.* 7.13 ft.

44. A Froude water-brake has recesses of 6 ins. internal and 18 ins. external diameter, the axis of the recesses being at 45° to the plane of the disk. The disk makes 80 revolutions per minute. The resistance to motion is balanced by 50 lbs. at the end of a 48-in. lever. Find the horse-power developed and the velocity of the water in feet per second in the direction of the recess axis. *Ans.* 3.05 H.P.; 11.4 ft.

45. A horizontal axle 10 ins. in diameter has a vertical load upon it of 20 tons and a horizontal pull of 4 tons. The coefficient of friction is 0.02. Find the heat generated per minute, and the horse-power wasted in friction, when making 50 revolutions per minute. *Ans.* 155 units; 3.63 H.P.

46. The vertical pressure upon a steel pivot of 100 mm. diameter is $2100k$ and the pivot makes 100 revolutions per minute. What is the work absorbed by friction per second, μ being .07? *Ans.* 51.34 km.

47. A shaft makes 20 revolutions per minute in a bearing of 0.25 m. diameter. If the load on the bearing is $8000k$, find the work consumed per second, μ being .07. *Ans.* 147 km.

48. A 6-ft. plank AB, hinged at B, has its middle point supported on a 24-in. grindstone and carries a weight of 100 lbs. at the end A. The grindstone weighs 600 lbs. and makes 175 revolutions per minute on a 1-in. axle. Taking the coefficient of plank friction and rolling friction to be .3 and .05 respectively, find in how many turns the grindstone will come to rest when the motive power ceases to act. *Ans.* 2.04 turns.

49. A 4-in. axle makes 400 revolutions per minute on anti-friction wheels

30 ins. in diameter, which are mounted on 3-in. axles. The load on the axle is 5 tons. Find the horse-power absorbed. ($\mu = 0.1$; $\theta = 30°$; $d = 0.01$ in.) *Ans.* 1.7.

50. Find the work absorbed by friction per revolution by a pivot 3 ins. long and carrying 6 tons, its upper face being 6 ins. in diameter, coefficient of friction .04, and 2α being 90°. *Ans.* .33936 in.-ton.

51. Calculate the horse-power absorbed by a footstep-bearing 8 ins. in diameter when supporting a load of 4000 lbs. and making 100 revolutions per minute, (*a*) with a flat end, (*b*) with a conical pivot $\alpha = 30°$, (*c*) with a Schiele pivot. (Take $\mu = .03$.)

Ans. (*a*) .51; (*b*) 1.02; (*c*) .76, if upper radius = length of tangent = 2 × lower radius.

52. The diameter of a solid cylindrical cast-steel pivot is $2\frac{1}{4}$ ins. Find the diameter of an equally efficient conical pivot.

53. The pressure upon a 4-in. journal making 50 revolutions per minute is 6 tons, the coefficient of friction being .05. Find the number of units of heat generated per second, Joule's mechanical equivalent of heat being 778 ft.-lbs.

54. A water-wheel of 20 ft. diameter and weighing 20,000 lbs. makes 10 revolutions per minute; the gudgeons are 6 ins. in diameter and the coefficient of friction is .1. Find the loss of mechanical effect due to friction. If the motive power is suddenly cut off, how many revolutions will the wheel make before coming to rest? *Ans.* $\frac{10}{21}$H.P.; 10.9.

55. A wedge with a taper of 1 in 8 is driven into a cottered joint with an estimated pressure of 600 lbs. Find the force with which the two parts of the joint are drawn together and the force required to withdraw the wedge, the coefficient of friction being .2. *Ans.* 1128 lbs.; 307 lbs.

56. A grindstone with a radius of gyration = 12 ins. and making 120 revolutions per minute is suddenly left to the influence of gravity and axle friction and comes to rest in 160 revolutions. Find the coefficient of axle friction, the diameter of the axle being $1\frac{1}{2}$ ins.

57. A fly-wheel weighing 8000 lbs. and having a radius of gyration of 10 ft. is disconnected from the engine at the moment it is making 27 revolutions per minute; it stops after making 17 revolutions. Find the coefficient of friction, the axle being 12 ins. in diameter. *Ans.* .2325.

58. A railway truck weighing 12 tons is carried on wheels 3 ft. in diameter; the journals are 4 ins. in diameter, the coefficient of friction $\frac{1}{20}$. Find the resistance of the truck so far as it arises from the friction of the journals.

Ans. $37\frac{1}{3}$ lbs.

59. A tramcar wheel is 30 ins. in diameter, the axle $2\frac{1}{2}$ ins.; the coefficient of axle friction .08, of rolling friction .09. Find the resistance per ton.

Ans. 28.37 lbs.

60. A bearing 16 ins. in diameter is acted upon by a horizontal force of 50 tons and a vertical force of 10 tons; the coefficient of friction is $\frac{1}{10}$. Find the horse-power absorbed by friction per revolution. *Ans.* .906 H.P.

61. A steel pivot 3 ins. in diameter and under a pressure of 5 tons makes 60 revolutions per minute in a cast-iron step-well lubricated with oil. How much work is absorbed by friction, the coefficient of friction being .08?

Ans. $.85\frac{1}{3}$ H.P.

62. A pair of spur-wheels are 4 ins. and 2 ins. in diameter; the flanks of the teeth are radial; the larger wheel has 16 teeth; the arc of approach = arc of recess = $\frac{3}{2}$ of the pitch. Show how to form the teeth, and find their efficiency. (Coefficient of friction = .11.) *Ans.* .97.

63. Find the work lost by the friction of a pair of teeth, the number of teeth in the wheels being 32 and 16, and the diameter of the larger wheel, which transmits 3 H.P. at 50 revolutions per minute, 3 ft. *Ans.* 3.646 ft.-lbs.

64. The driver of a pair of wheels has 120 teeth, and each wheel has an addendum equal to .28 time the pitch; the arcs of approach and recess are each equal to the pitch; the tooth-flanks are radial. Find the efficiency. (Coefficient of friction = .106.) *Ans.* .994.

CHAPTER VII.

ON THE TRANSVERSE STRENGTH OF BEAMS.

1. The Moment of Resistance.—Let the plane of the paper be a plane of symmetry with respect to the beam $PQRS$. If the beam is subjected to the action of external forces in this plane, $PQRS$ is bent and assumes a curved form $P'Q'R'S'$. The upper layer of fibres $Q'R'$ is extended, the lower layer $P'S'$ is compressed, while of the layers within the beam, those nearer $P'S'$ are compressed and those nearer $Q'R'$ are extended. Hence there must be a layer $M'N'$ between $P'S'$ and $Q'R'$ which is neither compressed nor extended. It is called the *neutral surface* (or *cylinder*), and its axis is perpendicular to the plane of flexure. In the present treatise it is proposed to deal with flexure in one plane only, and, in general, it will be found more convenient to refer to $M'N'$ as the *neutral line* (or *axis*), a term only used in reference to a *transverse* section.

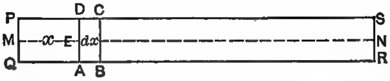


FIG. 404.

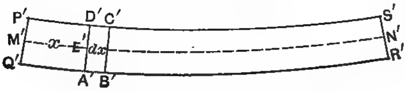


FIG. 405.

If a force act upon the beam in the direction of its length, the lower layer $P'S'$, instead of being compressed, may be stretched. In such a case there is no neutral surface *within* the beam, but theoretically it still exists somewhere *without* the beam.

Consider an indefinitely thin slice of the beam $ABCD$ at a distance x from an origin in the neutral axis and of thickness dx. If $A'B'C'D'$ is this element in the bent beam, the following assumptions are made:

(*a*) That the flexure is small;

(*b*) That the material of the beam is homogeneous;

(*c*) That any section AD which is plane *before bending* remains *plane after* bending;

(*d*) That the strains of the several layers are directly proportional to the stresses to which they are due, so that the layers stretch and shorten freely under the action of tensile and compressive forces, notwithstanding the connection between the different layers.

Let Figs. 406 and 407 represent enlarged views of the elements $ABCD$ and $A'B'C'D'$ in Figs. 404 and 405, and let the planes $A'D'$ and $B'C'$ intersect in O. The point O is the *centre of curvature* of the bent layers between $A'B'$ and $C'D'$.

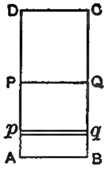


FIG. 406.

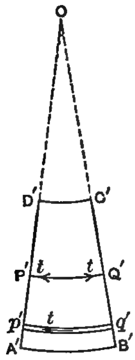


FIG. 407.

Let R be the *radius of curvature* of the layer $P'Q'$, which is neither lengthened nor shortened, and which is, therefore, subjected to *no stress* in the direction of its length.

Let t be the force developed along the layer $p'q'$ of sectional area da and at a distance y from $P'Q'$.

Without altering the conditions of equilibrium it may be assumed that two forces, opposite in direction but each equal in magnitude to t, act along $P'Q'$, and therefore the force along $p'q'$ is equivalent to

(*a*) a force t along $P'Q'$, together with

(*b*) a couple of moment ty.

The force along every other layer of the element is also equivalent to a similar force and a similar couple.

Hence the forces along all the layers are equivalent to

(*c*) a force Σt along $P'Q'$, together with

(*d*) a couple of moment Σty,

the symbol Σ denoting *algebraic* sum, as the forces change from tensions to compressions on passing from one side of the neutral surface to the other.

Again,
$$\frac{R+y}{R}=\frac{Op'}{OP'}=\frac{p'q'}{P'Q'},$$

or
$$\frac{y}{R}=\frac{p'q'-P'Q'}{P'Q'}=\text{the strain of the layer } p'q'$$
$$=\frac{1}{E}\frac{t}{da}.$$

Therefore
$$t=\frac{E}{R}yda.$$

Hence the total force along $P'Q'$
$$=\Sigma t=\frac{E}{R}\Sigma yda=0,$$

since $P'Q'$ remains unchanged in length.

Therefore $\Sigma y\cdot da=0$, and P' is the C. of G. of the face $A'D'$. *Thus the neutral axis is the locus of the centres of gravity of the transverse sections of the beams.* Also, the total moment of the couple acting on the element
$$=\Sigma ty=\frac{E}{R}\Sigma y^2da=\frac{E}{R}I=\frac{E}{R}Ak^2,$$

k being the radius of gyration and I the *moment of inertia* of the transverse section of the beam through $A'B'$ with respect to an axis at P' perpendicular to the plane of flexure.

The moment $\frac{E}{R}I$ is generally termed the *moment of resistance*, but is sometimes spoken of as the *elastic moment*. It must necessarily balance the bending moment M of the external forces which cause the flexure, and therefore

$$M=\frac{E}{R}I, \qquad \text{or} \qquad \frac{M}{I}=\frac{E}{R}.$$

Let f_y be the stress, i.e., the load per unit of area, in $p'q'$. Then

$$t=f_y \cdot da, \text{ and therefore } f_y=\frac{E}{R}y \text{ or } \frac{f_y}{y}=\frac{E}{R}.$$

Again, let z be the vertical deviation of E', Fig. 405, from MN, i.e., the *deflection* of E' with respect to the neutral axis. Then

$$\frac{1}{R}=\frac{\mp\frac{d^2z}{dx^2}}{\left\{1+\left(\frac{dz}{dx}\right)^2\right\}^{\frac{3}{2}}},$$

the upper or lower sign being taken according as the centre of curvature O falls above or below $M'N'$.

Now, $\frac{dz}{dx}$ is the tangent of the angle θ which the tangent to the neutral axis at E' makes with the straight line $M'N'$, and this angle is very small. Therefore, approximately,

$$\frac{dz}{dx}=\tan\theta=\theta,$$

and may be taken equal to *zero* in the expression for the curvature, so that

$$\frac{1}{R}=\mp\frac{d^2z}{dx^2}=\mp\frac{d\theta}{dx}.$$

Hence

$$\mp E\frac{d^2z}{dx^2}=\frac{E}{R}=\frac{M}{I}=\frac{f_y}{y}$$

are equations from which may be determined the deflection (z), the slope (θ), the curvature $\left(\frac{1}{R}\right)$ at any point of the neutral axis, and the stress developed at any distance y from the neutral axis. The **curvature** of the beam is $\frac{1}{R}=\frac{M}{EI}$. If the beam has an initial curvature $\frac{1}{R_0}$, the **change of curvature** is evidently $\frac{1}{R_0}-\frac{1}{R}=\frac{M}{EI}$.

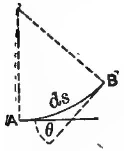


Fig. 408.

Curvature, Fig. 408, may be defined as the angular change (in radians) of the direction of the curve per unit of length $=\frac{\text{change in angle}}{AB}=\frac{\theta}{ds}$.

The beam is strained to the limit of safety when *either* of the extreme layers $A'B'$, $D'C'$ is strained to the limit of elasticity. In such a case the *least* of the values of $\frac{f_y}{y}$ for the layers in question is the greatest value consistent with the strength of the beam. If f_c and c are the corresponding stress and distance from the neutral axis, then

$$\frac{\mathbf{E}}{\mathbf{R}}\mathbf{I}=\mathbf{M}=\frac{\mathbf{f_c}}{\mathbf{c}}\mathbf{I}.$$

Again, Fig. 409 represents on an exaggerated scale the *transverse* section of the beam at $A'D'$, the upper and lower breadths of the beam, $A'A''$ and $D'D''$, being respectively contracted and stretched, and being also arcs of circles having a common centre at O'.

Let R' be the radius of the arc $P'P''$, whose length remains unchanged.

Let mE be the lateral coefficient of elasticity, m being a numerical coefficient. As before, for any layer at a distance y from $P'P''$,

$$\frac{mE}{R'}=\frac{t}{ay}=\frac{E}{R'}$$

and therefore $\quad R'=mR.$

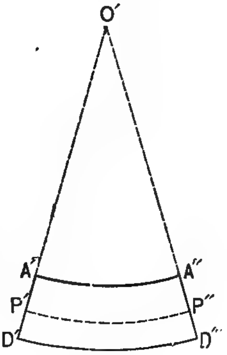


Fig. 409.

Thus, *within the limits of elasticity*, the curvature of the breadth is $\frac{1}{m}$ that of the length, and does not sensibly affect the resistance of the beam to bending. The influence, however, upon the bending may become sensible if the breadth is very large as compared with the depth, as, e.g., in the case of iron or steel plates.

2. Modulus of Section.—The ratio $\frac{I}{C}$ is called the strength *modulus* of the section and is usually denoted by the symbol Z.

The modulus may be easily determined as follows:

Divide any section, as, e.g., that shown by Fig. 410, into a number of thin layers by lines parallel to the neutral axis.

Let a_y be the area of one of these layers at y from the neutral axis, and let f_y be the stress developed in this layer.

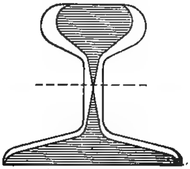


FIG. 410.

If the width, and therefore the area, of this layer are diminished in the ratio of $\frac{f_y}{f}$, f being the skin stress, then

$$\frac{f_y}{f}a_y = \text{diminished area} = a_y', \text{ suppose,}$$

and
$$f_y a_y = f a_y'.$$

Treating every layer in a similar manner, the *modulus figure*, shown shaded, is obtained and

$$\Sigma(f_y a_y) = \text{total force } \textit{on one side} \text{ of the neutral axis}$$

$$= f\Sigma(a_y') = fA',$$

A' being the area of *one side* of the modulus figure.

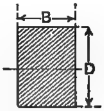
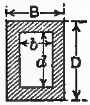
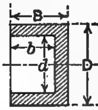
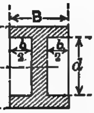
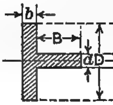
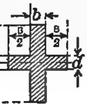
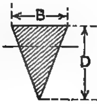
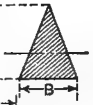
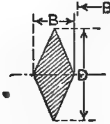
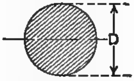
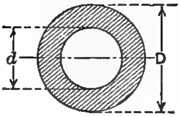
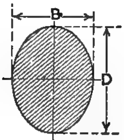
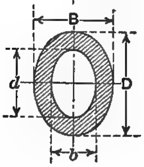
But this area is the same on each side since the neutral axis is at the C. of G. of the section.

Hence if $2A$ is the total area of the modulus figure, and if h is the *effective depth*, i.e., the distance between the centres of gravity of the modulus areas above and below the neutral axis,

$$fAh = \text{moment of resistance of section} = f\frac{I}{c} = fZ,$$

and
$$Ah = Z.$$

The table on page 422 gives the moments of inertia, strength modulus, and the square of the radius of gyration of various sections met with in ordinary practice.

Section.	Moment of Inertia I.	Section Modulus $\frac{I}{c}$.	Square of Radius of Gyration K^2.
FIG. 411. 	$\frac{BD^3}{12}$	$\frac{BD^2}{6}$	$\frac{D^2}{12}$
FIG. 412. 	$\frac{BD^3-bd^3}{12}$	$\frac{BD^3-bd^3}{6D}$	$\frac{BD^3-bd^3}{12(BD-bd)}$
FIG. 413.  FIG. 414. 	$\frac{BD^3-bd^3}{12}$	$\frac{BD^3-bd^3}{6D}$	$\frac{BD^3-bd^3}{12(BD-bd)}$
FIG. 415.  FIG. 416. 	$\frac{Bd^3+bD^3}{12}$	$\frac{Bd^3+bD^3}{6D}$	$\frac{Bd^3+bD^3}{12(Bd+bD)}$
FIG. 417.  FIG. 418. 	$\frac{BD^3}{36}$	$\frac{BD^2}{12}$ $\frac{BD^2}{24}$	$\frac{D^2}{18}$
FIG. 419. 	$\frac{BD^3}{48}$	$\frac{BD^2}{24}$	$\frac{D^2}{24}$
FIG. 420. 	$\frac{\pi D^4}{64}$	$\frac{\pi D^3}{32}$	$\frac{D^2}{16}$
FIG. 421. 	$\frac{\pi(D^4-d^4)}{64}$	$\frac{\pi(D^4-d^4)}{32D}$	$\frac{(D^2+d^2)}{16}$
FIG. 422. 	$\frac{\pi}{64}BD^3$	$\frac{\pi}{32}BD^2$	$\frac{D^2}{16}$
FIG. 423. 	$\frac{\pi}{64}(BD^3-bd^3)$	$\frac{\pi}{32}\frac{BD^3-bd^3}{D}$	$\frac{1}{16}\frac{BD^3-bd^3}{BD-bd}$

3. The Work of Flexure.—The work done stretching the layer $p'q'$ (Fig. 407)

$$=\frac{1}{2}t(p'q'-P'Q')=\frac{1}{2}t\frac{y}{R}dx=\frac{1}{2}\frac{dx}{R}ty,$$

and, therefore, the work done in distorting the element $ABCD$

$$=\frac{1}{2}\frac{dx}{R}\Sigma(ty)=\frac{1}{2}\frac{M}{R}dx=\frac{M^2dx}{2EI}.$$

Hence the total work of flexure between points defined by values x_1 and x_2 of x

$$=\frac{\mathrm{I}}{2\mathrm{E}}\int_{x_1}^{x_2}\frac{\mathrm{M}^2}{\mathrm{I}}\mathrm{dx}.$$

If on this portion of the beam loads are concentrated at different points, the integration must be taken between each pair of consecutive loads and the results superposed.

This expression is necessarily equal to the work of the external forces between the same limits, and is also the energy acquired by the beam in changing from its natural state of equilibrium.

If the proof load P is concentrated at one point of a beam, and if d is the proof deflection, the *resilience* $=\frac{P}{2}d$.

If a proof load of intensity w is uniformly distributed over the beam, and if y is the deflection at any point, the resilience $=\frac{1}{2}\int wydx$, the integration extending throughout the whole length of the beam.

4. Equalization of Stress.—The stress at any point of a beam under a transverse load is proportional to its distance from the neutral plane so long as the elastic limit is not exceeded. At this limit materials which have no ductility give way. In materials possessing ductility, the stress may go on increasing for some distance beyond the elastic limit without producing rupture, but the stress is no longer proportional to the distance from the neutral plane, its variation being much slower. This is due to the fact that the portion in compression acquires increased rigidity and so exerts a continually increasing resistance (Chapter IV) almost if not quite up to the point of rupture, while in the stretched portion a flow of metal occurs and an approximately constant resistance to the stress is developed. Thus there will be a more or less perfect equalization of stress throughout the section, accompanied by an increase of the elastic limit and of the apparent strength, the increase depending both upon the form of section and the ductility.

For example, if the tensile elastic limit is the same as the compressive, the shaded portion of Fig. 424 gives a graphical representation of the total stress in a beam of rectangular section when the straining is within the elastic limit. Beyond this limit it may be represented as in Fig. 425, and will be intermediate between Fig. 424 and the shaded rectangle of Fig. 426, which corresponds to a state of perfect equalization.

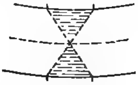


Fig. 424.

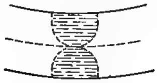


Fig. 425.

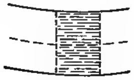


Fig. 426.

By means of a specially designed extensometer, the author has carried out a number of experiments with a view to determine the changes of fibre length, within the limit of elasticity, at different depths of a beam loaded transversely. For a full account of these experiments the reader is referred to the R.S.C. Trans., Vols. VII and VIII.

The loading was of two kinds, namely, (1) loads of increasing magnitude placed at the centre, and (2) equal loads of increasing magnitude concentrated at two points equidistant from the centre, the maximum B.M. in each case being the same as for the corresponding centrally placed load.

In all measurements the beams were placed on supports 60 ins. apart and the distance between the extensometer points was 8 ins.

Diagrams were plotted showing the "lengthenings" and "shortenings" of the fibres at different depths, and an inspection shows that they are approximately proportional to the distance from the line which does not apparently change in length. These experiments, therefore, seem to verify the assumption that the stress developed at any point in the beam is approximately proportional to the distance from the neutral surface, and the agreement of the assumption with fact becomes more marked as the homogeneousness of the material increases.

The results of the experiments justify the following general inferences:

With timber beams (Figs. 427 and 428):

(1) That when a beam is loaded at the centre, the position of the neutral surface under increasing loads remains practically unchanged and is a little nearer the compression than the tension side.

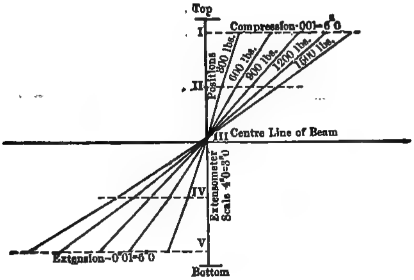


FIG. 427.

(2) That when loads are concentrated at points equidistant from the centre, the neutral surface under the smaller loads is at some considerable distance from the mid-depth on the compression side. This distance diminishes as the load increases, and under the heaviest loads the neutral surface seems to have gradually returned to nearly the same position as when the beam was loaded at the centre.

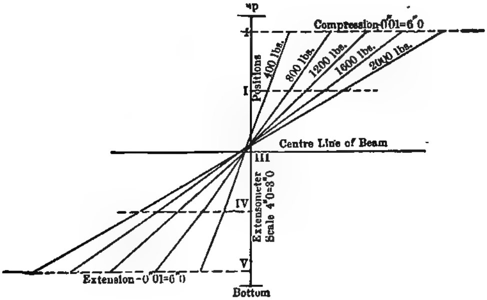


FIG. 428.

With a 7.1″×3.33″ cast-iron beam (Figs. 429 and 430):

(1) That the curves for the loads at the centre show less variation in the position of the neutral surface than when the loads are

concentrated at equidistant points. The axis is precisely at the mid-depth of the beam up to 2400 lbs., but beyond this load there is a perceptible movement towards the compression side.

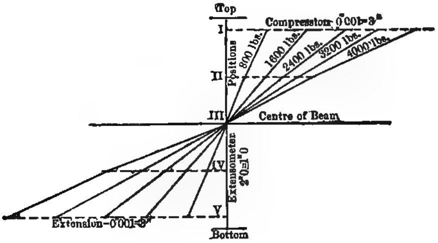


FIG. 429.

(2) That the diagram for the beam under loads 10 ins. from the centre shows a slight movement of the neutral surface towards

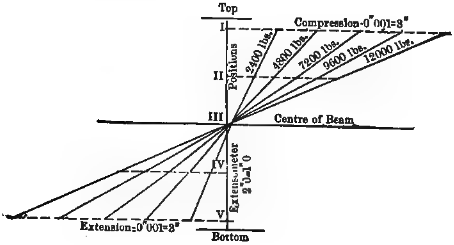
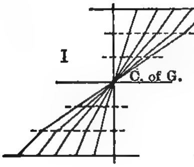


FIG. 430.

the tension side for the smaller loads, but under 3600 lbs. it suddenly moves to the compression side and then gradually returns towards the centre under still higher loads.



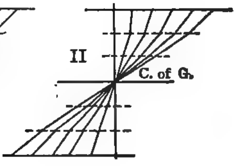


FIG. 431. FIG. 432.

(3) That loads at 20 ins. from the centre show that the neutral surface is much nearer the tension side for the smaller loads, but

gradually moves to a position slightly on the compression side for the higher loads.

With an 8-in. rolled joist (Figs. 431 to 434):

(1) That in Fig. 433, in which the loads are concentrated at 8-in. centres, the stress in the material is almost directly proportional to the distance from the neutral axis, which seems to be slightly above the centre of gravity.

(2) That the diagrams for the 30-in., 20-in., and 15-in. concentrations indicate that the stress in the material increases more rapidly than the distance from the neutral axis, while the increase is not so rapid for the 6-in. (Fig. 434) concentration and for the beam loaded at the centre. In the last case the neutral axis has moved very appreciably above the centre of gravity.

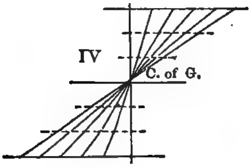


FIG. 433.

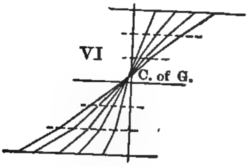


FIG. 434.

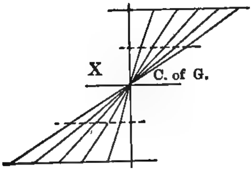


FIG. 435.

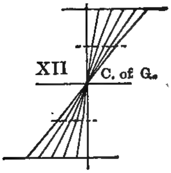


FIG. 436.

With a 7.85″×3.425″ cast-steel beam (Figs. 435 to 438):

3 That in all cases the stresses in the material are very approximately proportional to the distance from the neutral surface, and that this neutral surface very approximately coincides with the centre

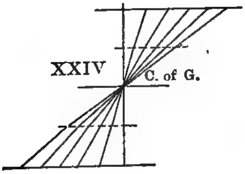


FIG. 437.

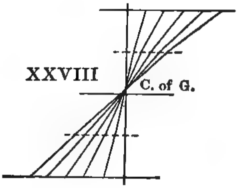


FIG 438.

of gravity, indicating that, in the case of the concentrated loads, the variation of stress in the beam in question is closely in accordance with theory.

(2) That the Figs. 435 to 438, plotted from the results obtained for a beam loaded at the centre, show that the neutral surface has very appreciably moved towards the compression side, but that the stress in the material is still proportional to the distance from the neutral surface in its changed position.

Ex. 1. *A timber beam* 6 *ins. square and* 20 *ft. long rests upon two supports, and is uniformly loaded with a weight of* 100 *lbs. per lineal foot. Determine the stress at the centre at a point distant* 2 *ins. from the neutral line.*

Also find the central curvature, E being 1,200,000 *lbs. per square inch.*

$$I=\frac{6\cdot 6^3}{12}=108, \quad M=\frac{100\cdot 20\cdot 20}{8}=5000 \text{ ft.-lbs.}=60{,}000 \text{ in.-lbs.}$$

Hence $$\frac{60000}{108}=\frac{f_y}{2}=\frac{1200000}{R}.$$

Therefore $f_y=1111\frac{1}{9}$ lbs./sq. in. and $R=2160$ ins. $=180$ ft.

Ex. 2. *A standpipe section* 33 *ft. in length and weighing* 5720 *lbs. is placed upon two supports in the same horizontal plane* 30 *ft. apart. The internal diameter of the pipe is* 30 *ins. and its thickness* $\frac{1}{2}$ *in. Determine the additional uniformly distributed load which the pipe can carry between the bearings, so that the stress in the metal may nowhere exceed* 2 *long tons per square inch.*

Let W be the required load in pounds.

The weight of the pipe between the bearings $=\frac{30}{33}\times 5720=5200$ lbs.

Thus the total distributed weight between the bearings $=(W+5200$ lbs.$)$.

The straining is evidently greatest at the centre, and at this point

$$M=\frac{W+5200}{8}30\times 12 \text{ in.-lbs.}=45(W+5200) \text{ in.-lbs.}$$

Also, $$I=\frac{22}{7}\cdot 15^3\cdot\frac{1}{2}=\frac{11}{7}\ 15^3.$$

Therefore $$\frac{45(W+5200)}{\frac{11}{7}15^3}=\frac{2\times 2240}{15}$$

and $$W=30{,}000 \text{ lbs.}$$

Ex. 3. *An iron bar is bent into the arc of a circle of* 500 *ft. diameter; the coefficient of elasticity is* 30,000,000 *lbs. Find the moment of resistance of a section of the bar and the maximum intensity of stress in the metal,* (*a*) *when* *the bar is round and* 1 *in. in diameter,* (*b*) *when the bar is square, having a side of* 1 *in.*

If the metal is not to be strained above 10,000 *lbs. per square inch, find* (*c*) *the diameter of the smallest circle into which the bar can be bent.*

$$\text{Moment of resistance} = \frac{E}{R}I = \frac{30000000}{250\times 12}I = 10{,}000I \text{ in.-lbs.}$$

For case (*a*), $$I = \frac{\pi(1)^4}{64}, \text{ and the moment} = 10{,}000\frac{\pi}{64}$$
$$= \frac{625}{4}\pi \text{ in.-lbs.}$$

Also, $$\frac{f}{\frac{1}{2}} = \frac{30000000}{250\times 12}, \quad \text{and} \quad f = 5000 \text{ lbs./sq. in.}$$

For case (*b*), $$I = \frac{1^4}{12}, \text{ and the moment} = 10{,}000\times\frac{1}{12}$$
$$= 833\tfrac{1}{3} \text{ in.-lbs.}$$

Also, $$\frac{f}{\frac{1}{2}} = \frac{30000000}{250\times 12}, \quad \text{and} \quad f = 5000 \text{ lbs./sq. in.}$$

For case (*c*), $$\frac{10000}{\frac{1}{2}} = \frac{30000000}{R\times 12}, \quad \text{and} \quad R = 125 \text{ ft.,}$$

or the diameter = 250 ft.

Ex. 4. *To cut out of an elliptic section, with its major axis,* 2*a*, *vertical, the rectangular section which has the greatest moment of resistance.*

Let $2b$ be the minor axis of the ellipse;

$2x$ and $2y$ be the depth and width respectively of the required rectangular section.

Its moment of resistance $= \frac{f_x}{x}\,\frac{4}{3}yx^3 = \text{a max.}$

But $\frac{f_x}{x}$ is constant, and therefore

$$yx^3 = \text{a max.},$$

or $$3ydx + xdy = 0.$$

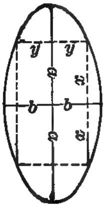


FIG. 439.

Again, $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and differentiating, $$\frac{xdx}{a^2} + \frac{ydy}{b^2} = 0.$$

Therefore $$\frac{x^2}{3a^2} = \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}.$$

Hence
$$x=\frac{a\sqrt{3}}{2},\quad y=\frac{b}{2},$$

and the corresponding moment of resistance $=\left(\frac{f_x}{x}\right)\frac{\sqrt{3}}{4}a^3b.$

Ex. 5. *The driving-wheels of a locomotive are d feet in diameter and the length of the crank-radius is r feet. At a speed of s miles per hour the stress developed in the connecting-rod is not to exceed f pounds per square inch. The connecting-rod weighs W pounds, is l feet long, and has a constant moment of inertia I from end to end.*

Assume that the connecting-rod may be considered a beam supported at the two ends. In addition to its own weight W it carries a uniformly distributed load of $\frac{W}{g}\frac{v^2}{r}$ developed by the centrifugal force, v being the linear velocity of the crank-pin in feet per second. Thus

$$v=s\times\frac{5280}{3600}\frac{2r}{d}=\frac{44}{15}\frac{rs}{d}\text{f/s},$$

and
$$W\left(1+\frac{v^2}{gr}\right)\frac{12l}{8}=\text{max. B.M. in in.-lbs.}$$
$$=\frac{f}{c}I,$$

c being the distance of the extreme fibres from the neutral axis.

Ex. 6. *The section of a hook taken at right angles to the direction of the pull is* 3 *ins. deep and* 1½ *ins. wide. Calculate the maximum and minimum stresses in the material when the hook is loaded with* 10 *tons and the distance of the line of pull from the inner edge of the section is* 1 *in.*

The max. stress/sq. in. = stress due to direct pull

± " " " bending

$$=\frac{10}{3\times1\frac{1}{2}}\pm\frac{10\times2\frac{1}{2}\times1\frac{1}{2}}{\frac{1}{12}\times1\frac{1}{2}\times3^3}$$

$$=\frac{20}{9}\pm\frac{100}{9}$$

= 13⅓ tons in tension and $8\frac{8}{9}$ tons in compression.

Ex. 7. *If the pin-holes for a bridge eye-bar were drilled out of truth sideways, and the main body of the bar were* 5 *ins. wide and* 2 *ins. thick, what proportion would the maximum stress bear to the mean over any cross-section of the bar at which the mean line of force was* ⅛ *in. from the middle of the section?*

Let P tons be the pull on the bar. Then

max. stress at section = stress due to direct pull ± stress due to bending

$$=\frac{P}{5\times2}\pm\frac{P\times\frac{1}{8}\times2\frac{1}{2}}{\frac{1}{12}\times2\times5^3}=\frac{P}{10}\left(1\pm\frac{3}{20}\right)$$

$=\frac{P}{10}\,\frac{23}{20}$ tons /sq. in. in tension and $\frac{P}{10}\,\frac{17}{20}$ tons/sq. in. in compression,

and the ratio of the maximum stress to the mean is 1.15 and 0.85 on the tension and compression sides respectively.

Ex. 8. *A steel boiler-plate tube* 36 *ft. long,* 30 *ins. inside diameter, weighs* 4200 *lbs. and rests upon supports* 33 *ft. apart. Find the maximum intensity of stress in the metal. What additional weight may be suspended from the centre, assuming that the stress is nowhere to exceed* 8000 *lbs. per square inch ?*

Let t = thickness of tube. Then

$$4200=\text{weight of tube in pounds}=\frac{1}{144}(\overline{15+t}^2-15^2)36\times490,$$

or

$$t^2+30t=\frac{120}{11}\quad\text{and}\quad t=.359\text{ in.}$$

$$I,\text{ the moment of inertia,}=\frac{22}{7}\left(\frac{\overline{15+t}^4-15^4}{4}\right)=\frac{11}{14}\times5028.1.$$

The weight of the unsupported length of tube $=\frac{33}{36}\times4200=3850$ lbs.

This is uniformly distributed, and therefore

$$\frac{3350}{8}\,33\times12=\text{max. B.M. in in.-lbs.}=\frac{f}{15.359}\left(\frac{11}{14}\times5028.1\right).$$

Hence, $$f=740.9\text{ lbs./sq. in.}$$

Next, let P be the weight required at the centre. Then

$$\frac{P}{4}\times33\times12+\frac{3850}{8}33\times12=\text{max. B.M. in in.-lbs.}$$

$$=\frac{8000}{15.359}\times\frac{11}{14}\times5028.1,$$

and therefore $$P=18{,}860\text{ lbs.}$$

Note. If r is the interior radius of a hollow tube of thickness t, the sectional area of the tube $=\pi(\overline{r+t}^2-r^2)=2\pi rt$, if t is so small as compared with r that t^2 may be disregarded without appreciable error. Also, the moment of inertia of the section $=\frac{\pi}{4}(\overline{r+t}^4-r^4)=\pi r^3t$, approximately. Hence the moment of resistance of a hollow tube whose thickness is small as compared with the radius

$$=\frac{f}{r}\pi r^3t=\pi fr^2t,\ \text{approximately.}$$

Ex. 9. *A cast-iron rectangular girder rests upon supports 12 ft. apart and carries a weight of 2000 lbs. at the centre. If the breadth is* one half *the depth, find the sectional area of the girder so that the intensity of stress may nowhere exceed 4000 lbs. per square inch.*

Also, find the depth of a wrought-iron girder 3 ins. wide which might be substituted for the cast-iron girder, the coefficient of strength for the wrought-iron being 8000 lbs. per square inch.

Take d the depth and $b(=\frac{1}{2}d)$ the breadth of the cast-iron girder. Then, disregarding the weight of the girder,

$$\frac{2000}{4}12\times12=\text{max. B.M. in in.-lbs.}=4000\frac{bd^2}{6}=\frac{4000}{12}d^3.$$

Therefore $d^3=216$ and $d=6$ ins., so that the sectional area $=bd=18$ sq. ins.

The weight of the girder $=\frac{bd}{144}12\times450=\frac{75}{4}d^2$ lbs. Taking this into account,

$$\frac{1}{8}\times\frac{75}{4}d^2\times12\times12+\frac{2000}{4}12\times12=\text{max. B.M. in in-lbs.}$$

$$=\frac{4000}{12}d^3,$$

or $$d^3-\frac{81}{80}d^2-216=0 \quad\text{and}\quad d=6.38 \text{ ins.}$$

Therefore $$\text{sectional area}=\frac{6.38}{2}\times6.38=20.353 \text{ sq. ins.}$$

Next, let D be the required depth of the wrought-iron girder. Disregarding the weights of the girders,

$$2000\frac{3\times6^2}{6}=8000\frac{3D^2}{6} \quad\text{and}\quad D^2=9, \quad\text{or}\quad D=3 \text{ in.}$$

The weight of the wrought-iron girder

$$=\frac{3D}{144}12\times480=120D \text{ lbs.}$$

Taking this into account,

$$\frac{120D}{8}12\times12+\frac{2000}{4}12\times12=\text{max. B.M. in in.-lbs.}$$

$$=8000\frac{3D^2}{6},$$

or $$D^2-.54D-18=0 \quad\text{and}\quad D=4.52 \text{ ins.}$$

Ex. 10. *Compare the strength modulus of a rectangular section* 6 *cm. wide* × 24 *cm. deep with that of a double-tee section of the same area, the flanges being* 18 *cm.*×3 *cm. and the web* 18 *cm.*×2 *cm.*

For the rectangular section,

$$\frac{I}{c}=\frac{1}{12}\left(\frac{6\times 24^3}{12}\right)=576.$$

For the double-tee section,

$$\frac{I}{c}=\frac{1}{12}\left(\frac{18\times 24^3-2\times 8\times 18^3}{12}\right)=1080.$$

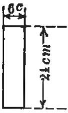


FIG. 440.

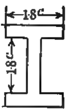


FIG. 441.

Thus the ratio of the two moduli $=\frac{1080}{576}=\frac{15}{8}$, so that the double-tee section is nearly twice as strong as a rectangular section of the same area.

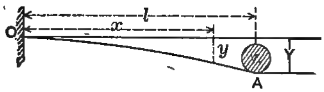


FIG. 442.

Ex. 11. *OA, Fig.* 442, *is the neutral axis of a cantilever fixed at O and carrying a uniformly distributed load of intensity w together with a weight W at the free end A.*

Let l be the length of the cantilever. Then, at any point (x, y) in the neutral axis with respect to O,

$$M_x=EI\frac{d^2y}{dx^2}=\frac{w}{2}(l-x)^2+W(l-x)$$

$$=\frac{w}{2}(l^2-2lx+x^2)+W(l-x).$$

Integrating,

$$EI\frac{dy}{dx}=\frac{w}{2}\left(l^2x-lx^2+\frac{x^3}{3}\right)+W\left(lx-\frac{x^2}{2}\right)+c_1,$$

c_1 being a constant of integration.

If the cantilever is so fixed that the neutral axis at O is horizontal, then $\frac{dy}{dx}=0$ at O, i.e., when $x=0$, and therefore $c_1=0$. Thus

$$EI\frac{dy}{dx}=EI\tan\theta=\frac{w}{2}\left(l^2x-lx^2+\frac{x^3}{3}\right)+W\left(lx-\frac{x^2}{2}\right)$$

an equation giving the slope θ of the axis at any point. Integrating again,

$$EIy=\frac{w}{2}\left(l^2\frac{x^2}{2}-l\frac{x^3}{3}+\frac{x^4}{12}\right)+W\left(l\frac{x^2}{2}-\frac{x^3}{6}\right),$$

an equation giving the deflection of any point in the axis.

There is no constant of integration, as x and y vanish together. The point A is evidently the most deflected point, and if Y is the value of y at A, i.e., when $x=l$,

$$\mathrm{EIY}=\mathrm{l}^3\left(\frac{\mathrm{wl}}{8}+\frac{\mathrm{W}}{3}\right)$$

or
$$\mathrm{EIY} = l^3\left(\frac{P}{8}+\frac{W}{3}\right),$$

if P is the uniformly distributed load $= wl$.

Again, let ΔY be the increment of Y corresponding to an increment ΔW of the weight W. Then

$$EI(Y+\Delta Y) = l^3\left(\frac{P}{8}+\frac{W+\Delta W}{3}\right),$$

and therefore
$$EI\Delta Y = \frac{l^3}{3}\Delta W,$$

or
$$\mathrm{E} = \frac{1}{3}\frac{\Delta \mathrm{W}}{\Delta \mathrm{Y}}\frac{l^3}{\mathrm{I}},$$

which is the equation commonly employed in determining the value of the modulus E.

If W is *nil*,
$$\mathrm{Y} = \frac{1}{8}\frac{wl^4}{\mathrm{EI}} = \frac{1}{8}\frac{\mathrm{P}l^3}{\mathrm{EI}},$$

and if $P(=wl)$ is small as compared with W,

$$\mathrm{Y} = \frac{1}{3}\frac{\mathrm{W}l^3}{\mathrm{EI}}.$$

The deflection may be approximately found by assuming that the neutral axis OA is bent into the form of a circular arc of radius R. Draw the horizontal AN, intersecting the vertical through O in N. The curvature is so small that the difference in the lengths of OA and AN may be disregarded. Taking $ON = Y =$ the maximum deflection, then

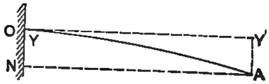


Fig. 443.

$$AN^2 = ON(2R-ON) = 2R\cdot ON, \text{ approximately.}$$

Therefore
$$l^2 = 2RY \quad \text{and} \quad Y = \frac{l^2}{2R}.$$

Also,
$$\frac{\mathrm{M_0}}{\mathrm{I}} = \frac{f_y}{y} = \frac{\mathrm{E}}{\mathrm{R}} = \frac{2\mathrm{EY}}{l^2}.$$

Ex. 12. *A beam supported at O and A in the same horizontal plane carries a uniformly distributed load of intensity w together with a weight W concentrated at the middle point.*

Let l be the distance between the supports.

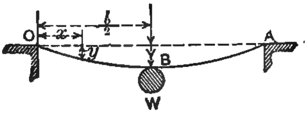


Fig. 444

First. Let the beam merely rest upon the supports (Fig. 444). Then, *between O and B*, at any point (x, y) of the neutral axis, with reference to O,

$$M = -EI\frac{d^2y}{dx^2} = \frac{W+wl}{2}x - \frac{wx^2}{2},$$

$\frac{W+wl}{2}$ being the reaction at O (or A). Integrating,

$$-EI\frac{dy}{dx} = \frac{W+wl}{4}x^2 - \frac{wx^3}{6} + c_1,$$

c_1 being a constant of integration.

At the middle point the neutral axis is horizontal, and therefore

$$\frac{dy}{dx} = 0, \quad \text{when} \quad x = \frac{l}{2},$$

so that

$$0 = \frac{Wl^2}{16} + \frac{wl^3}{24} + c_1.$$

Hence

$$-EI\frac{dy}{dx} = -EI \tan\theta$$

$$= \frac{W+wl}{4}x^2 - \frac{wx^3}{6} - \frac{l^2}{8}\left(\frac{W}{2} + \frac{wl}{3}\right),$$

an equation giving the slope of the neutral axis at any point between O and B

Integrating again,

$$-EIy = \frac{W+wl}{12}x^3 - \frac{wx^4}{24} - \frac{l^2}{8}\left(\frac{W}{2} + \frac{wl}{3}\right)x,$$

in which there is no constant of integration, as x and y vanish together.

This last equation defines the deflection curve and gives the deflection o any point of the neutral axis between O and B.

Let Y be the maximum deflection, i.e., the deflection at the middle point where $x = \frac{l}{2}$. Then

$$+EIY = \frac{l^3}{48}\left(\tfrac{5}{8}wl + W\right),$$

or

$$EIY = \frac{l^3}{48}\left(\tfrac{5}{8}P + W\right),$$

if P is the uniformly distributed load $= wl$.

Again, let ΔY be the increment of Y, corresponding to an increment ΔW of W. Then

$$EI(Y + \Delta Y) = \frac{l^3}{48}\left(\tfrac{5}{8}P + W + \Delta W\right),$$

and therefore

$$EI\Delta Y = \frac{l^3}{48}\Delta W,$$

or

$$E = \frac{1}{48}\frac{\Delta W}{\Delta Y}\frac{l^3}{I},$$

an equation also commonly employed in determining the value of the modulus E.

If W is *nil*,
$$Y=\frac{5}{384}\frac{wl^4}{EI}=\frac{5}{384}\frac{Pl^3}{EI},$$

and if P is small as compared with W,
$$Y=\frac{1}{48}\frac{Wl^3}{EI}.$$

The deflection may be approximately found by assuming that the neutral axis OA is bent into the arc of a circle of radius R. Draw the horizontal OA, intersecting the vertical, through the middle point B in N. The curvature is so small that the difference in the lengths of ON and OB may be disregarded.

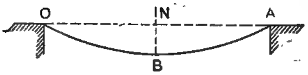


FIG. 445.

Take $BN=Y$, the maximum deflection. Then
$$DN^2=BN(2R-BN)=2R\cdot BN, \text{ approximately.}$$

Therefore
$$\frac{l^2}{4}=2RY \quad\text{and}\quad Y=\frac{1}{8}\frac{l^2}{R}.$$

Also,
$$\frac{M_B}{I}=\frac{f_y}{y}=\frac{E}{R}=\frac{8EY}{l^2}.$$

The deflection at any distance x from O is $Y-y$, y being given by
$$\left(\frac{l}{2}-x\right)^2=2Ry.$$

Second. Let both ends of the beam be fixed so that the neutral axis is horizontal at O and at A. Let M_1 be the *moment of fixture* at each end. The moment at O evidently tends to produce rotation from right to left. Then, at any point (x, y) *between O and B*,
$$M=-EI\frac{d^2y}{dx^2}=\left(\frac{W+wl}{2}\right)x-\frac{wx^2}{2}-M_1.$$

Integrating,
$$-EI\frac{dy}{dx}=-EI\tan\theta$$
$$=\frac{W+wl}{4}x^2-\frac{wx^3}{6}-M_1x,$$

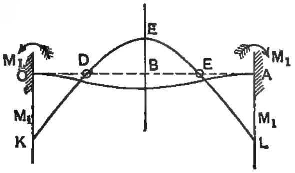


FIG. 446.

an equation giving the slope θ at any point.

There is no constant of integration, as $\frac{dy}{dx}$ and x vanish together.

Also,
$$\frac{dy}{dx}=0 \quad\text{when}\quad x=l, \quad\text{and therefore}$$
$$0=\frac{W+wl}{16}l^2-\frac{wl^3}{48}-M_1\frac{l}{2},$$

or $$M_1=\frac{Wl}{8}+\frac{wl^2}{12}=l\left(\frac{W}{8}+\frac{P}{12}\right).$$

Integrating again, $$-EIy=\frac{W+wl}{12}x^3-\frac{wx^4}{24}-M\frac{x^2}{2},$$

an equation defining the deflection curve.

There is no constant of integration, as x and y vanish together.

Let Y be the maximum deflection, i.e., the value of y at the middle point, where $x=\frac{l}{2}$. Then

$$EIY=\frac{l^3}{192}\left(W+\frac{wl}{2}\right)=\frac{l^3}{192}\left(W+\frac{P}{2}\right).$$

If W is *nil*, $$Y=\frac{1}{384}\frac{Pl^3}{EI} \quad\text{and}\quad M_1=\frac{wl^2}{12}=\frac{Pl}{12}.$$

If P $(=wl)$ is small as compared with W,

$$Y=\frac{1}{192}\frac{wl^3}{EI} \quad\text{and}\quad M_1=\frac{Wl}{8}$$

Again, the B.M. is *nil* when $M=0=-EI\frac{d^2y}{dx^2}$,

or $$0=\left(\frac{W+wl}{2}\right)x-\frac{wx^2}{2}-M_1,$$

a quadratic, giving two values of x and defining two points in the beam at which the B.M. is zero and at which, therefore, pins may be introduced so that the beam may be considered as consisting of two cantilevers *OD*, *AC*,

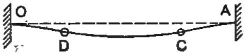


FIG. 447.

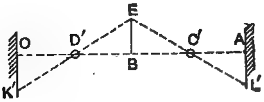


FIG. 448.

supporting an independent intermediate length *CD* (Fig. 447). The length *OD* $(=AC)$ is, of course, the least of the values of x given by the last equation.

At the middle point of the beam, i.e., when $x=\frac{l}{2}$, the

$$\text{B.M.}=\left(\frac{W+wl}{2}\right)\frac{l}{2}-\frac{w}{2}\left(\frac{l}{2}\right)^2-l\left(\frac{W}{8}+\frac{wl}{12}\right)$$

$$=l\left(\frac{W}{8}+\frac{wl}{24}\right)=l\left(\frac{W}{8}+\frac{P}{24}\right).$$

Take $OK=l\left(\frac{W}{8}+\frac{P}{12}\right)=AL$ and the vertical $BE=l\left(\frac{W}{8}+\frac{P}{24}\right)$.

The parabola $KDECL$ is the B.M. diagram, showing that the bending actions on the central span and on the two side lengths are opposite in character.

If it is assumed that W alone acts, then

$$M_1=\frac{Wl}{8},$$

and *between O and B*, $$M=\frac{Wx}{2}-\frac{Wl}{8},$$

so that the B.M. at B, i.e., when $x=\frac{l}{2}$, is also $\frac{Wl}{8}$.

Thus the two lines $K'D'E'$ and $E'C'L'$, Fig. 448, are the B.M. diagram, the value of $OD'(=AC')$ being the value of x when

$$0=\frac{Wx}{2}-\frac{Wl}{8} \quad \text{or} \quad x=\frac{l}{4}.$$

If it is assumed that $P(=wl)$ alone acts, then

$$M_1=\frac{wl^2}{12}$$

and $$M=\frac{wl}{2}x-\frac{wx^2}{2}-\frac{wl^2}{12}.$$

The B.M. at B, i.e., when $x=\frac{l}{2}$, is $\frac{wl^2}{24}$ or $\frac{Pl}{24}$.

Thus the parabola $K''D''E''C''L''$, Fig. 449, is the B.M. diagram, the lengths OD'', OC'' being the values of x in the quadratic

$$0=\frac{wl}{2}x-\frac{wx^2}{2}-\frac{wl^2}{12},$$

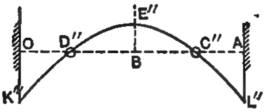


FIG. 449.

from which $$x=\frac{l}{2}\left(1\mp\frac{1}{\sqrt{3}}\right),$$

so that $$OD''=.211l \quad \text{and} \quad OC''=.789l.$$

The ordinate of the B.M. diagram in Fig. 446 is evidently the *algebraic sum* of the corresponding ordinates in Figs. 448 and 449.

Again, assuming I to be constant, the work done in bending the beam in the *first case*

$$=2\times\frac{1}{2EI}\int_0^{\frac{l}{2}}\left(\frac{W+P}{2}x-\frac{wx^2}{2}\right)^2dx$$

$$=\frac{l^3}{EI}\left\{\frac{(W+P)^2}{96}-\frac{(W+P)P}{128}+\frac{P^2}{640}\right\},$$

which becomes

$$\frac{1}{96}\frac{W^2l^3}{EI}\text{ if }P=0\quad\text{and}\quad\frac{1}{240}\frac{P^2l^3}{EI}\text{ if }W=0;$$

in the *second case*

$$=2\times\frac{1}{2EI}\int_0^{\frac{l}{2}}\left\{\left(\frac{W+P}{2}\right)x-\frac{wx^2}{2}-M_1\right\}^2dx$$

$$=\frac{l^3}{EI}\left(\frac{W^2}{384}+\frac{P^2}{1440}+\frac{PW}{384}\right),$$

which becomes $\frac{1}{384}\frac{w^2l^3}{EI}$ if $P=0$ and $\frac{1}{1440}\frac{P^2l^3}{EI}$ if $W=0$.

Third. Let the beam be fixed at O and merely rest upon the support at A. In the *first place* consider the effect of the uniformly distributed load and let

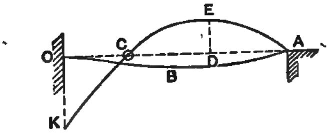


FIG. 450.

R_1, R_2 be the vertical reactions at the supports O and A respectively.

Then, at any point (x, y) of the neutral axis,

$$M=-EI\frac{d^2y}{dx^2}=R_1x-\frac{wx^2}{2}-M_1.$$

At A, $M=0$, and therefore

$$0=R_1l-\frac{wl^2}{2}-M_1,$$

or

$$R_1l-M_1=\frac{wl^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

Integrating, $$-EI\frac{dy}{dx}=R_1\frac{x^2}{2}-\frac{wx^3}{6}-M_1x,$$

an equation giving the slope of the neutral axis at any point. There is no constant of integration, as x and $\frac{dy}{dx}$ vanish together.

Integrating again, $$-EIy=R_1\frac{x^3}{6}-\frac{wx^4}{24}-M_1\frac{x^2}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

an equation giving the deflection of the neutral axis at any point. There is no constant of integration, as x and y vanish together. The deflection at A, i.e., when $x=l$, is also 0. Therefore

$$0=R_1\frac{l^3}{6}-\frac{wl^4}{24}-M_1\frac{l^2}{2}$$

or

$$R_1l-3M_1=\frac{wl^2}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

Hence, by (A) and (C),

$$R_1=\frac{5}{8}wl=\frac{5}{8}P \quad \text{and} \quad M_1=\frac{wl^2}{8}=\frac{1}{8}Pl.$$

Thus the fixture of one end throws *five eighths* of the load upon that end, while *three eighths* of the load is borne at A.

The B.M. is *nil* at points given by

$$0=R_1x-\frac{wx^2}{2}-M_1=-\frac{w}{8}(l-x)(l-4x),$$

i.e., when $x=\frac{l}{4}$ and when $x=l$.

Take $OK=\frac{Pl}{8}$ and $OC=\frac{l}{4}$ (Fig. 450). Then the parabola $KCEA$, with its vertex E at a vertical distance $DE=\frac{9}{128}Pl$ above the middle point of AC, is the B.M. diagram, and the bending actions on the portions OC and AC are evidently opposite in kind.

The maximum deflection is no longer at the centre, but its position may be found by putting $\frac{dy}{dx}=0$ in eq. (B). Then

$$0=R_1\frac{x^2}{2}-\frac{wx^3}{6}-M_1x,$$

or

$$0=\frac{5}{16}lx-\frac{x^2}{6}-\frac{l^2}{8},$$

from which $x=\frac{l}{16}(15-\sqrt{33})=.58l$, very nearly.

The corresponding value of y, i.e., the maximum deflection, can now be found by substituting this value of x in eq. (B).

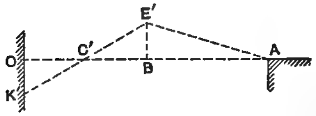


FIG. 451.

In the *second place*, consider the effect of the single weight W, R_1 and R_2 being again the reactions at O and A respectively. Then, *between O and B*,

$$M=-EI\frac{d^2y}{dx^2}=R_1x-M_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D)$$

Integrating,

$$-EI\frac{dy}{dx}=R_1\frac{x^2}{2}-M_1x,$$

an equation giving the slope at any point of the neutral axis *between O and B.*

Let α be the slope at B, then

$$-EI\tan\alpha = R_1\frac{l^2}{4} - M_1\frac{l}{2}. \quad \dots\dots\dots \quad (E)$$

Integrating again, $$-EIy = R_1\frac{x^3}{6} - M_1\frac{x^2}{2},$$

an equation giving the deflection of any point of the neutral axis *between O and B.*

Let y_B be the value of y at B, then

$$-EIy_B = R_1\frac{l^3}{48} - M_1\frac{l^2}{8}. \quad \dots\dots\dots \quad (F)$$

Between B and A,

$$M = -EI\frac{d^2y}{dx^2} = R_1x - M_1 - W\left(x - \frac{l}{2}\right).$$

At A the B.M. is *nil*, and therefore

$$0 = R_1l - M_1 - W\frac{l}{2},$$

or $$R_1l - M_1 = W\frac{l}{2}. \quad \dots\dots\dots \quad (G)$$

Integrating, $$-EI\frac{dy}{dx} = R_1\frac{x^2}{2} - M_1x - \frac{W}{2}\left(x - \frac{l}{2}\right)^2 + c_1,$$

c_1 being a constant of integration.

But $\frac{dy}{dx} = \tan\alpha$ at B, i.e., when $x = \frac{l}{2}$. Therefore

$$-EI\tan\alpha = R_1\frac{l^2}{4} - M_1\frac{l}{2} + c_1,$$

and, by eq. (E), $c_1 = 0$.

Hence $$-EI\frac{dy}{dx} = R_1\frac{x^2}{2} - M_1x - \frac{W}{2}\left(x - \frac{l}{2}\right)^2,$$

an equation giving the slope of the neutral axis at any point *between B and A.*

Integrating again, $$-EIy = R_1\frac{x^3}{6} - M_1\frac{x^2}{2} - \frac{W}{6}\left(x - \frac{l}{2}\right)^3 + c_2,$$

c_2 being a constant of integration.

But $y = y_B$ when $x = \frac{l}{2}$. Therefore

$$-EIy_B = R_1\frac{l^3}{48} - M_1\frac{l^2}{8} + c_2,$$

and, by eq. (F), $c_2 = 0$.

Hence $$-EIy=R_1\frac{x^3}{6}-M_1\frac{x^2}{2}-\frac{W}{6}\left(x-\frac{l}{2}\right)^3, \quad . \quad . \quad . \quad . \quad . \quad (H)$$

an equation giving the deflection of any point of the neutral axis *between B and A.*

But y is also *nil* at A, i.e., when $x=l$. Therefore

$$0=R_1\frac{l^3}{6}-M_1\frac{l^2}{2}-\frac{W}{48}l^3,$$

or $$R_1l-3M_1=\frac{Wl}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (K)$$

Hence, by eqs. (G) and (K), $R_1=\frac{11}{16}W$ and $M_1=\frac{3}{16}Wl$,

so that the fixture throws *eleven sixteenths* of the weight on O and *five sixteenths* of the weight on A.

The B.M. is *nil* at a point between O and B, defined by the value of x, given by making $M=0$ in eq. (D). Then

$$0=R_1x-M_1 \quad \text{or} \quad x=\frac{M_1}{R_1}=\frac{3}{11}l.$$

Take $OK'=\frac{Wl}{8}$ and $OC'=\frac{3}{11}l$ (Fig. 451). The two lines $KC'E'$ and $E'A$ are the B.M. diagram, and

$$BE'=\frac{5}{32}Wl.$$

The *maximum deflection* is evidently at some point between B and A, and its position may be found by making $\frac{dy}{dx}=0$ in eq. (H). Then

$$0=R_1\frac{x^2}{2}-M_1x-\frac{W}{2}\left(x-\frac{l}{2}\right)^2,$$

which becomes $$0=-5x^2+10lx-4l^2,$$

from which $$x=l\left(1-\frac{1}{\sqrt{5}}\right)=.55l, \text{ very nearly.}$$

When the beam carries the two loads, the B.M. at any point is the *algebraic* sum of the corresponding ordinates of the B.M. diagrams obtained by considering the loads separately.

So also the slope and the deflection are the algebraic sums of the corresponding slopes and deflections due to the separate loads.

If there are a number of weights concentrated at different points along a beam, each weight may be treated independently and the several results superposed.

5. Reinforced Concrete Beams.—In concrete beams, reinforced by the introduction of steel rods, expanded metal, etc., on the tension side, it is commonly considered good practice to assume that the whole of the tension is carried by the metal.

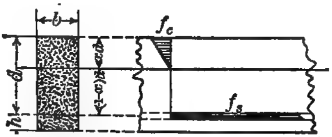


FIG. 452.

Let E_c, E_s be Young's moduli for concrete and steel, respectively, *in compression;*
f_c, f_s " the extreme fibre stresses in the concrete and steel;
b " " breadth of the beam;
d " " distance between the upper surface of the beam and the centre line of the steel;
xd " " distance between the upper surface and the neutral axis;
h " " distance between the centre line of the steel and the bottom surface of the beam;
a " " area of the steel reinforcement.

For equilibrium, the algebraic sum of the horizontal forces in the beam must be *nil.* Therefore

$$f_c\frac{bxd}{2}-af_s=0,$$

or

$$\frac{bxd}{2a}=\frac{f_s}{f_c}=\frac{E_s(1-x)d}{E_cxd}=\frac{1-x}{rx},$$

taking

$$r=\frac{E_c}{E_s}.$$

Hence

$$x^2+\frac{2a}{rbd}x=\frac{2a}{rbd},$$

and

$$x=\frac{1}{rbd}(\sqrt{a^2+2arbd}-a),$$

which determines the position of the neutral axis.

The *moment of resistance*

$$=\frac{f_cb}{3}(xd)^2+f_sa(1-x)d$$

$$=\frac{1}{2}f_cbd^2x\left(1-\frac{x}{3}\right),$$

or

$$=f_sad\left(1-\frac{x}{3}\right).$$

Ex. 13. *A concrete beam 4 ins. wide by 12 ins. deep is reinforced by a steel rod of $\frac{1}{2}$ in. diameter placed with its centre line 1 in. above the tension face of the beam. Young's moduli for concrete and steel in compression are 3,000,000 (E_c) and 30,000,000 (E_s) lbs. per square inch. The concrete may be subjected to a compression of 600 lbs. per square inch and the steel to a tension of 18,000 lbs. per square inch.*

Then $$a=\frac{1}{4}\pi\left(\frac{1}{2}\right)^2=.196 \text{ sq. in.},\quad r=\frac{E_c}{E_s}=\frac{1}{10},$$

and $$x=\frac{\sqrt{2\times.196\times\frac{1}{10}\times4\times11+(.196)^2}-.196}{\frac{1}{10}\times4\times11}=.256.$$

Hence the *moment of resistance,* so far as it depends upon the *compressive strength* of the concrete,

$$=\frac{1}{2}\times600\times4\times11^2\times.256\left(1-\frac{.256}{3}\right)=34,005 \text{ in.-lbs.},$$

and the moment of resistance, so far as it depends upon the *tensile strength* of the steel,

$$=18000\times.196\times11\left(1-\frac{.256}{3}\right)=35,497 \text{ in.-lbs.}$$

The smaller of the two results must be taken as the actual moment of resistance.

Ex. 14. *To design a concrete-steel slab to carry an external load of 100 lbs. per square foot, the slab to be placed between two steel beams 60 ins. apart. Take the same moduli as in the preceding example, and let 625 lbs. and 18,000 lbs. be the stresses which can be safely developed in the concrete and steel respectively.*

Try a slab 5 ins. thick, reinforced by steel rods of $\frac{1}{4}$ in. diameter, spaced 6 ins. apart and placed 4 ins. below the top of the slab. Consider a portion of the slab 6 ins. wide, containing *one* reinforcing rod, and let the weight of the concrete be 150 lbs. per cubic foot.

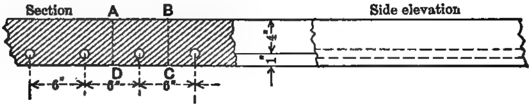


FIG. 453.

Weight of concrete in position per lineal foot	$=\frac{6\times5\times12}{1728}150=31.25$ lbs.
" " steel " " " " "	= .17 lb.
" " external load " " "	=50.00 lbs.
Total load " " "	=81.42 lbs.

or say 82 lbs. per lineal foot.

The maximum B.M. on the portion under consideration

$$=\frac{82\times5^2\times12}{8}=3075 \text{ in.-lbs.},$$

which must not exceed the *least moment of resistance* if the slab is to be safe.

Now, $$x=\frac{\sqrt{(.049)^2+2\times.049\times\frac{1}{10}\times6\times4}-.049}{\frac{1}{10}\times6\times4}=.183.$$

The *moment of resistance,* so far as it depends upon the *compressive strength* of the concrete,

$$=\frac{625\times6\times.183\times4^2}{2}\left(1-\frac{.183}{3}\right)=5156 \text{ in.-lbs.},$$

and the *moment of resistance,* so far as it depends upon the *tensile strength* of the steel

$$=18000\times.049\times4\left(1-\frac{.183}{3}\right)=3313 \text{ in.-lbs.}$$

As the smaller of these moments is greater than the max. B.M., the slab is amply safe.

A more exact adjustment might be made by diminishing the thickness of the concrete and spacing the rods a little further apart. The properties of concrete, however, are so variable that great nicety of adjustment seems unnecessary.

6. Formula $Wl=Cbd^2$.—In this formula b is a transverse dimension of a beam of length l, depth d, and carrying a load W.

It has been shown that

$$M=f\frac{I}{c}=f\frac{Ak^2}{c},$$

and if f is the max. stress,

$$M\propto\frac{Ak^2}{c}.$$

But A is proportional to bd,

k " " " d,

c " " " d,

and M " " " Wl.

Therefore
$$Wl \propto \frac{bd.d^2}{d} \propto bd^2 = Cbd^2,$$

C being a coefficient which depends upon the nature of the material, the character of the loading, and the method of supporting the beam. Its value must be determined by experiment.

This formula is sometimes used to determine the breaking weight of a beam, and C is then called the *coefficient of rupture.* Values of C for iron, steel, and timber beams are tabulated at the end of Chapter IV.

Ex. 15. *A* 10-*in.-deep*×6-*in.-wide red pine beam resting upon supports* 20 *ft. apart fails under a load of* 14,250 *lbs. concentrated at the centre. Find C. Disregarding the weight of the beam,*

$$14250 \times 20 \times 12 = C \cdot 6 \cdot 10^2.$$

Therefore
$$C = 5700.$$

7. **Beams of Uniform Strength.**—A beam of uniform strength is a beam so designed that the greatest stress developed under a given load is the same at every section from end to end of the beam. Let y be the depth and z the greatest width of a section of such a beam at any distance x from an origin in the neutral axis. Then f is the max. stress developed in this section. Its moment of resistance $= f\frac{2I}{y} \propto zy^2$, since f is to be constant.

Therefore
$$zy^2 \propto \text{the B.M. at the section,}$$
$$\propto M,$$

which may be written
$$zy^2 = cM,$$

c being a coefficient depending upon the value of f and the form of the section.

At points at which the B.M. is *nil*, $zy^2 = 0$, and therefore either z or y must be *nil*. *Theoretically*, then, a beam requires no sectional area, i.e., no material, at such points, but it is manifest that at *every* *point* the beam must have a sufficient sectional area to take up the *shearing force.*

CASE *a*. If the width (i.e., z) is constant,

$$y \propto \sqrt{M},$$

and the ordinates of the profile in elevation are proportional to the square root of the corresponding ordinates of the B.M. curve.

CASE *b*. If the depth (i.e., y) is constant,

$$z \propto M,$$

and the ordinates of the profile in plan are directly proportional to the corresponding ordinates of the B.M. curve.

CASE *c*. If the ratio of depth to width $\left(\text{i.e., } \frac{y}{z}\right)$ is constant,

$$y \propto z \propto \sqrt[3]{M},$$

and the ordinates of the profiles, both in elevation and in plan, are proportional to the cube root of the corresponding ordinates of the B.M. curve.

If the load on the beam consists of a number of concentrated loads $M \propto x$, and the profiles are evidently made up of cubical parabolas.

CASE *d*. If the sectional area (yz) is constant,

$$y \propto M,$$

and the ordinates of the profile in elevation are again proportional to the corresponding ordinates of the B.M. curve.

Again, when $M=0$, $y=0$, and, therefore, z is infinite. Hence, in a beam of this kind, the distribution of the material is most defective.

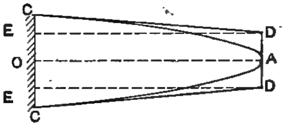


FIG. 454.

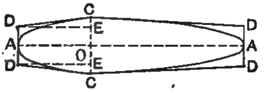


FIG. 455.

Beams of Approximately Uniform Strength.—The curved lines in Figs. 454 and 455 are the theoretical profiles of a cantilever and a beam of uniform strength. It is sometimes the practice to replace the curved boundary lines by the tangents CD at the section CC of greatest depth; a depth (or width) DD is thus provided at A which is sufficient for the S.F. at that point. Such a beam may be said to be of *approximately* uniform strength.

Let $y=f(x)$ be the equation to the curved profile. Then

$$\frac{dy}{dx}=f'(x)$$

is the tangent of the angle which the tangent at (x, y) makes with the neutral axis.

At C, i.e., when $x=0$,

$$f'(0)=\tan CDE=\frac{CE}{DE}=\frac{OC-AD}{OA},$$

and therefore
$$DD=2AD=2OC-2OAf'(0)$$
$$=CC\ \ -2OAf'(0).$$

Ex. 16. *A cantilever OA of length l and constant width z carries a weight W_1 at the free end A and a uniformly distributed load W_2.*

At any point distant x from O,

$$zy^2=cM=c\left\{W_1(l-x)+\frac{1}{2}\frac{W_2}{l}(l-x)^2\right\},$$

which may be written in the form

$$\frac{\left(\frac{W_1+W_2}{W_2}l-x\right)^2}{\left(\frac{W_1}{W_2}l\right)^2}-\frac{y^2}{\frac{cl}{2z}\frac{W_1{}^2}{W_2}}=1,$$

so that, *theoretically*, the profile of the cantilever in elevation is an hyperbola with its centre at a distance $\frac{W_1+W_2}{W_2}l$ from O, and its semi-axes equal to $\frac{W_1}{W_2}l$ and $\sqrt{\frac{cl}{2z}\frac{W_1{}^2}{W_2}}$.

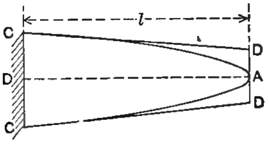


FIG. 456.

A sufficient area must, however, be provided at A to carry the S.F. at that point due to W_1. This can be done by substituting for the curved profile the tangents CD, which will give a cantilever of approximately uniform strength.

At O, i.e., when $x=0$,

$$zOC^2 = cl\left(W_1+\frac{W_2}{2}\right) = z\frac{CC^2}{4}.$$

Also, $$2zy\frac{dy}{dx} = -c\left\{W_1+\frac{W_2}{l}(l-x)\right\},$$

and therefore at O, i.e., when $x=0$,

$$2zOC \times -f'(0) = -c(W_1+W_2)$$

or $$-f'(0) = \frac{c(W_1+W_2)}{zCC}.$$

Hence $$DD = CC - 2OAf'(0)$$

$$= CC\left(1-2lc\frac{W_1+W_2}{zCC^2}\right) = CC\frac{W_1}{2W_1+W_2}.$$

The dotted lines define the cantilever of uniform strength when OA is the lower face.

Ex. 17. *A beam AA of constant width z and of length $2l$ carries a uniformly distributed load W_2 and a weight W_1 concentrated at the middle point O.*

At any point distant x from O,

$$zy^2 = cM = c\left\{\frac{W_1}{2}(l-x)+\frac{1}{4}\frac{W_2}{l}(l^2-x^2)\right\},$$

which may be written

$$\frac{\left(x+\frac{W_1}{W_2}l\right)^2}{\frac{l^2}{W_2^2}(W_1+W_2)^2}+\frac{y^2}{\frac{cl}{4W_2z}(W_1+W_2)^2}=1,$$

so that, *theoretically*, the profile of the beam in elevation consists of two elliptic arcs CA, the ellipses having their centres at the points O_1, O_2, where

$$OO_1 = OO_2 = \frac{W_1}{W_2}l.$$

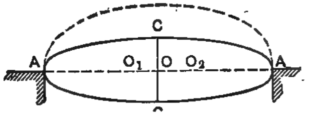


FIG. 457.

At O, i.e., when $x=0$,

$$zOC^2 = \frac{cl}{2}\left(W_1+\frac{W_2}{2}\right) = z\frac{CC^2}{4}.$$

Also, $$z2y\frac{dy}{dx} = -\frac{c}{2}\left(W_1+W_2\frac{x}{l}\right).$$

Therefore at O $$-f'(0)\times 2zOC = -\frac{cW_1}{2},$$

or
$$f'(0)=\frac{cW_1}{4zOC}=\frac{cW_1}{2zCC},$$

and
$$DD=CC-2OA\times f'(0)$$

$$=CC\left(1-2l\,\frac{cW_1}{2zCC^2}\right)$$

$$=CC\left\{1-\frac{clW_1}{2cl\left(W_1+\frac{W_2}{2}\right)}\right\}$$

$$=CC\frac{W_1+W_2}{2W_1+W_2}.$$

The dotted lines show the beam of uniform strength when AA is the lower face.

Ex. 18. *Design a cantilever of approximately uniform strength from the following data: Length* = 11 *ft.; circular section—load at free end* = 1960 *lbs.; working stress* = 7680 *lbs./sq. in.; disregard the weight of the beam.*

At x feet from the fixed end, let y be the radius. Then

$$\frac{7680}{y}\,\frac{\pi y^4}{4}=M=1960(11-x)12,\quad\text{or}\quad y^3=\frac{343}{88}(11-x)$$

is the curve of the theoretical profile and is a cubical parabola.

At the fixed end, $x=0$, and

$$\text{the radius}=\sqrt[3]{\frac{343}{88}11}=3\tfrac{1}{2}\text{ ins.}$$

Again,
$$3\frac{dy}{dx}y^2=-\frac{343}{88},$$

or
$$\frac{dy}{dx}=-\frac{343}{264}\,\frac{1}{y^2}.$$

Hence, if the tangent at C, Fig 454, replaces the curved profile,

$$\frac{dy}{dx}\text{ at }C=-\frac{343}{264}\,\frac{1}{(3\frac{1}{2})^2}=-\tan CDE=-\frac{CE}{11}$$

and
$$CE=11\times\frac{343}{264}\,\frac{4}{49}=\frac{7}{6}.$$

Therefore the depth DD of the cantilever of approximately uniform strength

$$=2AD=2(OC-CE)=2(3\tfrac{1}{2}-1\tfrac{1}{6})=4\tfrac{2}{3} \text{ ins.}$$

8. Stiffness.—If D is the maximum deflection of a girder of span l, $\frac{D}{l}$ is a measure of the *stiffness* of the girder.

In practice the deflection of an iron or a steel girder under the working load should lie between $\frac{l}{1200}$ and $\frac{l}{600}$, i.e., it is limited to 1 or 2 ins. per 100 ft. of span, and rarely exceeds $\frac{l}{1000}$, or 1.2 ins. per 100 ft. of span.

A timber beam should not deflect more than $\frac{l}{360}$, or 1 in. per 30 ft. of span.

The proper stiffness of a girder is sometimes secured by requiring the central depth to lie between $\frac{l}{14}$ and $\frac{l}{8}$, its value depending upon the material of which the girder is composed, its sectional form, and the work to be done.

In all the cases of Exs. 11 and 12, the maximum deflection Y is proportional to $\frac{l^3}{EI}$, and therefore

$$Y=\frac{pl^3}{EI},$$

p being a coefficient which depends upon the character of the loading and the nature of the supports. Thus

$$\frac{\mathbf{Y}}{\mathbf{l}}=\textbf{the stiffness}=\frac{\mathbf{pl^2}}{\mathbf{EI}},$$

Also, if $M_{\text{max.}}$ is the max. B.M. on the beam and f the corresponding safe stress in the material,

$$M_{\text{max.}}=\frac{f}{c}I=\frac{f}{qd}I,$$

d being the depth of the beam and q a coefficient depending upon its sectional form.

Therefore
$$\frac{Y}{l}=\frac{1}{E}\frac{p}{q}\left(\frac{l}{d}\right)\frac{fl}{M_{\text{max.}}}.$$

Ex. 19. *A cast-iron beam of rectangular section and of 20 ft. span carries a uniformly distributed load of 20 tons; the coefficient of working strength is 2 tons per square inch; the stiffness is 1 in. in 100 ft.; E is 8000 tons. Find the dimensions of the beam, viz., b the breadth and d the depth.*

$$M=\frac{20\times20}{8}12=\frac{f}{c}I=2\frac{bd^2}{6}=\frac{bd^2}{3};$$

therefore
$$bd^2=1800.$$

Also,
$$\frac{Y}{l}=\frac{1}{1200}=\frac{5}{384}\frac{20(240)^2 12}{8000bd^3}=\frac{45}{2bd^3},$$

and therefore
$$bd^3=27,000.$$

Hence
$$\frac{bd^3}{bd^2}=\frac{27000}{18000}=15 \text{ ins.}=d$$

and
$$b=\frac{1800}{15^2}=8 \text{ ins.}$$

Ex. 20. *Compare the strength and stiffness of two similarly loaded beams of the same material of equal lengths and equal sectional areas, the one being round and the other square.*

Let r be the radius of the round beam, f_r the intensity of the surface stress.

Let a be a side of the square beam, f_a the intensity of the surface stress.

Then $\pi r^2=a^2$; I, for round bar, $=\frac{\pi r^4}{4}$, and for square bar $=\frac{a^4}{12}$.

Also, since the beams are similarly loaded, the bending moments at corresponding points are equal, and therefore

$$\frac{f_r}{r}\frac{\pi r^4}{4}=M=\frac{f_a}{\frac{a}{2}}\frac{a^4}{12},$$

so that
$$\frac{f_r}{f_a}=\frac{2}{3}\frac{a^3}{\pi r^3}=\frac{2}{3}\sqrt{\frac{22}{7}}=\sqrt{\frac{88}{63}}.$$

Thus, under the same load, the round beam is strained to a greater extent than the square beam, and the latter is the stronger in the ratio of $\sqrt{88}$ to $\sqrt{63}$.

Again, the stiffness of a beam is $\frac{pl^2}{EI}$. In the present case, p, E, and l are the same for the two beams and therefore the stiffness is inversely proportional to the moment of inertia. Hence

$$\frac{\text{the stiffness of the round beam}}{\text{the stiffness of the square beam}}=\frac{\frac{a^4}{12}}{\frac{\pi r^4}{4}}=\frac{1}{3\pi}(\pi)^2=\frac{22}{21}.$$

Ex. 21. *A horizontal beam of a length l equal to twenty times the depth d rests upon supports at the ends and carries a uniformly distributed load P. If the neutral axis is at mid-depth, and if E is* 1,200,000 *lbs. per square inch, determine the stiffness of the beam, so that the maximum stress may nowhere exceed* 400 *lbs. Also find the work of flexure.*

Let Y be the maximum deflection. Then

$$\frac{Pl}{8}=800\frac{I}{d}, \text{ or } \frac{Pl^2}{I}=6400\left(\frac{l}{d}\right)=128,000;$$

also, $$\textit{the stiffness}=\frac{Y}{l}=\frac{5}{384}\frac{Pl^2}{EI}=\frac{5}{384}\frac{128000}{1200000}=\frac{1}{720}.$$

Again, by Ex. 12,

$$\textit{the work of flexure}=\frac{1}{240}\frac{P^2l^3}{EI}=Pl\frac{128000}{240\times 1200000}=\frac{Pl}{2250}.$$

9. Distribution of Shearing Stress.—Let Figs. 458 and 459 represent a slice of a beam bounded by two consecutive sections AB, $A'B'$, transverse to the horizontal neutral axis OO'.

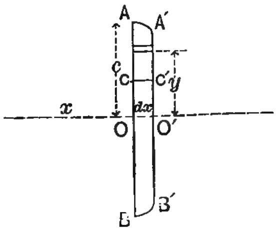


FIG. 458.

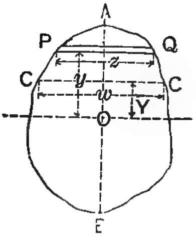


FIG. 459.

Let the abscissæ of these sections with respect to an origin in the neutral axis be x and $x+\Delta x$, so that the thickness of the slice is Δx.

In the limit, since Δx becomes indefinitely small, corresponding linear dimensions in the two sections are the same.

Let I be the moment of inertia of the section AB (or $A'B'$ in the limit) with respect to the neutral axis.

Let c be the distance of A (or A' in the limit) from the neutral axis.

Let f_1, f_2 be the unit stresses at A and A' respectively.

Consider the portion $ACC'A'$ of the slice, CC' being parallel to and at a distance Y from the neutral axis. Since it is in equilibrium, the algebraic sum of the horizontal forces acting upon it must be *nil.*

The stress at P distant y from $OO' = f_1 \frac{y}{c}$.

Therefore the horizontal force on the slice PQ of thickness dy

$$= f_1 \frac{y}{c} z dy,$$

and the total horizontal force on the area ACC

$$= \frac{f_1}{c} \int_y^c yzdy = \frac{f_1}{c} A\bar{y},$$

A being the area ACC and y the distance of its C. of G. from the neutral axis.

Similarly, the total horizontal force on the face $A'C'C'$

$$= \frac{f_2}{c} A\bar{y},$$

and the resultant horizontal force on the element under consideration

$$= \frac{f_1}{c} A\bar{y} - \frac{f_2}{c} A\bar{y} = A\bar{y}\left(\frac{f_1}{c} - \frac{f_2}{c}\right).$$

But $\qquad M_1$, the B.M. at $A, = \frac{f_1}{c} I$,

and $\qquad M_2$, " " " $A', = \frac{f_2}{c} I$.

Therefore $\qquad M_1 - M_2 = \Delta M = \frac{f_1 - f_2}{c} I$.

ΔM being the change of B.M. in passing from A to A'.

In the limit Δx and ΔM become indefinitely small, so that $\frac{\Delta M}{\Delta x}=\frac{dM}{dx}=S$, the S.F. at the section AB.

Thus
$$\frac{f_1-f_2}{c}=\frac{\Delta M}{I}=\frac{S}{I}dx,$$

and the resultant horizontal force on the element $ACC'A'$

$$=A\bar{y}\frac{\Delta M}{I}=\frac{S}{I}A\bar{y}dx.$$

This must be balanced by the shear developed over the surface $CC'C'C$, when the thickness becomes indefinitely small.

Let q be the average intensity of shear over this surface.

" w " " width CC.

Then
$$qwdx=\text{shear developed}$$
$$=\frac{S}{I}A\bar{y}dx.$$

Hence
$$qw=\frac{S}{I}A\bar{y}.$$

Ex. 22. *A solid rectangular section of width b and depth d.*

Area between CC and surface $=b\left(\frac{d}{2}-Y\right)$.

Distance of centre of gravity of this area, i.e., $\bar{y}$ from the neutral axis

$$=\frac{1}{2}\left(\frac{d}{2}+Y\right),$$

$$I=\frac{bd^3}{12} \quad \text{and} \quad w=b.$$

Therefore
$$qb=\frac{S}{\frac{bd^3}{12}}b\left(\frac{d}{2}-Y\right)\frac{1}{2}\left(\frac{d}{2}+Y\right)$$

and
$$q=\frac{6S}{bd^3}\left(\frac{d^2}{4}-Y^2\right).$$

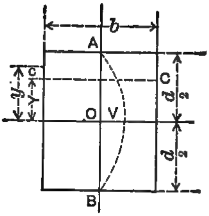


FIG. 460.

This is evidently greatest and $=\frac{3}{2}\frac{S}{bd}$ when $Y=0$, i.e., at the neutral surface.

The intensity of shear at any point may be represented by the horizontal distance of the point from the parabola AVB having its vertex at the point V where $OV=\frac{3}{2}\frac{S}{bd}=q_{\text{max.}}$

If q_{av} is the average intensity of S, the S.F. at the cross-section,

$$\frac{q_{\text{max.}}}{q_{av}}=\frac{\frac{3}{2}\frac{S}{bd}}{\frac{S}{bd}}=\frac{3}{2}.$$

Ex. 23. *A solid circular section of diameter d.*

Area of element of thickness dy at distance y from the neutral axis $=\left(2\sqrt{\frac{d^2}{4}-y^2}\right)dy$. Therefore

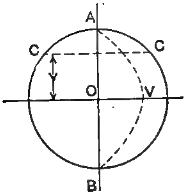


FIG. 461.

$$A\bar{y}=\int_{Y}^{\frac{d}{2}}2y\sqrt{\frac{d^2}{4}-y^2}\,dy=\frac{2}{3}\left(\frac{d^2}{4}-Y^2\right)^{\frac{3}{2}}$$

The width $w=CC=2\sqrt{\frac{d^2}{4}-Y^2}$

and $$I=\frac{\pi d^4}{64}.$$

Hence $$q2\sqrt{\frac{d^2}{4}-Y^2}=\frac{S}{\frac{\pi d^4}{64}}\frac{2}{3}\left(\frac{d^2}{4}-Y^2\right)^{\frac{3}{2}}$$

and $$q=\frac{64S}{3\pi d^4}\left(\frac{d^2}{4}-Y^2\right),$$

which is evidently greatest when $Y=0$.

Thus $$q_{\text{max.}}=\frac{16S}{3\pi d^2},$$

and the intensity of shear at any point may be represented by the horizontal distance of the point from the parabola AVB, where $OV=\frac{16S}{3\pi d^2}$.

Also, $$\frac{q_{\text{max.}}}{q_{\text{av.}}}=\frac{16S}{3\pi d^2}\div\frac{S}{\frac{\pi d^2}{4}}=\frac{4}{3}.$$

Ex. 24. *A triangular section of depth d and with a base of width b.*

$$\text{Area of section} = \frac{bd}{2};$$

$$\text{" " } ACC = \frac{bd}{2}\frac{(\frac{2}{3}d - Y)^2}{d^2} = \frac{1}{2}\frac{b}{d}\left(\frac{2}{3}d - Y\right)^2.$$

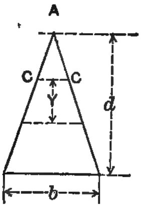


FIG. 462.

Distance of centre of gravity of ACC from the neutral axis

$$= Y + \frac{1}{3}\left(\frac{2}{3}d - Y\right) = \frac{2}{3}\left(\frac{1}{3}d + Y\right);$$

$$\text{the width } CC(=w) = \frac{b}{d}\left(\frac{2}{3}d - Y\right).$$

Therefore $$q\frac{b}{d}\left(\frac{2}{3}d - Y\right) = \frac{S}{I}\frac{1}{2}\frac{b}{d}\left(\frac{2}{3}d - Y\right)^2\frac{2}{3}\left(\frac{1}{3}d + Y\right)$$

and $$q = \frac{1}{3}\frac{S}{I}\left(\frac{2}{3}d - Y\right)\left(\frac{1}{3}d + Y\right),$$

which is a maximum when

$$0 = -dY\left(\frac{1}{3}d + Y\right) + dY\left(\frac{2}{3}d - Y\right)$$

or $$Y = \frac{1}{6}d,$$

i.e., at half the depth of the beam.

Thus $$q_{\text{max.}} = \frac{1}{12}\frac{S}{I}d^2.$$

Also $$q_{\text{av.}} = S \div \frac{1}{2}bd.$$

Therefore $$\frac{q_{\text{max.}}}{q_{\text{av.}}} = \frac{1}{24}\frac{bd^3}{I} = \frac{36}{24} = \frac{3}{2},$$

since $$I = \frac{bd^3}{36}.$$

10. Deflection Due to Shear.—The strain energy per unit of volume due to shear $= \frac{q^2}{2G}$. Therefore the strain energy of the elementary volume PQ (Figs. 458 and 459)

$$= \frac{1}{2G}\left(\frac{S}{I}\frac{A\bar{y}}{z}\right)^2 z\,dx\,dy,$$

and the total strain energy due to shear

$$=\frac{1}{2G}\int\int\left(\frac{S}{I}A\bar{y}\right)^2\frac{1}{z}dxdy,$$

the integrations extending over the whole length and depth of the beam.

Ex. 25. *A cantilever of length l, depth d, width b, and loaded with W at the free end. Then*

$$S=W,\quad I=\frac{bd^3}{12},\quad z=b,\quad\text{and}\quad A\bar{y}=\frac{b}{2}\left(\frac{d^2}{4}-y^2\right).$$

Therefore the total shear strain energy

$$=\frac{18W^2}{Gbd^6}\int_0^l\int_{-\frac{d}{2}}^{\frac{d}{2}}\left(\frac{d^2}{4}-y^2\right)^2dxdy=\frac{3}{5}\frac{W^2l}{Gbd},$$

and if δ is the corresponding deflection,

$$\frac{W}{2}\delta=\frac{3}{5}\frac{W^2l}{Gbd},$$

so that

$$\delta=\frac{6}{5}\frac{Wl}{Gbd}.$$

Hence the total deflection due to bending and to shear

$$=\frac{1}{3}\frac{Wl^3}{EI}+\frac{6}{5}\frac{Wl}{Gbd}.$$

If Y is the deflection due to bending,

$$\frac{\delta}{Y}=\left(\frac{6}{5}\frac{Wl}{Gbd}\right)\div\left(\frac{1}{3}\frac{Wl^3}{EI}\right)=\frac{3}{10}\frac{E}{G}\left(\frac{d}{l}\right)^2.$$

Taking $G=\frac{2}{5}E$ and $\frac{d}{l}=\frac{1}{15}$, this ratio becomes $\frac{1}{300}$, and the deflection due to shear is only $\frac{1}{3}$ per cent of that due to bending.

It appears therefore, that the deflection due to shear, in the case of solid cantilevers and beams, is a very small quantity and may be disregarded without sensible error. Its magnitude, however, may, and often does, become appreciable in the case of plate girders, built beams, rolled joists, etc., and should not be neglected.

11. Curves of Maximum Normal and Tangential Stress.—As already shown in Chapter V, if a shear stress is induced on any plane in a strained solid, an equal shear stress is developed upon a second plane at right angles to the first. Consider an indefinitely small triangular element *abc* (Fig. 463) of a horizontal beam bounded by a plane *bc* inclined at θ to the vertical, the horizontal plane *ab*, and the vertical plane *ac*.

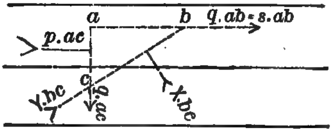


FIG. 463.

The element *abc* is kept in equilibrium by the stress $p \cdot ac$ upon *ac*, the shear $s \cdot ab (= q \cdot ab)$ along *ab*, the shear $q \cdot ac$ along *ac*, and the stress developed in the plane *bc*. The *weight* of the element is neglected as being indefinitely small as compared with the forces to which it is subjected. Let the stress upon *bc* be decomposed into two components, the one $X \cdot bc$ normal and the other $Y \cdot bc$ tangential to *bc*.

Resolving perpendicular and parallel to *bc*,

$$X \cdot bc = p \cdot ac \cos \theta + q \cdot ab \cos \theta + q \cdot ac \sin \theta$$

and
$$Y \cdot bc = -p \cdot ac \sin \theta - q \cdot ab \sin \theta + q \cdot ac \cos \theta,$$

or
$$X = p \cos^2 \theta + q \sin 2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and
$$Y = -p \frac{\sin 2\theta}{2} + q \cos 2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The value of θ for which X is a maximum is given by

$$\frac{dX}{d\theta} = 0 = -p \sin 2\theta + 2q \cos 2\theta, \quad \text{or} \quad \tan 2\theta = +\frac{2q}{p}. \quad . \quad (3)$$

Substituting this value of θ in eq. (1),

$$\text{the max. value of } X = \frac{p}{2} \pm \sqrt{\frac{p^2}{4} + q^2}. \quad . \quad . \quad . \quad . \quad (4)$$

The upper sign gives the maximum stress of the *same* kind as p, while the maximum stress of the *opposite* kind is given by the lower sign, if the total stress is negative.

The value of θ for which Y is a maximum is given by

$$\frac{dY}{d\theta} = 0 = -p \cos 2\theta - 2q \sin 2\theta, \quad \text{or} \quad \tan 2\theta = -\frac{p}{2q}. \quad . \quad (5)$$

Substituting this value of θ in eq. (2),

$$\text{the max. value of } Y=\sqrt{\frac{p^2}{4}+t^2}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

which is the maximum intensity of shear.

The values of X and Y given by eqs. (4) and (6) have been obtained by another method in Chap. V. It was also shown in the same chapter that the planes of principal stress are defined by the relation $\tan 2\theta=\frac{2t}{p}$.

Let θ_1, θ_2 be the values of θ for which X and Y are respectively greatest. Then

$$\tan 2\theta_1 \tan 2\theta_2=-\frac{2t}{q}\frac{q}{2t}=-1,$$

and therefore

$$\theta_1-\theta_2=45^\circ.$$

Hence, at any point, the angle between the plane upon which the normal intensity of stress is a maximum and the plane upon which the tangential intensity of stress is a maximum is equal to 45°.

Again, q is zero when $\theta_1=90^\circ$ or 0°, and p is zero when $\theta_1=45^\circ$.

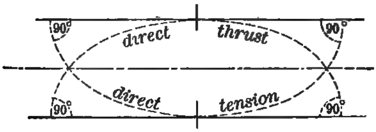


FIG. 464.

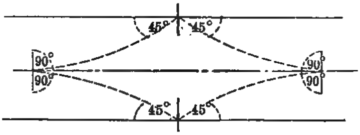


FIG. 465.

Thus the *curve of greatest normal intensity* cuts the neutral axis at an angle of 45°, one skin surface at 90° and the opposite at 0°, while the *curve of greatest tangential intensity* cuts the skin surfaces at 45° and touches the neutral axis.

Fig. 464 serves to illustrate the curves of greatest *normal* intensity. There are evidently two sets of these curves, referring respectively to direct *thrust* and direct *tension.*

Fig. 465 illustrates the curve of greatest *tangential* intensity.

Ex. 26. A pitch-pine beam, 14 ins. wide, 15 ins. deep, and weighing 45 lbs. per cubic foot, is placed upon supports 10 ft. 9 ins. apart, and carries a load of 20 long tons at the centre. Find the deflection and curvature, E being 1,270,000 lbs. Determine the *stiffness* of the beam and the uniformly distributed load which will produce the same deflection. Also find the maximum intensities of thrust, tension, and shear at points (a) half-way between the neutral axis and the outside skin in the central transverse section; (b) at one third of the depth of the beam, in a transverse section at one of the quarter-spans. Also find the inclinations of the planes of principal stress at these points.

If Y is the maximum deflection,

$$EIY = \frac{l^3}{48}\left(\frac{5}{8}P + W\right).$$

In the present case $E = 1{,}270{,}000$ lbs./sq. in.; $I = \frac{14 \times 15^3}{12}$; $l = 129$ ins.; $P = \text{wt. of beam} = \frac{14 \times 15}{144} 10\tfrac{3}{4} \times 45 = \frac{22575}{32}$ lbs. $=$ say 700 lbs.; $W = 20$ tons $= 44{,}800$ lbs. Therefore

$$\text{the stiffness} = \frac{Y}{l} = \frac{\overline{129}^2}{48}\left(\frac{5}{8}\frac{22575}{32} + 44800\right)\frac{12}{1270000 \times 14 \times 15^3} = \frac{1}{319},$$

and

$$\text{the deflection} = Y = \frac{129}{319} = .4 \text{ in.}$$

Again,

$$\left(\frac{Pl}{8} + \frac{Wl}{4}\right)12 = \left(\frac{P}{2} + W\right)\frac{129}{4} = \text{max. B.M. in in.-lbs.}$$

$$= \text{moment of resistance} = \frac{E}{R}I.$$

Therefore

$$\left(\frac{700}{2} + 44800\right)\frac{129}{4} = \frac{1270000}{R}\,\frac{14 \times 15^3}{12},$$

and $\frac{1}{R} = \text{curvature at the centre} = \frac{1}{3434}$, so that the radius of curvature at the centre is 3434 ins. or 286 ft.

To find p and q.

At the central section

$$\left(\frac{700}{2} + 44800\right)\frac{129}{4} = \frac{p}{3\tfrac{3}{4}}\,\frac{14 \times 15^3}{12}.$$

Therefore $p = 1386\tfrac{3}{4}$ lbs.

Also,

$$q = \frac{S}{I}A\bar{y} = \frac{22400 \times 12}{14 \times 15^3}\,14 \times 3\tfrac{3}{4}\left(1\tfrac{7}{8} + 3\tfrac{3}{4}\right) = 1680 \text{ lbs.}$$

Hence the max. shear stress $=\sqrt{\frac{p^2}{4}+q^2}=1817.5$ lbs.,

and the max. normal stresses $=1386.75\pm1817.5$

$=3204.25$ lbs. and -430.75 lbs.

At the quarter-span B.M. $=\frac{1}{2}(700+44800)\frac{129}{4}-\frac{700}{4}\times\frac{129}{8}$

$$=\frac{179900}{8}\,\frac{129}{4}\text{ in.-lbs.}$$

Also, S.F. $=22400-\frac{700}{4}=22,225$ lbs.

Therefore
$$\frac{179900}{8}\,\frac{129}{4}=\frac{p}{2\frac{1}{2}}\,\frac{14\times15^3}{12},$$

and
$$p=460\tfrac{11}{24}\text{ lbs.}$$

Also,
$$14q=\frac{22225\times12}{14\times15^3}\,14\times5\times5,$$

and
$$q=141\tfrac{1}{9}\text{ lbs.}$$

Hence max. shear stress $=\sqrt{\frac{(460\frac{11}{24})^2}{4}+(141\frac{1}{9})^2}=270$ lbs.,

and max. normal stresses $=460.5\pm270=730.5$ and $190.5.$ lbs.,

For the principal plane,

in the 1st case, $\tan 2\theta=\frac{2\times1680}{1386.75}=2.422$ and $\theta=31^\circ\,30'$;

" " 2d " $\tan 2\theta=\frac{2\times141\frac{1}{9}}{460\frac{11}{24}}=.613$ and $\theta=15^\circ\,45'$.

12. Moment of Inertia Variable.—In the preceding investigations the moment of inertia I has been assumed to be constant.

From the general equations, at any point x, y in the neutral axis,

$$\pm EI\frac{d^2y}{dx^2}=\frac{f}{c}I,$$

and therefore
$$\pm E\frac{d^2y}{dx^2}=\frac{f}{c},$$

c being proportional to the depth of the girder at a transverse section distant x from the origin.

Hence, for beams of *uniform strength*, the value of c in terms of x may be substituted in the last equation, which may then be integrated.

Ex. 27. *A girder of uniform strength, of length l, of rectangular section, rests upon two supports and carries a uniformly distributed load of w lbs. per unit of length, which produces a maximum stress of f lbs. at every section of the beam. Show that the central deflection is* $\frac{\pi-2}{2}\frac{f^{\frac{3}{2}}}{E}\left(\frac{b}{3w}\right)^{\frac{1}{2}}l$ *when the breadth (b) is constant and the depth variable. Find the deflection when the depth (d) is constant and the breadth variable.*

Let $2z$ be the depth of the beam at x from the middle point, and let y be the deflection of the neutral axis at the same point. Then

$$\frac{w}{2}\left(\frac{l^2}{4}-x^2\right)=\text{B.M.}=\frac{f}{z}\frac{(2z)^3b}{12}=\frac{2}{3}fz^2b$$

and
$$z=\left(\frac{3}{4}\frac{w}{fb}\right)^{\frac{1}{2}}\left(\frac{l^2}{4}-x^2\right)^{\frac{1}{2}}.$$

Hence
$$-\frac{d^2y}{dx^2}=\frac{f}{Ez}=\frac{f}{E}\left(\frac{4fb}{3w}\right)^{\frac{1}{2}}\left(\frac{l^2}{4}-x^2\right)^{-\frac{1}{2}}=a\left(\frac{l^2}{4}-x^2\right)^{-\frac{1}{2}},$$

where
$$a=\frac{f}{E}\left(\frac{4fb}{3w}\right)^{\frac{1}{2}}.$$

Integrating,
$$-\frac{dy}{dx}=a\int_0^x\left(\frac{l^2}{4}-x^2\right)^{-\frac{1}{2}}dx=a\ \sin^{-1}\frac{2x}{l}.$$

Integrating again,

$$-y=a\int\sin^{-1}\frac{2x}{l}dx=\frac{la}{2}(\theta\sin\theta+\cos\theta)+k,$$

k being a constant of integration and $\sin\theta=\frac{2x}{l}$.

When $x=\frac{l}{2}$, $y=0$ and $\theta=\frac{\pi}{2}$. Therefore

$$0=\frac{la}{2}\frac{\pi}{2}+k.$$

When $x=0$, y, i.e., the deflection, is greatest and $\theta=0$. Therefore

$$-y_{\text{max.}}=\frac{la}{2}+k.$$

Hence
$$y_{\text{max.}}=\frac{la}{2}\left(\frac{\pi}{2}-1\right)=\frac{\pi-2}{2}\frac{f^{\frac{3}{2}}}{E}\left(\frac{b}{3w}\right)^{\frac{1}{2}}.$$

Again, let Fig. 466 represent a cantilever of length l, specific weight w, circular section, and with a parabolic profile, the vertex of the parabola being at A.

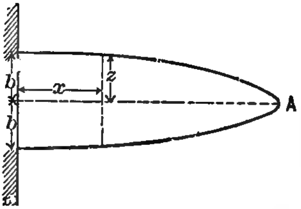


FIG. 466.

Let $2b$ be the depth of the cantilever at the fixed end.

Let the cantilever also carry a uniformly distributed load of intensity p.

Consider a transverse section of radius z at a distance x from the fixed end.

Let x, y be the co-ordinates of the neutral axis at the same section. Then

$$EI\frac{d^2y}{dx^2}=\frac{w\pi}{6}\frac{b^2}{l}(l-x)^3+\frac{p}{2}(l-x)^2=E\frac{\pi z^4}{4}\frac{d^2y}{dx^2}.$$

But $z^2=\frac{b^2}{l}(l-x)$.

Therefore
$$E\frac{\pi}{4}\frac{b^4}{l^2}(l-x)^2\frac{d^2y}{dx^2}=\frac{w\pi}{6}\frac{b^2}{l}(l-x)^3+\frac{p}{2}(l-x)^2,$$

or
$$\frac{\pi E}{4}\frac{b^4}{l^2}\frac{d^2y}{dx^2}=\frac{w\pi}{6}\frac{b^2}{l}(l-x)+\frac{p}{2}.$$

Integrating,
$$\frac{\pi E}{4}\frac{b^4}{l^2}\frac{dy}{dx}=\frac{w\pi}{6}\frac{b^2}{l}\left(lx-\frac{x^2}{2}\right)+\frac{px}{2}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

There is no constant of integration, as $\frac{dy}{dx}=0$ when $x=0$. Integrating again,

$$\frac{\pi E}{4}\frac{b^4}{l^2}\frac{dy}{dx}=\frac{w\pi}{6}\frac{b^2}{l}\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)+\frac{px^2}{4}. \quad . \quad . \quad . \quad . \quad (2)$$

There is no constant of integration, as x and y vanish together. Thus equation (1) gives the slope at any point, and equation (2) defines the neutral axis.

The slope at the free end $(x=l)=\frac{l^3}{Eb^2}\left(\frac{w}{3}+\frac{2p}{\pi b^2}\right)$.

The deflection " " " $=\frac{l^4}{Eb^2}\left(\frac{2}{9}w+\frac{p}{\pi b^2}\right)$.

13. Springs.—(*a*) *Flat Springs.*—If two forces, each equal to P but acting in opposite directions in the same straight line, are applied to the ends of a straight uniform strip of flat steel spring, the spring will assume one of the forms shown below, known as the *elastic curve.* This curve is also the form of the linear arch best suited to withstand a fluid pressure, Chap. XII.

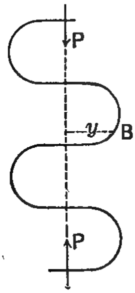


FIG. 467.

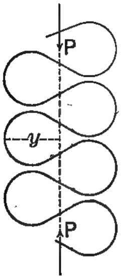


FIG. 468.

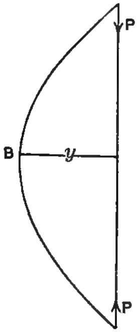


FIG. 469.

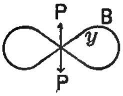


FIG. 470.

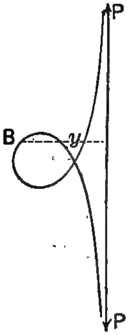


FIG. 471.

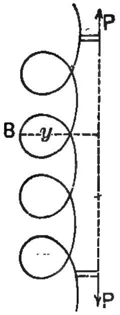


FIG. 472.

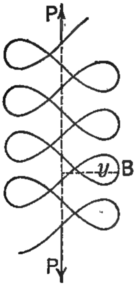


FIG. 473.

Consider a point B of the spring distant y from the line of action of P. Then

$$Py = \text{bending moment at } B = \frac{EI}{R},$$

R being the radius of curvature at B, and I the moment of inertia of the section.

If E and I are both constant,

$$Ry = \text{a constant}$$

is the equation to the elastic curve.

(*b*) *Spiral Springs* (as, e.g., in a watch).—Let Fig. 474 represent a spiral spring fixed at C and to an arbor at A, and subjected at every point of its length to a bending action only.

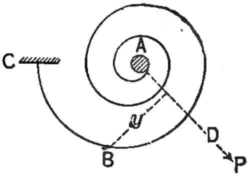


FIG. 474.

Consider the equilibrium of any portion AB of the spring.

The forces at A are equivalent to a couple of moment M, and to a force P acting in some direction AD.

This couple and force must balance the elastic moment at B.

Therefore $M+Py=EI\times$ change of curvature at B, y being the distance of B from the line of action of P, or

$$M+Py=EI\left(\frac{1}{R}-\frac{1}{R_0}\right),$$

R_0 being the radius of curvature at B *before* winding, and R that *after* winding.

Let ds be an elementary length of the spring at B.

Then, for the whole spring,

$$\Sigma(M+Py)ds=EI\Sigma\left(\frac{ds}{R}-\frac{ds}{R_0}\right)=EI\Sigma(d\theta-d\theta_0),$$

or $M\Sigma ds+P\Sigma yds=EI\times$ total change of curvature between A and C,

and $$Ms+Ps\bar{y}=EI(\theta-\theta_0),$$

s being the length of the spring, $\bar{y}$ the distance of its C. of G. from AD, θ the angle through which the spring is wound up, and θ_0 the "unwinding" due to the fixture at C. With a large number of coils the distance between the C. of G. and A may be assumed to be *nil* and then $\bar{y}=0$.

Also, if the spring is so secured that there is no change of direction relatively to the barrel.

$$\theta_0=0, \quad \text{and} \quad Ms=EI\theta.$$

Let the winding-up be effected by a couple of moment $Qq=M$, Q being a tangential force at the circumference of a circle of radius q.

The distance through which Q moves (or *deflection of* Q)

$$=q\theta=\frac{qf}{cE}s, \quad \text{since} \quad M=\frac{f}{c}I,$$

f being the skin stress, and c the distance of the neutral axis of the spring from the skin.

Thus, if b is the width of a spring of circular or rectangular section, $c=\frac{b}{2}$, and hence

$$\text{the deflection}=\frac{2qf}{bE}s.$$

The *work done* $=\frac{1}{2}Q\times\text{deflection}=\frac{1}{2}\frac{M}{q}q\theta=\frac{M\theta}{2}$

$$=\frac{f^2}{2}\frac{sI}{Ec^2}=\frac{f^2}{2}\frac{sAk^2}{Ec^2}=\frac{f^2V}{2E}\frac{k^2}{c^2},$$

k^2 being the square of the radius of gyration, A the sectional area of the spring and V its volume.

In case of spring of rectangular section $\frac{k^2}{c^2}=\frac{1}{3}$.

" " " " " circular " $\frac{k^2}{c^2}=\frac{1}{4}$.

Again, the spiral spring in Fig. 475 is wholly subjected to a *bending* action by means of a *twisting couple* of moment $M=Qq$ in a plane perpendicular to the axis of the spring. Any torsion in the spring itself is now due to the coils not being perfectly flat.

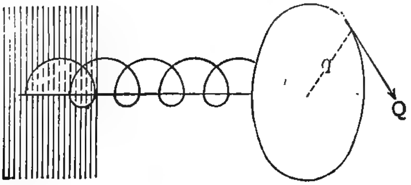


FIG. 475.

Let R_0 = radius of a coil before the couple is applied.
" R = " " " " after " " " "

$$Qq=M=EI\left(\frac{1}{R}-\frac{1}{R_0}\right)=EI\frac{\theta}{s},$$

θ being the angle of twist; or

$$\frac{Qqs}{EI}=\frac{Ms}{EI}=\frac{s}{R}-\frac{s}{R_0}=(N-N_0)2\pi,$$

N being the number of coils before the couple is applied, and
N_0 " " " " " after " " " "

The distance through which Q acts, i.e.,

$$\text{the "deflection,"}=q\theta=\frac{fqs}{Ec},$$

and

$$\text{the work done}=\frac{M\theta}{2}=\frac{fV}{2E}\frac{k^2}{c^2},$$

$$=\frac{1}{6}\frac{f^2V}{E} \text{ for spring of rectangular section,}$$

$$=\frac{1}{8}\frac{f^2V}{E} \text{ " " " circular "}$$

c. Simple rectangular spring.

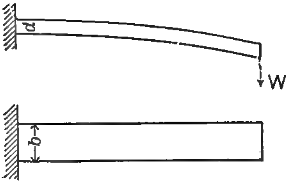


FIG. 476.

By Ex. 11,

$$\Delta=\frac{1}{3}\frac{Wl^3}{EI}=\frac{2}{3}\frac{fl^2}{Ed}, \quad . \quad . \quad . \quad (1)$$

since

$$\frac{Wl}{I}=\frac{M}{I}=\frac{2f}{d}=\frac{12Wl}{bd^3}.$$

$$\text{Also, } W\Delta=\frac{bd^2f}{6l}\frac{2}{3}\frac{fl^2}{Ed}=\frac{1}{9}\frac{f^2bdl}{E}=\frac{f^2V}{9E}.$$

Hence

$$V=9\frac{W\Delta E}{f^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\text{the work done}=\frac{W\Delta}{2}=\frac{f^2V}{18E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

d. Spring of constant depth but triangular in plan.

Let b_x be the breadth at a distance x from the fixed end. Then

$$\frac{b_x}{b}=\frac{l-x}{l},$$

and I at the same point

$$=\frac{b_xd^3}{12}=\frac{1}{12}\frac{l-x}{l}bd^3.$$

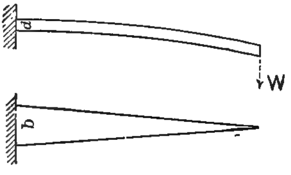


FIG. 477.

Therefore

$$\frac{d^2y}{dx^2}=\frac{W}{EI}(l-x)=\frac{12Wl}{Ebd^3}.$$

Integrating twice, $\frac{dy}{dx}=\frac{12Wl}{Ebd^3}x,$

and $y=\frac{6Wl}{Ebd^3}x^2.$

Therefore $\varDelta=\frac{6Wl^3}{Ebd^3}=\frac{fl^2}{Ed}$. . . . . (4)

Also, $W\varDelta=\frac{bd^2f}{6l}\frac{fl^2}{Ed}=\frac{f^2bdl}{6E}=\frac{f^2V}{3E},$

or $V=\frac{3W\varDelta E}{f^2},$ . . . . (5)

and the work done $=\frac{W\varDelta}{2}=\frac{f^2V}{6E}$. . . . . (6)

N.B.—The results (1) to (6) are the same if the springs are compound; i.e., if the rectangular spring is composed of n simple rectangular springs laid one above the other, and if the triangular spring is composed of n triangular springs laid one above the other.

(*e*) *Spring of constant width but parabolic in elevation.*

Let d_x be the depth at a distance x from the fixed end. Then

$$\left(\frac{d_x}{d}\right)^2=\frac{l-x}{l},$$

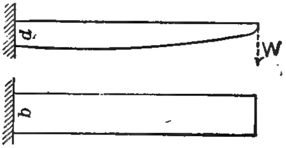


FIG. 478.

and I at the same point

$$=\frac{bd_x{}^3}{12}=\frac{db^3}{12}\left(\frac{l-x}{l}\right)^{\frac{3}{2}}.$$

Therefore $\frac{d^2y}{dx^2}=\frac{W}{EI}(l-x)=\frac{12W}{E}\frac{l^{\frac{3}{2}}}{bd^3}(l-x)^{-\frac{1}{2}}.$

Integrating twice and remembering that $\frac{dy}{dx}$ and y are each *nil* when $x=0$,

$$y=\frac{12W}{E}\frac{l^{\frac{3}{2}}}{bd^3}\left\{\tfrac{4}{3}(l-x)^{\frac{3}{2}}-2l^{\frac{1}{2}}(l-x)+\tfrac{2}{3}l^{\frac{3}{2}}\right\},$$

and hence $\varDelta=\frac{8W}{E}\frac{l^3}{bd^3}=\frac{4fl^2}{3Ed}$. . . . . (7)

Also, $W\varDelta=\frac{bd^2f}{6l}\frac{4}{3}\frac{fl^2}{Ed}=\frac{2}{9}\frac{f^2}{E}bdl=\frac{1}{3}\frac{f^2}{E}V,$

and therefore $$V=\frac{3W\Delta E}{f^2}.$$

The work done $=\frac{W\Delta}{2}=\frac{1}{6}\frac{f^2V}{E}$.

In the examples *a* to *e* on springs it will be observed that in each case the energy expended per unit of volume in bending a spring is proportional to $\frac{f^2}{2E}$, and if the energy is just sufficient not to produce a permanent set it is called the *resilience* of the spring. In this case f is the greatest stress, tensile or compressive, which the material can stand without taking a set.

Again, in Chap. V it was shown that the resilience of a cylindrical spring subjected to pure torsion is proportional to $\frac{q^2}{2G}$, q being the greatest shear stress which the material will take without permanent set, and G its coefficient of rigidity.

A table at the end of the chapter gives the values of these resiliences for different materials, but it is important to bear in mind that the value of a material for a spring depends not only upon its resilience but also upon other characteristics, such as its magnetic properties, its hardening and tempering properties, its deterioration by rust, the effect of time, etc.

(*f*) *Carriage-spring.*—Assume a uniform resilience and suppose the spring to be made of n strips each of thickness t.

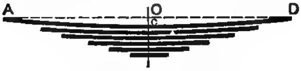


FIG. 479.

Let l be the length of the top strip;

x " " " overlap $=\frac{l}{2n}$.

First, considering the strips *free,*

let $\frac{1}{R}$ be the curvature of the top strip when unloaded;

$\frac{1}{R_1}$ " " " " " " loaded;

$\frac{1}{R_s}$ " " " sth " " unloaded.

The curvature of the sth strip when loaded $=\frac{1}{R_1+st}$.

Hence, if f is the maximum stress,

$$\frac{1}{R}-\frac{1}{R_1}=\frac{2f}{Et}=\frac{1}{R_s}-\frac{1}{R_1+st}.$$

Next, let the strips be connected together so as to form a spring, and let $\frac{1}{r_1}$ be the curvature of the top strip when the spring is unloaded.

The curvature of the sth strip $=\frac{1}{r_1+st}$.

Hence, if F_s is the force at the end of the sth overlap which will develop a B.M. $F_s x$ sufficient to change the curvature from $\frac{1}{R_s}$ to $\frac{1}{r_1+st}$, then

$$F_s x=EI\left(\frac{1}{R_s}-\frac{1}{r_1+st}\right)$$

$$=EI\left(\frac{1}{R}-\frac{1}{R_1}+\frac{1}{R_1+st}-\frac{1}{r_1+st}\right)$$

$$=\frac{2I}{t}\left\{f+\frac{Et}{2}\left(\frac{1}{R_1+st}-\frac{1}{r_1+st}\right)\right\}.$$

But $\Sigma F_s=0$ and therefore

$$\Sigma\left\{f+\frac{Et}{2}\left(\frac{1}{R_1+st}-\frac{1}{r_1+st}\right)\right\}=0,$$

Σ denoting the algebraic sum of the several values of the expression inside the brackets obtained by putting $s=1, 2, 3, \ldots n-1$.

If st is small as compared with R_1 and r_1, the portion between brackets may be written

$$f+\frac{Et}{2}\left\{\frac{1}{R_1}\left(1+\frac{st}{R_1}\right)^{-1}-\frac{1}{r_1}\left(1+\frac{st}{r_1}\right)^{-1}\right\}$$

$$=f+\frac{Et}{2}\left\{\frac{1}{R_1}-\frac{1}{r_1}-st\left(\frac{1}{R_1^2}-\frac{1}{r_1^2}\right)\right\}, \textit{ approximately.}$$

Hence
$$\Sigma\left[f+\frac{Et}{2}\left\{\frac{1}{R_1}-\frac{1}{r_1}-st\left(\frac{1}{R_1^2}-\frac{1}{r_1^2}\right)\right\}\right]$$
$$=nf+\frac{nEt}{2}\left(\frac{1}{R_1}-\frac{1}{r_1}\right)-\frac{n(n-1)}{2}\frac{Et^2}{2}\left(\frac{1}{R_1^2}-\frac{1}{r_1^2}\right)=0$$

or
$$r_1^2\left(\frac{2f}{Et}+\frac{1}{R_1}-\frac{n-1}{2}\frac{t}{R_1^2}\right)-r_1+\frac{n-1}{2}t=0,$$

a quadratic equation giving r_1.

Suppose that the plates of which the spring is made are to have the same strength throughout as the overlap.

Let W be the weight at the centre of the spring, and let b be the width of each plate.

The moment of resistance of each plate $=\frac{fbt^2}{6}$;

" " " " " the n plates $=\frac{nfbt^2}{6}$.

Then
$$\frac{Wl}{4}=\frac{nfbt^2}{6} \quad \text{and} \quad f=\frac{3}{2}\frac{Wl}{nbt^2}.$$

Also,
$$\frac{E}{R}=\frac{2f}{t},$$

and the deflection
$$Y=\frac{l^2}{8R}=\frac{f}{4}\frac{l^2}{Et}=\frac{3}{8}\frac{W}{E}\frac{l^3}{nbt^3}.$$

The resilience of a well-made carriage-spring is $\frac{f_0^2}{6E}$ inch-pounds f_0 being the proof-stress.

14. Allowance for the Weight of a Beam.—A beam is sometimes of such length that its weight becomes of importance as compared with the load it has to carry, and must be taken into account in determining the dimensions of the beam.

The necessary provision may be made by increasing the *width* of the beam designed to carry the external load alone, the width being a dimension of the first order in the expression for the elastic moment.

Assume that the weight of the beam and the external load are reduced to equivalent uniformly distributed loads.

Let W_e be the external load;
b_e " " breadth of a beam designed to support this load only;
B_e " " weight of the beam;
W " " total load, the weight of the beam being taken into account;
b " " corresponding breadth of the beam;
B " " " weight " " ".

Then $$W-B=W_e,$$

and $$\frac{b}{b_e}=\frac{B}{B_e}=\frac{W}{W_e}=\frac{W-B}{W_e-B_e}=\frac{W_e}{W_e-B_e}.$$

Ex. 28. *Apply the preceding results to a cast-iron girder of rectangular section resting upon two supports* 30 *ft. apart. The girder is* 12 *ins. deep and carries a uniformly distributed load of* 30,000 *lbs.*

Take 4 as a factor of safety; b_e is given by

$$\frac{120000}{2}=C\frac{b_e d^2}{l},$$

where $C=30{,}000$ lbs., $d=12$ ins., and $l=360$ ins.,

and therefore $$b_e=5 \text{ ins.}$$

Hence $$B_e=\frac{5\times 12}{144}\times 30\times 450=5625 \text{ lbs.},$$

$$W_e-B_e=30000-5625=24375 \text{ lbs.},$$

$$b=\frac{30000\times 5}{24375}=6\tfrac{2}{13} \text{ ins.},$$

$$B=\frac{30000}{24375}\times 5625=6923\tfrac{1}{13} \text{ lbs.},$$

$$W=W_e+B=36{,}923\tfrac{1}{13} \text{ lbs.}$$

15. Beam Acted upon by Forces Oblique to its Direction, but Lying in a Plane of Symmetry.—In discussing the equilibrium of such a beam the forces may be resolved into components parallel and perpendicular to the beam, and their respective effects superposed.

Let AB be the beam, P_1, P_2, P_3, . . . the forces, and α_1, α_2, α_3, . . . their respective inclinations to the neutral axis.

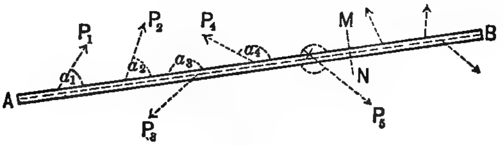


Fig. 480.

Divide the beam into any two segments by an imaginary plane MN perpendicular to the beam, and consider the segment AMN.

It is kept in equilibrium by the external forces on the left of MN, and by the elastic reaction of the segment BMN upon the segment AMN at the plane MN.

The resultant force along the beam is the algebraic sum of the components in that direction, of P_1, P_2, P_3, . . . ,

$$=P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots = \Sigma(P \cos \alpha) = H, \text{ suppose.}$$

If this force acts at a distance h from the neutral axis, it is equivalent to a couple of moment Hh and a single force H with its line of action coincident with the neutral axis. Thus the corresponding maximum and minimum stresses developed in the beam

$$= \pm Hh\frac{y}{I} + \frac{H}{A} = \frac{H}{A}\left(\pm\frac{hy}{k^2}+1\right),$$

A being the sectional area of the beam, I its moment of inertia, k the radius of gyration, and y the distance of the extreme fibres (or skin) from the neutral axis. The upper or lower sign of the first term, which is the stress due to bending, is to be taken according as it is the same or opposite kind of stress as that represented by the second term, which is the stress due to the direct load.

Again, $P_1 \sin \alpha_1$, $P_2 \sin \alpha_2$, . . . are the components of the forces at right angles to the beam, and if P_1, P_2, . . . are respectively the distances of these forces from MN, they are equivalent to a couple of moment $\Sigma(Pp \sin \alpha) = M$ and a single force $\Sigma(P \sin \alpha) = S$, which is evidently the *shearing force.* This force develops a *mean* tangential stress in the section at MN, which necessarily distorts the beam, but, generally speaking, the distortion is sufficiently small to be disregarded without appreciable error.

The skin stress due to $M = \pm M\frac{y}{I} = \pm\frac{M}{A}\frac{y}{k^2}$.

Hence the total *maximum stresses* developed in the beam are the *algebraic* sum of

$$\frac{H}{A}\left(\pm\frac{hy}{k^2}+1\right) \quad \text{and} \quad \frac{M}{A}\frac{y}{k^2}.$$

It will be observed that this result involves *two* intensities, the one due to a direct pull or thrust, the other due to a bending action. The latter is proportional to the distance of the unit area under consideration from the neutral axis. It is sometimes assumed that the same law of variation of stress holds true over the real or imaginary joints of masonry and brickwork structures, e.g., in piers, chimney-stacks, walls, arches, etc. In such cases the loci of the centres of pressure correspond to the neutral axis of a beam, and the maximum and minimum values of the intensity occur at the edges of the joint.

Ex. 29. *A horizontal beam of length l and sectional area A rests upon supports at the ends and carries a weight W at its middle point. It is also acted upon by a force H in the direction of its length.*

If the line of action of H coincides with the axis of the beam, the maximum and minimum stresses developed at the middle point

$$=\frac{H}{A}\pm\frac{M}{A}\frac{y}{k^2}=\frac{H}{A}\left(1\pm\frac{W}{H}\frac{ly}{4k^2}\right).$$

Thus the *minimum* stress will be *nil* and the *maximum* stress *doubled* if

$$\frac{W}{H}\frac{ly}{4k^2}=1 \quad \text{i.e., if} \quad \frac{W}{H}=\frac{4k^2}{ly}.$$

For a circular section of diameter d, $\frac{k^2}{y}=\frac{d}{8}$ and $\frac{W}{H}=\frac{d}{2l}$.

“ “ rectangular section of depth d, $\frac{k^2}{y}=\frac{d}{6}$ and $\frac{W}{H}=\frac{2}{3}\frac{d}{l}$

Ex. 30. Let the beam be acted upon, in the direction of its length only by a force H with its line of action at a distance h from the neutral axis.

The maximum and minimum stresses developed

$$=\frac{H}{A}\left(\pm\frac{hy}{k^2}+1\right).$$

Thus the minimum stress is *nil* and the maximum stress is doubled when $\frac{hy}{k^2}=1$ or $h=\frac{k^2}{y}$, i.e., when h is *one eighth* of the diameter for a *circular* section, and *one sixth* of the depth for a *rectangular* section.

Ex. 31. *A straight wrought-iron bar is capable of sustaining as a strut a weight w_1, and as a beam a weight w_2 at the middle point, the deflection being small as compared with the transverse dimensions. If the bar has simultaneously to sustain a weight w as a strut and a weight w' as a beam, the weight being placed at the middle of the pan, show that the beam will not break if*

$$w+\frac{w_1}{w_2}w'<w_1.$$

Let A = sectional area of bar. Then

$$\frac{w_1}{A}=\text{max. allowable compressive stress}$$

$$=\text{ " \quad " \quad " \quad " \quad developed by bending}$$

$$=\frac{w}{4}\frac{ly}{I}.$$

Therefore
$$\frac{w_1}{w_2}=\frac{ly}{4}\frac{A}{I}=\frac{ly}{4k^2}.$$

Again, the total maximum stress due to w and w'

$$=\frac{w}{A}+\frac{w'}{4}\frac{ly}{I}.$$

Therefore
$$\frac{w}{A}+\frac{w'}{4}\frac{ly}{I}<\frac{w_1}{A},$$

or
$$w+w'\frac{w_1}{w_2}<w_1.$$

Ex. 32. *The inclined beam OA,* 30 *ft. in length and carrying a uniformly distributed load of* 100 *lbs. per foot of length, is supported at A and rests against a smooth vertical surface at O.*

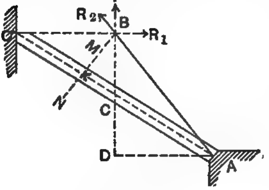


FIG. 481.

The resultant weight is vertical and acts through the centre C of OA; the reaction R_1 at O is horizontal.

Let the directions of these two forces meet in B. For equilibrium the reaction R_2 at A must also pass through B.

Let the vertical through C meet the horizontal through A in D.

The triangle ABD is a triangle of forces for the three forces which meet at B, and

$$\frac{R_1}{2000}=\frac{AD}{BD}=\frac{AD}{2DC}=\frac{1}{2}\cot 30^\circ=\frac{\sqrt{3}}{2},$$

the angle *OAD* being 30°. Therefore

$$R_1 = 1000\sqrt{3} = 1732 \text{ lbs.}$$

Consider a section *MN*, perpendicular to the beam, at a distance x from *O*.

The only forces on the left of *MN* are R_1 and the weight upon *OK*. This last is $100x$ lbs., and its resultant acts at the centre of *OM*, i.e., at a distance $\frac{x}{2}$ from *MN*.

The component of R_1 along the beam

$$= R_1 \cos 30° = \frac{2000}{2}\,\frac{\cos^2 30}{\sin 30} = 1500 \text{ lbs.}$$

The component of R_1 perpendicular to the beam

$$= R_1 \sin 30° = \frac{2000}{2} \cos 30° = 500\sqrt{3} = 866 \text{ lbs.}$$

The component of $100x$ lbs. along the beam $= 100x \sin 30° = 50x$ lbs.;
“ “ “ “ “ perpendicular to the beam $= 100x \cos 30°$
$= 86.6x$ lbs.

Hence the total compression in pounds at $NM = 1500 + 50x = C_x$,
“ “ shear “ “ “ “ $= 866 - 86.6x = S_x$.
The B.M. at K $= 866x - 43.3x^2 = M_x$,

and the maximum stress developed in the beam $= \dfrac{C_x}{A} \pm \dfrac{y}{I} M_x$,

A being the sectional area of the beam and I its moment of inertia.

These expressions may be interpreted graphically as already described, C_x, S_x being represented by the ordinates of straight lines, and M_x, f_y by the ordinates of parabolas.

f_y, for example, consists of two parts which may be treated independently. Draw *OE* and *AF* perpendicular to *OA*, and respectively equal or proportional to

$$\frac{1500}{A} \quad \text{and} \quad \frac{2500}{A}.$$

Join *EF*. The unit stress at any point of the beam due to direct compression is represented by the ordinate (drawn parallel to *OE* or *AF*) from that point to *EF*.

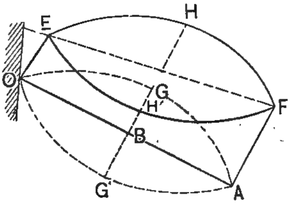


Fig. 482.

Upon the line *GG′* drawn through the middle point *B* perpendicular to *OA*, take $BG = BG'$, equal or proportional to $\frac{y}{I}\,\frac{2000}{8} 20 \cos 30° = 4330\frac{y}{I}$. According as the stress due to the bending action at any point of the beam is compressive or tensile, it is represented by the ordinate (drawn parallel to *OE* and *AF*) from that point to the parabola *OGA* or *OG′A*; *G* and *G′* respectively being the vertices, and *GG′* a common axis.

By superposing these results, the parabolas EHF, $EH'F$ are obtained, the ordinates of these curves being respectively proportional to the values of f_y for the compressed and stretched parts of the beam, i.e., for the parts above and below the neutral surface.

16. Beam Acted upon by a Bending Moment in a Plane which is not a Principal Plane.—Let XOX, YOY be the principal axes of the plane section of the beam.

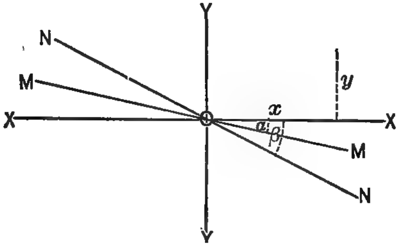


FIG. 483.

Let the axis MOM of the bending moment M make an angle α with OX.

M may be resolved into two components, viz.,

$$M\cos\alpha = X \quad\text{and}\quad M\sin\alpha = Y.$$

These components may be dealt with separately and the results superposed.

Thus the total stress, f, at any point (x, y)

$$=\text{stress due to } X + \text{stress due to } Y = \frac{Xy}{I_x} + \frac{Yx}{I_y} = f,$$

I_x, I_y being the moments of inertia with respect to the axes XOX, YOY, respectively.

If the point (xy) is on the neutral axis, then

$$\frac{Xy}{I_x} + \frac{Yx}{I_y} = 0,$$

or

$$\tan\beta = \frac{y}{x} = -\frac{YI_x}{XI_y} = -\frac{I_x}{I_y}\tan\alpha,$$

β being the angle between the neutral axis and XOX.

17. Flanged Girders, etc.—Beams subjected to forces, of which the lines of action are at right angles to the direction of their length, are usually termed *Girders;* a *Semi-girder,* or *Cantilever,* is a girder with one end fixed and the other free.

It has been shown that the stress in the different layers of a beam increases with the distance from the neutral surface, so that the most effective distribution of the material is made by withdrawing it from the neighborhood of the neutral surface and concentrating it in those parts which are liable to be more severely strained. This consideration has led to the introduction of *Flanged Girders,* or *Trusses,* i.e., girders consisting of *one* or *two flanges* (or chords), united to *one* or *two webs,* and designated *Single-webbed* or *Double-webbed* (*Tubular*) accordingly.

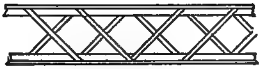


FIG. 484.

FIG. 485.

The web may be open like lattice-work (Fig. 484), or closed and continuous (Fig. 485).

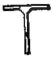


FIG. 486.

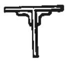


FIG. 487.

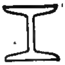


FIG. 488.

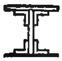


FIG. 489.

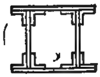


FIG. 490.

The principal sections adopted for flanged girders are:

The *Tee* (Figs. 486 and 487), the *I* or *Double-tee* (Figs. 488 and 489), the *Tubular* or *Box* (Fig. 490).

Classification of Flanged Girders.—Generally speaking, flanged girders may be divided into two classes, viz.:

I. Girders with Horizontal Flanges. In these the flanges can only convey horizontal stresses, and the shearing force, which is vertical, must be wholly transmitted to the flanges through the medium of the web.

If the web is open, or lattice-work, the flange stresses are transmitted through the lattices, or diagonals.

If the web is continuous, the distribution of stress, arising from the transmission of the shearing force, is indeterminate, and may lie in certain curves; but the stress at every point is resolvable into vertical and horizontal components. Thus the portion of the web adjoining the flanges bears a part of the horizontal stresses, and aids the flanges to an extent dependent upon its thickness.

With a thin web this aid is so trifling in amount that it may be disregarded without serious error.

II. Girders with one or both Flanges Curved. In these the shearing stress is borne in part by the flanges, so that the web has less duty to perform and requires a proportionately less sectional area.

Equilibrium of Flanged Girders.—AB is a girder in equilibrium under the action of external forces, and has its upper flange compressed and its lower flange extended. Suppose the girder to be divided into two segments by an imaginary vertical plane MN. Consider the segment AMN. It is kept in equilibrium by the external forces on the left of MN, by the flange compression at N $(=C)$, by the flange tension at M $(=T)$, and by the vertical and horizontal web forces along MN. The horizontal web forces may be disregarded *if the web is thin*, while the vertical web forces pass through M and N and consequently have no moments about these points.

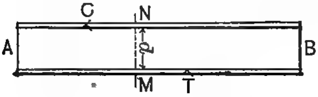


FIG. 491.

Let d be the *effective* depth of the girder, i.e., the distance between the points of application of the resultant flange stresses in the plane MN.

Take moments about M and N successively. Then

$$Cd = \text{the algebraic sum of the moments about } M \text{ of the external forces upon } AMN$$

$$= \text{the B.M. at } MN = M.$$

Similarly, $$Td = M.$$

Therefore $$\mathbf{Cd = M = Td, \quad and \quad C = T.}$$

Hence the flange forces at any vertical section of a girder with horizontal flanges are equal in magnitude but opposite in kind. The flange force, whether compressive or tensile, will be denoted by F.

Let f_1, f_2 be the unit stresses at MN in the lower and upper flanges respectively;
a_1, a_2 " the sectional areas at MN of the lower and upper flanges respectively.

Then

$$\frac{\mathbf{f}_y}{\mathbf{y}}\mathbf{I}=\frac{\mathbf{E}}{\mathbf{R}}\mathbf{I}=\mathbf{M}=\mathbf{Fd}=\mathbf{a}_1\mathbf{f}_1\mathbf{d}=\mathbf{a}_2\mathbf{f}_2\mathbf{d},$$

and the sectional areas are inversely proportional to the unit stresses.

This assumes that F is uniformly distributed over the areas a_1, a_2, so that the effective depth is the vertical distance between centres of gravity of these areas. Thus the flange forces at the centres of gravity are taken to be equal to the maximum forces, and the resistance offered by the web to bending is disregarded. The error due to the former may become of importance, and it may be found advisable to make the effective depth a geometric mean between the depths from outside to outside and from inside to inside of the flanges.

Thus, if these latter depths are h_1, h_2, the effective depth $=\sqrt{h_1h_2}$.

Ex. 33. *A flanged girder, of which the effective depth is* 10 *ft., rests upon two supports* 80 *ft. apart, and carries a uniformly distributed load of* 2500 *lbs. per lineal foot. Determine the flange force at* 10 *ft. from the end, and find the area of the flange at this point, so that the load on the metal per square inch may not exceed* 10,000 *lbs. in tension and* 8000 *lbs. in compression.*

The vertical reaction at each support

$$=\frac{80\times 2500}{2}=100{,}000 \text{ lbs.}$$

Therefore $\quad F\times 10=M=100000\times 10-2500\times 10\times 5=875{,}000$ ft.-lbs.

and $\quad F=87500$ lbs.

Thus the sectional area of tension flange $=\dfrac{87500}{10000}=$ 8.75 sq. ins.

and " " " " comp. " $=\dfrac{87500}{8000}=10.94$ sq. ins.

Ex. 34. *A continuous lattice-girder is supported at four points, each of the side spans being* 140 *ft.* 11 *in. in length,* 22 *ft.* 3 *in. in depth, and weighing* .68 *ton per lineal foot. On one occasion an excessive load lifted the end of one of* *the side spans off the abutment. Find the consequent intensity of stress in the bottom flange at the pier, where its sectional area is* 127 *sq. ins.*

$$f \times 127 \times 22\tfrac{1}{4} = 140\tfrac{11}{12} \times .68 \times \tfrac{1}{2}(140\tfrac{11}{12}).$$

Therefore $$f = 2.39 \text{ tons/sq. in.}$$

18. Examples of Moments of Resistance of Flanged Girders. (*a*) *Double-tee* section.

First, suppose the web to be so thin that it may be disregarded without sensible error.

Let the neutral axis pass through *G*, the centre of gravity of the section.

Let a_1, a_2 be the sectional areas of the lower and upper flanges respectively, and *assume that each flange is concentrated at its centre line.*

Let h_1, h_2 be the distances of these centre lines from *G*.

Let $h_1 + h_2 = d$.

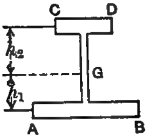


FIG. 492.

Approximately, $$\mathbf{I} = \mathbf{a_1 h_1^2 + a_2 h_2^2}.$$

Also, $$(a_1 + a_2)h_1 = a_2 d \quad \text{and} \quad (a_1 + a_2)h_2 = a_1 d.$$

Therefore $$\mathbf{I} = a_1\left(\frac{a_2 d}{a_1 + a_2}\right)^2 + a_2\left(\frac{a_1 d}{a_1 + a_2}\right)^2 = \frac{\mathbf{a_1 a_2 d^2}}{\mathbf{a_1 + a_2}}.$$

Again, if f_1, f_2 are respectively the unit stresses in the metal of the lower and upper flanges,

$$\text{the } \textit{moment of resistance} = \frac{\mathbf{f_1}}{\mathbf{h_1}}\mathbf{I} = \mathbf{f_1 a_1 d} = \frac{\mathbf{f_2}}{\mathbf{h_2}}\mathbf{I} = \mathbf{f_2 a_2 d}.$$

$$= \textit{the B.M.} \text{ at the section.}$$

If $a_1 = a_2 = a$, then $f_1 = f_2 = f$, suppose, and

$$\text{the } \textit{moment of resistance} = \mathbf{fad}.$$

ond. Let the web be too thick to be neglected.

As before, let the neutral axis pass through *G*, the centre of gravity of the section.

Let a_1, a_2 be the sectional areas of the lower and upper flanges respectively, and assume that each flange is concentrated at its centre line.

Let a_3, a_4 be the sectional areas of the portions of the web below and above G respectively;

Let h_1, h_2 be the distances from G of the lower and upper flange centre lines.

Let $h_1+h_2=d$.

Approximately,

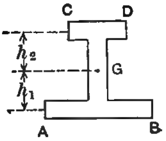


FIG. 493.

$$\mathbf{I}=a_1h_1^2+a_3\left(\frac{h_1^2}{12}+\frac{h_1^2}{4}\right)+a_2h_2^2+a_4\left(\frac{h_2^2}{12}+\frac{h_2^2}{4}\right)$$

$$=\left(\mathbf{a}_1+\frac{\mathbf{a}_3}{3}\right)\mathbf{h}_1^2+\left(\mathbf{a}_2+\frac{\mathbf{a}_4}{3}\right)\mathbf{h}_2^2,$$

and the *moment of resistance* $=\dfrac{f_1}{h_1}I=\dfrac{f_2}{h_2}I$

$=$ *the B.M.* at the section.

Again, let A' be the sectional area of the web and take moments about G. Then

$$\mathbf{a}_1\mathbf{h}_1+\frac{\mathbf{A}'}{2}(\mathbf{h}_1-\mathbf{h}_2)=\mathbf{a}_2\mathbf{h}_2,$$

or

$$\mathbf{a}_1\mathbf{f}_1+\frac{\mathbf{A}'}{2}(\mathbf{f}_1-\mathbf{f}_2)=\mathbf{a}_2\mathbf{f}_2.$$

If $a_1=a_2=A$, and $a_3=a_4=\dfrac{A'}{2}$, then $h_1=h_2=\dfrac{d}{2}$,

and $f_1=f_2=f$, suppose.

Hence

$$\mathbf{I}=\left(\mathbf{A}+\frac{\mathbf{A}'}{6}\right)\frac{\mathbf{d}^2}{4}+\left(\mathbf{A}+\frac{\mathbf{A}'}{6}\right)\frac{\mathbf{d}^2}{4}$$

$$=\left(\mathbf{A}+\frac{\mathbf{A}'}{6}\right)\frac{\mathbf{d}^2}{2},$$

and the *moment of resistance* $=\dfrac{f}{\frac{1}{2}d}\left(A+\dfrac{A'}{6}\right)\dfrac{d^2}{2}$

$$=\mathbf{f}\left(\mathbf{A}+\frac{\mathbf{A}'}{6}\right)\mathbf{d},$$

f being the stress in either flange.

It is important to remember that f_1 and f_2 are the stresses developed at the centre lines of the two flanges and that the greatest stresses are developed at the points most distant from the neutral axis. Thus the maximum stresses in the section are $f_1\left(1+\frac{1}{2}\frac{t_1}{h_1}\right)$ and $f_2\left(1+\frac{1}{2}\frac{t_2}{h_2}\right)$, t_1 and t_2 being the thickness of the lower and upper flanges respectively.

Thus the web aids the girder to an extent equivalent to the increase which would be derived by adding *one sixth* of the web area to each flange. In practice the web is usually considered as aiding the flange to the extent of *one eighth* instead of *one sixth* of its area. Approximately this makes allowance for the rivet-holes cut out of the web in making connections with the flanges and stiffeners. Some specifications altogether disregard the effect of the web in resisting bending and a somewhat higher flange stress is then allowable. The former is probably the preferable plan, as it encourages the use of thicker webs, which may add considerably, especially in exposed situations, to the life of the structure.

Again, if the weight of the material in such a beam remains constant, M increases with d. At the same time the thickness of the web diminishes, its minimum value being limited by certain practical considerations (Art. 19). Hence it follows that the distribution of material is most effective when it is concentrated as far as possible from the neutral axis.

The principles of economic construction require a beam or girder to be designed in such a manner as to be of uniform strength, i.e., equally strained at every point. An exception, however, is usually made in the case of *timber* beams or girders. The fibres of this material are real fibres and offer the most effective resistance in the direction of their length, so that if they are cut, their remaining strength is due only to cohesion with the surrounding material. Besides, there is no economy to be gained by removing a lateral portion, as the waste is of little, if any, practical value.

The correct moment of inertia (I_G) with respect to the neutral axis of the section shown by Fig. 493 may be conveniently found in the following manner:

Let t_1 and t_2 be the thickness of the bottom and top flanges respectively;

h be the depth of the web;

I be the moment of inertia of the section with respect to any main bounding line, say AB. Then

I = moment of inertia of bottom flange with respect to AB
+ " " " " " web " " " "
+ " " " " top flange " " " "

$$=\frac{a_1t_1^2}{3}+A'\left\{\frac{h^2}{12}+\left(\frac{h}{2}+t_1\right)^2\right\}+a_2\left\{\frac{t_2^2}{12}+\left(\frac{t_2}{2}+h+t_1\right)^2\right\};$$

and if y is the distance between AB and the parallel axis through G,

$$\mathbf{I_G=I-(a_1+A'+a_2)y^2.}$$

If the web, instead of being rectangular, gradually increases in width from top to bottom (Fig. 494), it may be subdivided into a rectangle of area a' and two triangles each of area a''.

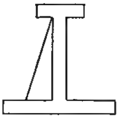


FIG. 494.

Then the moment of inertia of the web with respect to AB

= moment of inertia of rectangle with respect to AB
+2 × moment of inertia of a triangle with respect to AB

$$=a'\left\{\frac{h^2}{12}+\left(\frac{h}{2}+t_1\right)^2\right\}+2a''\left\{\frac{h^2}{18}+\left(\frac{h}{3}+t_1\right)^2\right\}.$$

Built Beams.—The moment of inertia (I) of a built beam symmetrical with respect to the neutral axis may be determined as follows:

Let Fig. 495 represent the section of such a beam, composed of equal flanges connected with the web by four equal angle-irons.

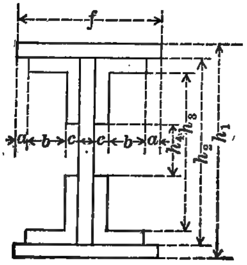


FIG. 495.

Let the dimensions be as shown on the figure.

Then I = the moment of inertia of the rectangle fh_1, diminished by *twice* the sum of the moments of inertia of the rectangles ah_2, bh_3, and ch_4

$$=\frac{fh_1^3}{12}-2\left(\frac{ah_2^3}{12}+\frac{bh_3^3}{12}+\frac{ch_4^3}{12}\right)$$

$$=\frac{\mathbf{1}}{\mathbf{12}}\{\mathbf{fh_1^3-2(ah_2^3+bh_3^3+ch_4^3)}\}.$$

In this value of I the weakening effect due to the rivet-holes in the tension flange has been disregarded. If it is to be taken into account, let p be the diameter of the rivets.

The centre of gravity of the section is now moved towards the compression flange from its original position through a distance

$$x=\frac{1}{4}\frac{p}{A'}(h_1^2-h_3^2),$$

and the moment of inertia of the net section with respect to the axis through the new C. of G. is

$$I-A'x^2-\frac{p}{12}\left(h_1^3-h_3^3\right),$$

A' being the *net* area of the section.

If the web is of the *open* type and if e is the thickness and h_5 the depth of the open part, then $\frac{eh_5^3}{12}$ must be subtracted from the value already obtained for I.

If the beam is unsymmetrical with respect to the neutral axis, its I_G may be determined by first of all finding the moment of inertia I' with respect to an axis coincident with one of the main bounding lines, as already described. Then

$$\mathbf{I_G=I'-Ay^2},$$

A being the area of the section and y the distance between the two axes.

Effective Length and Depth.—The effective length of a girder may be taken to be the distance from centre to centre of bearings.

The effective depth depends in part upon the character of the web, but in the calculation of flange stresses the following approximate rules are sufficiently accurate for practical purposes:

If the web is continuous and very thin, the effective depth is the full depth of the girder.

For plate girders the effective depth is usually taken as the distance between the centres of gravity of the flanges. In the event of cover-plates being used this depth should not exceed the distance back to back of the flange angles. The rivet-holes in the tension flange are sometimes deducted in finding the C. of G. of that flange.

For open-webbed, pin-connected and riveted girders the effective depth is the distance between the centres of gravity of the upper and lower chords. In the former this should always be the same as the distance between the centres of the pins.

If the flanges are cellular, the effective depth is the distance between the centres of the upper and lower cells.

Ex. 34. *The flanges of a girder are of equal sectional area, and their joint area is equal to that of the web. What must be the sectional area to resist a bending moment of* 300 *in.-tons, the effective depth being* 10 *ins. and the limiting inch-stress* 4 *tons?*

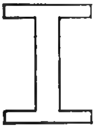


FIG. 496.

Let A' = area of web;

" d = depth of web = 10 ins.

Then
$$\text{area of each flange} = \frac{A'}{2}$$

and
$$I = A'\frac{d^2}{12} + 2\frac{A'}{2}\left(\frac{d}{2}\right)^2 = \frac{1}{3}A'd^2, \textit{ approximately.}$$

Hence
$$300 \text{ in.-tons} = \text{moment of resistance}$$
$$= \frac{4}{\frac{1}{2}d}\,\frac{1}{3}A'd^2 = \frac{8}{3}A' \times 10,$$

and
$$A' = 11\tfrac{1}{4} \text{ sq. ins.}$$

Therefore
$$\text{area of section} = 2A' = 22\tfrac{1}{2} \text{ sq. ins.}$$

Ex. 35. *The thickness of the web of an equal-flanged I beam is a certain fraction of the depth. Show that the greatest economy of material is realized when the area of the web is equal to the joint area of the flanges, and that the moment of resistance to bending is* $\frac{1}{3}fAh$, *f being the coefficient of strength, A the total sectional area, and h the depth.*

Let a = area of each flange;

" A' = " " web $= mh^2$,

mh being the thickness and m a coefficient less than unity. Then

$$2a + A' = \text{a minimum} = 2a + mh^2.$$

Therefore, differentiating,

$$2da + 2mhdh = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, the moment of resistance $= f\left(a + \frac{A'}{6}\right)h = f\left(a + \frac{mh^2}{6}\right)h.$

Differentiating, $$hda + \left(a + \frac{mh^2}{2}\right) dh = 0. \quad \text{. . . . .} \quad (2)$$

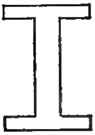


Fig. 497.

Hence, from eqs. (1) and (2),

$$\frac{h}{2} = \frac{a + \dfrac{mh^2}{2}}{2mh},$$

or $$a = \frac{mh^2}{2} = \frac{A'}{2} \quad \text{and} \quad A' = 2a.$$

Also, $$A = A' + 2a = 4a,$$

and $$\text{the moment of resistance} = f\left(a + \frac{2a}{6}\right)h = \frac{1}{3}fAh.$$

Ex. 36. *The lower and upper flanges of the section of a girder are 1 in. and 1½ ins. thick respectively, and are each 24 ins. wide; the effective depth of the girder is 48 ins. and the web is ½ in. thick. Determine the position of the neutral axis, and also find the flange unit stresses when the bending moment at the given section is 580 ft.-tons. Using the preceding notation,*

$$a_1 = 24 \text{ sq. ins.}, \quad a_2 = 36 \text{ sq. ins.}, \quad \text{and} \quad a_3 + a_4 = 24 \text{ sq. ins.}$$

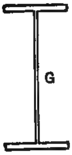


Fig. 498.

The centre of gravity of the web is at its middle point. Thus

$$24h_1 + 24(h_1 - 24) = 36(48 - h_1).$$

Therefore $h_1 = \frac{192''}{7}$ and $h_2 = \frac{144''}{7}$, defining the position of G.

Again $$a_3 = \frac{192}{7}\,\frac{1}{2} = \frac{96}{7} \text{ sq. ins.}, \quad a_4 = \frac{144}{7}\,\frac{1}{2} = \frac{72}{7} \text{ sq. ins.},$$

and $$I = \left(24 + \frac{32}{7}\right)\left(\frac{192}{7}\right)^2 + \left(36 + \frac{24}{7}\right)\left(\frac{144}{7}\right)^2 = \frac{267264}{7}.$$

Hence $$580 \times 12 = \text{moment of resistance in in.-tons}$$
$$= \frac{f_1}{\dfrac{192}{7}}\,\frac{267264}{7} = \frac{f_2}{\dfrac{144}{7}}\,\frac{267264}{3},$$

and therefore $f_1 = 5$ tons/sq. in. and $f_2 = 3\frac{3}{4}$ tons/sq. in.

Ex. 37. *In a double-flanged cast-iron beam the thickness of the web is a certain fraction of the depth h, and the maximum tensile and compressive intensities of stress are in the ratio of 2 to 5. Show that the greatest economy of material is realized when the areas of the bottom flange, web, and top flange are in the ratio of 25 to 20 to 4, and that the moment of resistance to bending is* $\frac{1}{3}fAh$, *where* $f = \frac{10}{7} \times$ *maximum tensile intensity of stress and A is the sectional area.*

Let the neutral axis divide the depth h into the segments h_1 and h_2. Then

$$\frac{h_1}{h_2}=\frac{f_1}{f_2}=\frac{2}{5} \quad \text{and} \quad h_1+h_2=h.$$

Therefore $$h_1=\frac{2}{7}h, \quad h_2=\frac{5}{7}h, \quad a_3=\frac{2}{7}mh^2, \quad a_4=\frac{5}{7}mh^2,$$

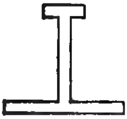


FIG. 499.

and $$I=\left(a_1+\frac{2}{21}mh^2\right)\frac{4}{49}h^2+\left(a_2+\frac{5}{21}mh^2\right)\frac{25}{49}h^2$$

$$=\frac{h^2}{49}\left(4a_1+25a_2+\frac{19}{3}mh^2\right).$$

Hence

$$\text{the moment of resistance}=\frac{f_1}{h_1}I=\frac{2}{\frac{2}{7}h}I=\frac{h}{7}\left(4a_1+25a_2+\frac{19}{3}mh^2\right).$$

Differentiating,

$$4h\cdot da_1+25h\cdot da_2+(4a_1+25a_2+19mh^2)dh=0. \quad . \quad . \quad . \quad . \quad (1)$$

Taking moments about the centre of gravity,

$$\left(a_1+\frac{a_3}{2}\right)h_1=\left(a_2+\frac{a_4}{2}\right)h_2,$$

or $$\left(a_1+\frac{mh^2}{7}\right)\frac{2}{7}h=\left(a_2+\frac{5}{14}mh^2\right)\frac{5}{7}h,$$

or $$2a_1-5a_2-\tfrac{3}{2}mh^2=0.$$

Differentiating, $$2\cdot da_1-5\cdot da_2-3mh\cdot dh=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also, $$a_1+a_2+mh^2=\text{a min.}$$

Therefore $$da_1+da_2+2mhdh=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence, by eqs. (1), (2), and (3),

$$da_1=da_2=-mhdh$$

and $$4a_1+25a_2=10mh^2=10A',$$

A' being the area of the web.

But $$2a_1-5a_2=\tfrac{3}{2}A'.$$

Therefore $$a_1=\tfrac{5}{4}A'=\tfrac{25}{4}a_2.$$

Again, $$A=a_1+A'+a_2=\tfrac{49}{20}A'.$$

Hence

$$\text{the moment of resistance} = \frac{h}{7}\left(5A' + 5A' + \frac{19}{3}A'\right)$$
$$= \tfrac{7}{3}A'h = Ah\tfrac{20}{21}$$
$$= \tfrac{1}{3}fAh,$$

where $f = \frac{10}{7} \times 2$.

Ex. 38 *A beam 36 ft. between bearings is a hollow tube of rectangular section and consists of a 24″×½″ top plate, a 24″×½″ bottom plate, and two side plates each 35″×½″. The plates are riveted together at the angles of the interior rectangle by means of four 6″×4″×½″ angle-irons, the 6-in. side being horizontal.*

First. Disregard effect of rivet-holes. Then

$$I = 24\frac{36^3}{12} - \frac{2}{12}\left(\frac{1}{2}\frac{27^3}{12} + \frac{5\frac{1}{2}\times 34^3}{12}\right) - \frac{11\times 35^3}{12} = 16341,$$

and the moment of resistance

$$= \frac{4\frac{1}{2}}{18} \times 16341 = 4085\tfrac{1}{4} \text{ in.-tons.}$$

Also, if W is the safe uniformly distributed load,

$$\frac{W\times 36}{8} \times 12 = \text{B.M. in in.-tons} = 4085\tfrac{1}{4}$$

and $$W = 75.653 \text{ tons.}$$

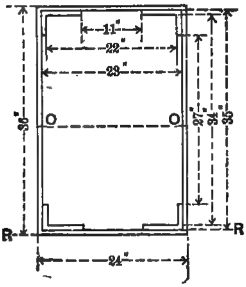


FIG. 500.

Second. Take the effect of the rivet-holes into account and assume them to be of 1 in. diameter,

The *gross* area A of the section = 78 sq. in.
" *net* " A' " " " = 76 sq. in.

The distance x through which the centre of gravity moves towards the compression flange is given by

$$x = \tfrac{1}{4}\tfrac{1}{76}(36^2 - 34^2) = \tfrac{35}{76} \text{ in.} = .46 \text{ in.}$$

The I with respect to the new centre of gravity

$$= 16341 - 76\left(\tfrac{35}{76}\right)^2 \tfrac{1}{12}(36^3 - 34^3) = 15712.2.$$

The new minimum moment of resistance

$$= \frac{4\frac{1}{2}}{18.46} \times 15712.2 = 3830 \text{ in.-tons,}$$

and if W' is the new safe uniformly distributed load,

$$\frac{W'\times 36}{8} \times 12 = \text{B.M. in in.-tons} = 3830.$$

Therefore $$W' = 70.93 \text{ tons.}$$

The pitch of the rivets connecting the angles with the upper plate is 4 ins., and it is assumed that there is an effective width of 5½ ins. in shear for each rivet.

The distance between the neutral axis and the upper face

$$=36-18.46=17.54 \text{ ins.}$$

At the surface *RR* the intensity of shear q is given by

$$q\times 11=\frac{S}{I}(24\tfrac{1}{2}\times 17.29)$$

or

$$q=18.862\times\frac{S}{I} \text{ tons/sq. in.}$$

At the neutral surface *OO*, where the intensity of shear is greatest and equal to 3½ tons per square inch,

$$3\tfrac{1}{2}\times 1=\frac{S}{I}\left(24\times\frac{1}{2}\times 17.29+2\times 6\times\frac{1}{2}\times 16.79+2\times 3\tfrac{1}{2}\times\frac{1}{2}\times 14.79\right.$$
$$\left.+2\times 17.04\times\frac{1}{2}\times\frac{17.04}{2}\right)$$
$$=\frac{S}{I}\times 505.1658.$$

Therefore

$$S=\frac{3\tfrac{1}{2}I}{505.1658}=\frac{3\tfrac{1}{2}\times 15712.2}{505.1658}=108.86 \text{ tons,}$$

and

$$q=18.862\times\frac{108.86}{15712.2}=.13 \text{ ton/sq. in.}$$

Hence if the plate and angle faces are close together,

$$\tfrac{22}{7}\,\tfrac{1}{4}d^2\,3\tfrac{1}{2}=.13\times 4\times 5\tfrac{1}{2}$$

and

$$d=1.02 \text{ ins.};$$

if the plate and angle faces are not close together so that the rivets are subject to a bending action by Ex. 23, Art. 9,

$$\tfrac{3}{4}(\tfrac{22}{7}\,\tfrac{1}{4}d^2\,3\tfrac{1}{2})=.13\times 4\times 5\tfrac{1}{2}$$

and

$$d=1.18 \text{ ins.}$$

Ex. 39. *Find the moment of resistance of a section composed of two equal flanges, each consisting of two* 600-*mm.*×7-*mm. plates riveted to a* 1200-*mm.*×8-*mm. web plate by means of two* 100-*mm.*×100-*mm.*×12-*mm. angle-irons; two* 70-*mm.* ×70-*mm.*×9-*mm. angles are also riveted to the inner faces of the flanges, the ends of the horizontal arms being* 24 *mm. from the outside flanges; the total depth of the section* =3.228 *m., and the interval between the two web plates, which is open, is* 2 *m.; coefficient of strength* =6 *k. per millimetre*^{2}.

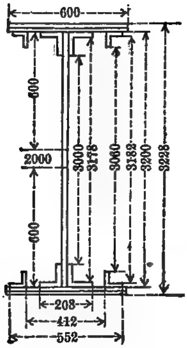


FIG. 501.

Disregarding the effect of rivet-holes,

$$I=\tfrac{1}{12}600.\overline{3228}^3-\tfrac{2}{12}(24.\overline{3200}^3+61.\overline{3182}^3+9.\overline{3060}^3+102.\overline{3200}^3+88.\overline{3176}^3+12.\overline{3000}^3)-\tfrac{1}{12}8.\overline{2000}^3$$

$$=\frac{1130950789328}{12},$$

and the moment of resistance

$$=\frac{6}{1614}\,\frac{1130950789328}{12\times(1000)^2}\text{km.}=350.3565\text{ km.}$$

Ex. 40. *A girder of 21 ft. span has a section composed of two equal flanges each consisting of two 3½"×5"×½" angles riveted to a 39"×⅜" web; the cover-plates on the flanges are each 12"×⅜", and the rivets in the covers alternate with those connecting the angles and web; the pitch of the rivets is 3½ ins. Find the diameter and also find the maximum flange stresses, (a) disregarding the weakening effect of the rivet-holes in the tension flange; (b) taking this effect into account.*

The load upon the girder is a uniformly distributed load of 20,800 lbs. (including weight of girder) and a load of 50,000 lbs. concentrated at each of the points distant 4½ ft. from the middle point of the girder.

(*a*) Disregarding the effect of riveting, the neutral axis is at *O*, the middle point of the depth. Then

$$I_0=\frac{12(39\tfrac{3}{4})^3}{12}-\frac{2}{12}\left(\frac{13}{16}39^3+4\tfrac{1}{2}\times38^3+\frac{1}{2}32^3\right)=10890\,{}_3\tfrac{7}{92}.$$

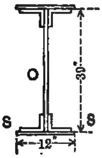


FIG. 502.

If q is the intensity of shear at the surface SS,

$$q10\tfrac{3}{8}=\frac{S}{I}\times12\times\tfrac{3}{8}\times19\tfrac{11}{16}=\frac{S}{I}\,\frac{2835}{32}.$$

$$\text{At } O,\ 3\tfrac{1}{2}\,\frac{3}{8}=\frac{S}{I_0}(12\tfrac{3}{8}\times19\tfrac{11}{16}+2\times4\tfrac{1}{2}\times\tfrac{1}{2}\times19\tfrac{1}{4}+2\times3\tfrac{1}{2}\times\tfrac{1}{2}\times17\tfrac{3}{4})=\frac{S}{I_0}\,\frac{19753}{64}.$$

$$\text{Therefore}\quad\frac{q}{3\tfrac{1}{2}}\,\frac{10\tfrac{3}{8}}{\tfrac{3}{8}}=\frac{2835}{32}\,\frac{64}{19753},\quad\text{or}\quad q=.0363\text{ tons/sq. in.}$$

There is *one* rivet to each (½×3½×10⅜) sq. ins.

First. Assume that there is close contact along *SS*, so that the rivets are in shear only. Then

$$\frac{\pi d^2}{4}3\tfrac{1}{2}=\text{working strength of rivet of diam. } d$$

$$=\tfrac{1}{2}\times3\tfrac{1}{2}\times10\tfrac{3}{8}\times.0363$$

and $$d=.48\text{ in., say }\tfrac{1}{2}\text{ in.}$$

Second. If the contact along *SS* is not close, the rivets are subject to a bending action, and then

$$\frac{3}{4}\left(\frac{\pi d^2}{4}3\tfrac{1}{2}\right)=\tfrac{1}{2}\times 3\tfrac{1}{2}\times 10\tfrac{3}{8}\times .0363 \text{ (Ex. 23, Art. 9)}$$

and $$d=.56 \text{ in., say } \tfrac{9}{16} \text{ in.}$$

(*b*) Taking into account the effect of riveting, and using a $\frac{9}{16}$-in. rivet, the neutral axis is moved to *G*, where

$$OG=\frac{1}{4}\frac{\frac{9}{16}}{\frac{2473}{64}}(39\tfrac{3}{4}{}^2-38^2)=\frac{19593}{39568}=.494 \text{ in., } \frac{2473}{64} \text{ being the } net \text{ area of the section.}$$

Then

$$19\tfrac{7}{8}-OG=19.381 \text{ ins.} \quad \text{and} \quad 19\tfrac{7}{8}+OG=20.369 \text{ ins.,}$$

Also, $$I_G=I_0-\frac{2473}{64}(.494)^2-\frac{1}{12}\frac{9}{16}(39\tfrac{3}{4}{}^3-38^3)=10508.6.$$

The maximum B.M. is at the middle point, and therefore

$$\text{B.M.}_{\text{max.}}=\left(\frac{20{,}800}{8}21+50000\times 6\right)12=4{,}255{,}200 \text{ in.-lbs.}$$

$$=\text{moment of resistance in in.-lbs.}$$

Hence if f_t and f_c are the flange tensile and compressive stresses respectively,

in case (*a*), $4255200=\frac{f_t(=f_c)}{19.875}10890_{\frac{7}{192}}$ and $f_t=f_c=7766$ lbs./sq. in.;

in case (*b*), $4255200=\frac{f_t}{20.369}\times 10508.6=\frac{f_c}{19.381}\times 10508.6$,

so that $f_t=7847$ lbs./sq. in. and $f_c=8248$ lbs./sq. in.

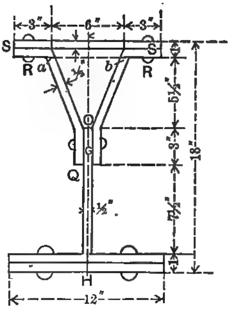


FIG. 503.

Ex. 41. *Determine the position of the neutral axis, the moment of resistance, the ratio of the maximum to the average intensity of shear of the section shown by Fig.* 503, *the coefficients of strength per square inch being* 4½ *tons for tension and compression and* 3½ *tons for shear. Also find the diameter d of the rivets R, disregarding the weakening effect of the rivet-holes and taking the pitch to be* 4 *inches. Neutral axis is at G, the centre of gravity of the section; ab*$=5\tfrac{1}{2}\tfrac{3}{4}$ *ins.* Therefore

$$GH(12\times 1-\tfrac{1}{2}\times\tfrac{1}{2}\times\overline{5\tfrac{1}{2}\tfrac{3}{4}+6}+2\times\tfrac{1}{2}\times 5\tfrac{1}{2}+2\times 3\times\tfrac{1}{2}+\tfrac{1}{2}\times 10\tfrac{1}{2}$$
$$+12\times 1)=12\times 1\times 17\tfrac{1}{2}-5\tfrac{1}{2}\tfrac{3}{4}\times\tfrac{1}{2}\times 17\tfrac{1}{4}-\tfrac{1}{2}\times\tfrac{1}{2}\tfrac{1}{4}\times\tfrac{1}{2}\times 17\tfrac{1}{3}+23\times\tfrac{1}{2}\times 10$$
$$+10\tfrac{1}{2}\times\tfrac{1}{2}\times 6\tfrac{1}{4}+12\times\tfrac{1}{2}+2\times 5\tfrac{1}{2}\times\tfrac{1}{2}\times 14\tfrac{1}{4},$$

and $GH = 8.8171$ ins.; also $OG = OH - GH = 2.6829$ and $GK = 9.1829$.

Again,

$$I_0 = \tfrac{1}{12}\times 12\times 1^3 + 12\times 1\times 6^2 - \tfrac{1}{12}\times 5\tfrac{13}{24}(\tfrac{1}{2})^3 - 5\tfrac{13}{24}\times \tfrac{1}{2}(5\tfrac{3}{4})^2 - 2\times \tfrac{11}{48}\times\tfrac{1}{36}(\tfrac{1}{2})^3 - \tfrac{11}{48}\times\tfrac{1}{2}\times 5\tfrac{1}{3}^2 + \tfrac{1}{3}\times 2\times\tfrac{1}{2}\times(5\tfrac{1}{2})^3 + \tfrac{1}{3}\times 2\times\tfrac{1}{2}\times 3^3 + \tfrac{1}{3}\times\tfrac{1}{2}(10\tfrac{1}{2})^3 + \tfrac{1}{12}\times 12\times 1^3 + 12\times 1(11)^2 = 2047.827.$$

Hence $$I_G = 2047.827 - \frac{3347}{96}(2.6829)^2 = 1797,$$

$$\frac{3347}{96} = 17.4323 \text{ sq. ins. being the area of the section.}$$

The *least* moment of resistance $= \dfrac{4\frac{1}{2}}{9.1829}\times 1797 = 880.54$ in.-tons.

Let q be the shear stress at the surface SS. Then

$$q6 = \frac{S}{I_G}12\times\tfrac{1}{2}(8.9329).$$

The shear stress is greatest at Q. Therefore

$$3\tfrac{1}{2}\times\tfrac{1}{2} = \frac{S}{I_G}(\tfrac{1}{2}\times 7\tfrac{1}{2}\times 3\tfrac{3}{4} + 12\times 1\times 8) = \frac{1761}{16}\;\frac{S}{I_G}.$$

Hence $$S = \frac{7}{4}\;\frac{16}{1761}\;I_G = 28.574 \text{ tons, and}$$

$$q = \frac{1}{6}\;\frac{7}{4}\;\frac{16}{1761}12\times\tfrac{1}{2}\times 8.9329 = .142 \text{ ton/sq. in.}$$

Also, $$\frac{\textit{Average}\text{ shear stress}}{\textit{Maximum}\text{ shear stress}} = \frac{\dfrac{28.574}{17.4323}}{3.5} = .234.$$

Assuming the contact along SS to be close,

$$\frac{\pi d^2}{4}3\tfrac{1}{2} = 3\times 4\times .142 \quad\text{and}\quad d = .787 \text{ in., say } \tfrac{13}{16} \text{ in.}$$

If the contact is not close,

$$\frac{3}{4}\left(\frac{\pi d^2}{4}3\tfrac{1}{2}\right) = 3\times 4\times .142 \quad\text{and}\quad d = .9, \text{ say } \tfrac{15}{16} \text{ in.}$$

Ex. 42. *An aqueduct for a span of 20 feet consists of a cast-iron channel beam 30 ins. wide and 20 ins. deep. Find the thickness of the metal so that the water may safely rise to the top of the channel, the safe coefficient of strength being 1 ton per square inch. Find the safe limiting span of the channel under its own weight.*

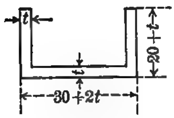


FIG. 504.

Let t'' = thickness of metal. Then

the weight W_1 of the aqueduct in tons $=\frac{(30+2t)(20+t)-600}{144}20\frac{450}{2000}=\frac{35t+t^2}{16}$;

" " W_2 " " water " " $=\frac{30\times20}{144}20\times62\frac{1}{2}\frac{1}{2000}=2\frac{29}{48}$.

If G is the centre of gravity of the section,

$$OG(2\times20\times t+t\,\overline{30+2t})=2\times20\times t\times10-\frac{t}{2}\times t\,\overline{30+2t}$$

and $$OG=\frac{400-15t-t^2}{70+2t};\quad OG+t=\frac{400+55t+t^2}{70+2t}.$$

Also,

$$I_0=\frac{1}{12}\times2\times20^3t+2\times20\times t\times10^2+\frac{1}{12}(30+2t)t^3+(30+2t)t\frac{t^2}{4}=\frac{16000}{3}t+\frac{30+2t}{3}t^3$$

and

$$I_G=\frac{16000}{3}t+\frac{30+2t}{3}t^3-\frac{t(400-15t-t^2)^2}{70+2t}.$$

Hence

Moment of resistance in in.-tons $=\frac{I_G}{OG+t}=$ max. B.M. in in.-tons $=\frac{W_1+W_2}{8}20\times12$.

The proper value of t can now be determined by trial as shown in the following table:

t-ins.	W_1 in tons.	B.M. in in.-tons.	Moment of Resistance in in.-tons.	I_G
$\frac{1}{8}$	.2744	86.36	66.414	384.674
$\frac{1}{4}$	.5508	94.56	132.35	776.88
$\frac{1}{6}$	.3663	89.215	88.4	514.566
.17	.3693	89.203	89.14	518.736

Hence .17 in. is the thickness required.

The total area of the section $=11.9578$ sq. ins.

If L feet is the limiting length of the aqueduct under its own weight,

$$\frac{11.9578}{144}L\frac{450}{2000}\frac{L}{8}12=89.14\quad\text{and}\quad L=56.4\text{ ft.}$$

Ex. 43. *A cast-iron girder* 139 *ins. between supports and* $12\frac{1}{8}$ *ins. deep had a top flange* $2\frac{1}{2}''\times\frac{7}{8}''$, *a bottom flange* $10''\times1\frac{1}{4}''$, *and a web* $\frac{3}{4}''$ *thick. The girder failed under loads of* $17\frac{1}{2}$ *tons placed at the two points distant* $3\frac{3}{4}$ *ft. from each support. What were the central flange stresses at the moment of rupture? What was the central deflection when the load at each point was* $7\frac{1}{2}$ *tons?* ($E=18{,}000{,}000$ *lbs.*; *weight of girder* $=3368$ *lbs.*; *ton* $=2240$ *lbs.*)

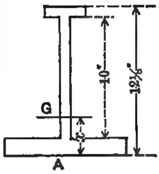


FIG. 505.

Area of top flange $=2\frac{1}{2}\times\ \frac{7}{8}=\frac{35}{16}$ sq. in.

" " web $=10\times\ \frac{3}{4}=\frac{15}{2}$ " "

" " bottom flange $=10\times1\frac{1}{4}=\frac{25}{2}$ " "

Let x = distance of centre of gravity from A. Then

$$x\left(\tfrac{35}{16}+\tfrac{15}{2}+\tfrac{25}{2}\right)=\tfrac{35}{16}\times 11\tfrac{11}{16}+\tfrac{15}{2}\times 6\tfrac{1}{4}+\tfrac{25}{2}\times\tfrac{5}{8} \quad\text{and}\quad x=\tfrac{4109}{1136}=3.617\text{ ins.}$$

Also, moment of inertia, I_A, with respect to lowest face of section is given by

$$I_A=\tfrac{1}{3}10\left(\tfrac{5}{4}\right)^3+\tfrac{1}{12}\,\tfrac{3}{4}(10)^3+\tfrac{15}{2}\left(6\tfrac{1}{4}\right)^2+\tfrac{1}{12}\,\tfrac{5}{2}\left(\tfrac{7}{8}\right)^3+\tfrac{35}{36}\left(11\tfrac{11}{16}\right)^2=660.92573.$$

Therefore $\qquad I_G=660.92573-\tfrac{355}{16}\left(\tfrac{4109}{1136}\right)^2=370.642.$

Again,

$$\text{max. B.M.}=\left(17\tfrac{1}{2}\times 2240\times 3\tfrac{3}{4}\times 12+\frac{3368}{8}\times 139\right)\text{ in.-lbs.}$$

$$=1822519\text{ in.-lbs.}=\frac{f_t}{3.617}\times 370.642=\frac{f_c}{8.508}\times 370.642.$$

Hence $\qquad f_t=17786$ lbs./sq. in. $\quad$ and $\quad f_c=41836$ lbs./sq. in.

These results are of course based on the hypothetical assumption that the elastic theory of the transverse strength of beams holds good up to the point of failure.

Again, by Ex. 12,
the deflection due to the 7½-ton concentrations

$$=\frac{7\tfrac{1}{2}\times 2240\times 3\tfrac{3}{4}\times 12}{18000000\times 370.642}\left(\frac{\overline{139}^2}{8}-\frac{\overline{45}^2}{6}\right)=.23554\text{ in.},$$

the deflection due to the weight of the girder

$$=\frac{5}{384}\,\frac{3368\times\overline{139}^3}{18000000\times 370.642}=.01765,$$

and the total central deflection $\qquad = .2532$ in.

19. Design of a Girder of an I Section with Equal Flange Areas, to Carry a Given Load.

At any point distant x from the middle of the girder, let y be the depth of the girder, A the sectional area of each flange, A' the sectional area of the web, M_x the B.M. and S_x the S.F. Then

$$f\left(A+\frac{A'}{6}\right)y=M_x,$$

f being the safe unit stress in tension or compression.

It is in accordance with good practice to assume that the flanges are assisted by the web to the extent of $\dfrac{A'}{8}$ instead of $\dfrac{A}{6}$.

Web.—Assume that the web transmits the *whole* of the shearing force. This is not strictly correct if the flange is curved, as the flange then bears a portion of the shearing force. The error, however, is on the safe side.

Theoretically, the web should contain no more material than is absolutely necessary.

Let f_s be the safe unit stress in shear. Then

$$A' = \frac{S_x}{f_s},$$

and the sectional area is, therefore, independent of the depth.

$$\text{The } \textit{thickness} \text{ of the web} = \frac{A'}{y} = \frac{S_x}{f_s y},$$

but this is often too small to be of any practical use.

Experience indicates that the minimum thickness of a plate which has to stand ordinary wear and tear is about $\frac{1}{4}$ or $\frac{5}{16}$ in., while if subjected to saline influence its thickness should be $\frac{3}{8}$ or $\frac{1}{2}$ in. Thus the weight of the web rapidly increases with the depth, and the greatest economy will be realized for a certain definite ratio of the depth to the span.

The thickness of the web in a cast-iron girder usually varies from 1 to 2 ins.

In the case of riveted girders with plate webs of medium size all practical requirements are effectively met by specifying that the shearing stress is not to exceed *one half* of the flange tensile stress, and that stiffeners are to be introduced at intervals not exceeding *twice* the depth of the girder when the thickness of the web is less than *one eightieth* of the depth. Again, it is a common practical rule to stiffen the web of a plate girder at intervals approximately equal to the depth of the girder, whenever the shearing stress in pounds per square inch exceeds $12000 \div \left(1 + \frac{H^2}{3000}\right)$, H being the ratio of the depth of the web to its thickness.

Flanges.—First. Assume that the flanges have the same sectional area from end to end of girder.

If the effect of the web is neglected, and taking f as the coefficient of strength,

$$y=\frac{M_x}{fA},$$

and the depth of the beam at any point is proportional to the ordinate of the bending-moment curve at the same point.

For example, let the load be uniformly distributed and of intensity w, and let l be the span. Then

$$M_x=\frac{w}{2}\left(\frac{l^2}{4}-x^2\right),$$

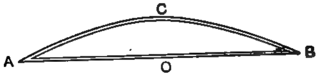


FIG. 506.

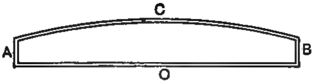


FIG. 507.

and the beam in elevation is the parabola ACB, having its vertex at C and a central depth $CO=\frac{wl^2}{8Af}$. The depths thus determined are a little greater than the depths more correctly given by the equation

$$y=\frac{M_x}{f\left(A+\frac{A'}{6}\right)}.$$

Second. Assume that the depth y of the girder is constant. Then

$$A+\frac{A'}{6}=\frac{M_x}{fy},$$

and neglecting the effect of the web, the area of the flange at any point is proportional to the ordinate of the curve of bending moments at the same point.

Let the load be uniformly distributed and of intensity w; also, let the flange be of the same uniform width b throughout.

The flange, in elevation, is then the parabola ACB, Fig. 508, having its vertex at C and its central thickness $CO=\frac{wl^2}{8fyb}$. Such beams are usually of wrought iron or steel, and are built up by means of plates. It is impracticable to cut these plates in such a manner as to make the curved boundary of the flange a true parabola (or any other curve). Hence the flange is generally constructed as follows:

Draw the curve of bending moments to any given scale. By altering the scale, the ordinates of the same curve will represent the flange thicknesses. Divide the span into segments of suitable lengths.

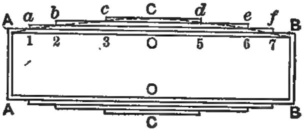


FIG. 508.

From A to 1 and B to 7 the thickness of the flange is $1a=7f$; from 1 to 2 and 7 to 6 the thickness is $2b=6e$; from 2 to 3 and 6 to 5 the thickness is $3c=5d$; and from 3 to 5 the thickness is CO.

The more correct value of $A\left(=\frac{M}{fy}-\frac{A'}{6}\right)$ is somewhat less than that now determined, but the error is on the safe side.

Again, at any section

$$\frac{E}{R}=\frac{2f}{y},\quad \text{and hence}\quad R\propto y \text{ the depth.}$$

Thus the curvature diminishes as the depth increases, so that a girder with horizontal flanges is superior in point of *stiffness* to one of the parabolic form. The amount of metal in the web of the latter is much less than in that of the former. If great flexibility is required, as in certain dynamometers, the parabolic form is of course the best.

Ex. 44. *Design of 50-ft. plate-girder span.*

The data to be used are as follows:

Live load 4900 lbs. per lineal foot of span, *or* an axle concentration of 44,000 lbs.

Percentage to be added to live load for impact $=\frac{400}{L+500}$,

where L = length of span over which the load giving the maximum stress is distributed.

Distance centre to centre of bearings =50 ft.
" " " " " girders = 7 "
Allowable flange stress =14,000 lbs. per sq. in.
" shearing stress in webs and rivet =10,000 " " " "
" bearing " in rivets =20,000 " " " "
" fibre stress in timber = 2,000 " " " "

We may assume that the rail distributes the axle concentration equally over three ties; hence each tie will be loaded as shown in Fig. 519. As the ties will not greatly exceed 1 ft. centre to centre, the span for three ties may be taken as 2 ft. and the impact allowance 80 per cent. Therefore

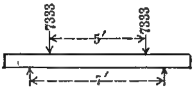


FIG. 509.

$$B.M. = 7333 \times 12 \times 1.8 = 158{,}400 \text{ in.-lbs.}$$

Making the tie 7 ins. wide to secure a good bearing,

$$158400 = \frac{2000 \times 7 \times d^2}{6},$$

where d is the depth, so that

$$d = 8.2''.$$

We shall use $7'' \times 9''$ ties 10 ft. long, set on edge, spaced at 13-in. centres and dapped ½ in. on the girders; also two $7'' \times 8''$ longitudinal guard timbers. The dead load on the span may now be estimated as follows:

Ties and guard-rails	218	lbs. per lineal foot of span
Track and fastenings	62	" " " " " "
Bracing	40	" " " " " "
Main girders	600	" " " " " "
Total dead load	920	" " " " " "
Live load	4900	" " " " " "
Impact, 73 per cent	3571	" " " " " "
Total load	9391	" " " " " "

or say 4700 lbs. per lineal foot of each girder.

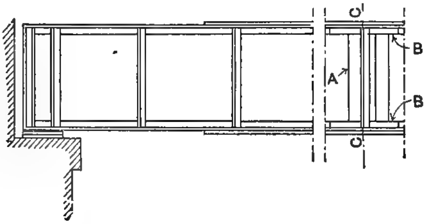


FIG. 510.

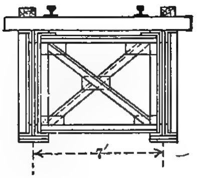


FIG. 511.

In railway structures the thickness of the web, for durability, should not be less than ⅜ inch. The depth generally depends on economical considerations, which will usually be satisfied by making it from one eighth to one tenth of the span, or in this case say 66 ins. Then

$$A' = 66 \times \tfrac{3}{8} = 24.75 \text{ sq. ins. } \textit{gross}, \text{ or about } 18 \text{ sq. ins. } \textit{actual.}$$

This will resist in shear $18 \times 10000 = 180{,}000$ lbs.

Greatest shear in girder $= 4700 \times 25 = 117{,}500$ lbs.

(The web section is therefore more than sufficient to carry the shear, but the thickness cannot be diminished, and a decrease in the depth would probably cause a more than corresponding increase in the flange area.)

The effective depth y, or distance between the centres of gravity of the top and bottom flanges, cannot be known exactly until the flange section is determined. It may be assumed, however, to be 5.35 ft. Hence if A is the area of the flange,

$$M=\left(A+\frac{A'}{8}\right)fy.$$

Now $M=\frac{4700\times 50^2}{8}=1{,}468{,}750$ ft.-lbs. Therefore

$$1468750=\left(A+\frac{24.75}{8}\right)\times 14000\times 5.35,$$

and $A=16.5$ sq. ins., net.

One eighth instead of *one sixth* of the web area is taken as assisting the flange, to allow approximately for the rivet-holes.

Use two angles $6''\times 6''\times \frac{9}{16}''$ . . . . $=12.86$ sq. ins.
One cover-plate $14\times\frac{1}{2}$. . . . . $=\ 7.00$ " "

Total gross area. $=19.86$ " "

Seven-eighth-inch rivets will be used, but a hole 1 inch in diameter will be deducted to allow for possible injury to the metal in punching. Therefore

$$\text{area to be deducted from each flange}=2\times\tfrac{17}{16}+2\times\tfrac{9}{16}=3.25 \text{ sq. ins.},$$

and $$\text{the net area of flange}=19.86-3.25=16.61 \text{ sq. ins.},$$

which is sufficient.

To allow for irregularities in the edge of the web the flange angles will be placed $66\frac{1}{2}$ ins. back to back. The effective depth may now be checked, and will be found to agree with the assumption made above.

To find the length of the cover-plate:
Flange angles. . . . . $=12.86$ sq. ins.
One eighth web. . . . . $=\ 3.10$ " "

Total area beyond end of cover-plate. . . . . . . $=15.96$ " "
Deducting four rivet-holes $(4\times\frac{9}{16})$. . . . . $=\ 2.25$ " "

Net section. . . . . $=13.71$ " "

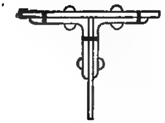
Effective depth beyond end of cover-plate $=5.25$ feet. Therefore
B.M. which flange can resist $=13.71\times 14000\times 5.25=1{,}007{,}685$ ft.-lbs.
Let $x=$ distance from centre of girder at which this B.M. is developed.

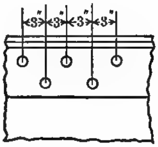
Then $$M_x=\frac{w}{2}\left(\frac{l^2}{4}-x^2\right),\quad \text{or}\quad x=\pm\sqrt{\frac{l^2}{4}-\frac{2M_x}{w}}$$

$$=\sqrt{\frac{50^2}{4}-\frac{2\times 1007685}{4700}}=14 \text{ ft.},$$

and the theoretical length of cover-plate $=2\times 14=28$ ft.

We shall, however, extend it 18 ins. further at each end, so as to get room for sufficient rivets to develope its strength, making the total length 31 ft.





Detail of Flange

FIG. 512. FIG. 513.

In designing the compression flange it would not be necessary to deduct the rivet-holes. It acts as a column, however, and so it is customary to make both flanges alike.

To determine the pitch of the rivets connecting the flange angles to the web, we may assume that the entire vertical shear is carried by the web and is distributed uniformly over the web. At the ends the vertical shear is 117,500 lbs., or $\frac{117500}{66}=1780$ lbs. per inch of depth.

Horizontal shear $=1780$ lbs. per inch of length.

The load from the ties must also be communicated to the web through the rivets.

The amount of this is $7333\times\frac{1}{13}=564$ lbs. per lineal inch.

Thus the total stress per lineal inch $=\sqrt{1780^2+564^2}=1867$ lbs.

Now the value of a $\frac{7}{8}$-in. rivet in bearing on a $\frac{3}{8}$-in. plate is

$$20000\times\tfrac{7}{8}\times\tfrac{3}{8}=6562 \text{ lbs.},$$

which is less than the value in double shear.

Therefore the maximum pitch $=\frac{6562}{1867}=3.5$ ins. We shall, however, space them with a 3-in. pitch, as shown in Fig. 513. Towards the centre of the span the shear decreases, but in this case we may retain the same spacing throughout.

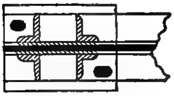
The flanges need not be spliced, as sufficiently long angles are obtainable. The web, however, must have a splice at mid-span, which should contain enough rivets to develope the strength of its entire net section. Two splice-plates (*A*) $13\frac{1}{2}''\times\frac{3}{8}''\times 54''$, one on each side of the web, with two vertical rows of rivets on each side of the splice, Fig. 510, will be used. Also, since these plates do not extend to the edge of the web and are therefore ill-suited to resist bending, we shall use two plates (*B*) $5''\times\frac{5}{8}''\times 36''$ on each flange, whose net section in accordance with good practice is 50 per cent greater than the web section counted as assisting the flange.

Web stiffeners cannot be designed rationally. It will be in keeping with good practice, however, to use $4''\times 3\frac{1}{2}''\times\frac{3}{8}''$ angles riveted in pairs to each side of the web at intervals not exceeding the depth of the girder, or say 5 ft.

Wind and lateral bracing for spans so small need not be calculated, as the smallest angles permitted by most standard specifications will be sufficient. We may use 3½″×3½″×⅜″ angles throughout.

Bearing Plates.—The girder will be about 52 ft. long over all, so that the weight supported at each end will be 4700×26 = 122,220 lbs.

Taking 300 lbs. per square inch as the safe load on good granitoid or limestone, the area required will be 122,200÷300 = 407 sq. ins. We shall use for the fixed end two 24″×18″×¾″ plates riveted to the girder and for the sliding end one 24″×18″×1″ plate riveted to the girder and a similar ½-in. plate on the masonry, the contiguous surfaces being planed so as to slide readily with changes of temperature.



Plan of End Stiffeners and Bearing

FIG. 514.

To distribute the load evenly, end stiffeners of sufficient section to carry the above load as a column will be placed as shown in Fig. 514.

20. Relations between the Deflection, Slope, and B.M. Curves.—Consider an element $KLL'K'$ of thickness dx and bounded by the vertical planes KL and $K'L'$. It is kept in equilibrium by S and M, the S.F. and B.M. at the section KL, by $S+dS$ and $M+dM$, the S.F. and B.M. at the section $K'L'$, and by the load wdx (which includes the weight of the element), w being the intensity of the load at K. Since dx is indefinitely small, it may be assumed that the load wdx is uniformly distributed and that its line of action is the middle line vv.

Taking moments about v,

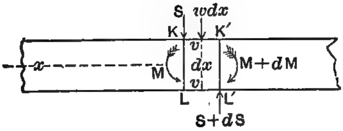


FIG. 515.

$$-M-S\frac{dx}{2}+M+dM-(S+dS)\frac{dx}{2}=0,$$

or

$$\frac{dM}{dx}=S,$$

disregarding the term $dS\frac{dx}{2}$, which is indefinitely small compared with the remaining terms.

Also,

$$-S-wdx+S+dS=0,$$

or

$$\frac{dS}{dx}=w.$$

Therefore

$$\frac{d^2M}{dx^2}=\frac{dS}{dx}=w. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

If the beam is loaded with a number of weights concentrated at different points, then *between any two consecutive weights*, w is merely the weight of the beam per lineal unit of length, and, as far as the concentrated loads are concerned, $\frac{dS}{dx}$ is *nil*, so that S is constant between such weights.

Again, the deflection (y), the slope $\left(\theta = \tan\theta = \frac{dy}{dx}\right)$, and the bending moment (M) are connected by the equations

$$\mathbf{\frac{d^2y}{dx^2} = \frac{d\theta}{dx} = \frac{M}{EI}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (B)$$

Comparing equations (A) and (B) it is at once observed that the values of y, θ, and $\frac{M}{EI}$ are obtained from one another by a process of graphical integration precisely similar to that by which the relative values of M, S, and w are found. Thus any property connecting the last three quantities must also be true for the first three. In other words, the mutual relations between curves drawn to represent the *deflection, slope,* and *bending moment* must be the same, *mutatis mutandis,* as those between the curves of bending moment, shearing force, and load.

Thus, divide the *effective* bending-moment area into a number of elementary areas, Fig. 516, by drawing vertical lines at convenient distances apart, and take the vertical lines 12, 23, . . . $n-1, n$, Fig. 517, to represent these areas.

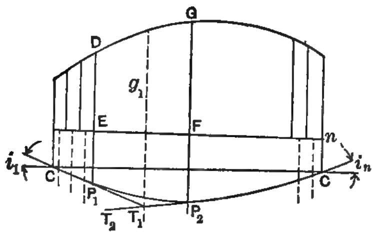


FIG. 516.

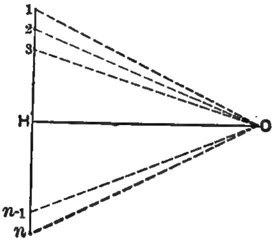


FIG. 517.

Let the pole O be at the distance EI from the line $1n$.

If the widths of the areas are made sufficiently small a funicular deflection curve CP_1P_2C can be drawn in exactly the same manner as the B.M. curve was drawn in Art. 6, Chap. II.

It was shown that any two tangents to the B.M. curve intersect in a point which is vertically below the C. of G. of the corresponding load curve. Hence it must follow that *any two tangents* P_1T_1, P_2T_2, *intersect in a point* T_1 *which is vertically below the* C. of G., g_1, *of the effective* B.M. *area* $DEFG$.

It was also shown that the B.M. at any point is the intercept of the vertical through the point between the closing line CC and the B.M. curve. Hence it follows that the deflection at any point is the intercept between the closing-line and the deflection curve, of the vertical through that point.

Again, through the pole O draw OH parallel to the closing-line intersecting $1n$ in H. Then, since $EI\theta=\int M dx$, the angle $P_1T_1T_2$ between any two tangents is equal to the corresponding B.M. area divided by EI. It is also evident that the

$$\text{slope } i_1 \text{ at } 1=\frac{1H}{EI},$$

and that the

$$\text{slope } i_n \text{ at } n=\frac{nH}{EI}.$$

Therefore

$$i_1+i_n=\frac{1n}{EI},$$

which gives the total change of slope.

In the case of a cantilever, the last side of the funicular polygon is obviously the closing-line.

21. Graphical Determination of the Slope and Deflection.—If ρ is the radius of curvature at any point of the deflected neutral axis,

$$\frac{1}{\rho}=\frac{d^2y}{dx^2}=\frac{d\theta}{dx}=\frac{M}{EI}.$$

Therefore

$$\int\frac{dx}{\rho}=\frac{dy}{dx}=\theta=\int\frac{Mdx}{EI}$$

and $$\int\int\frac{dx^2}{\rho}=y=\int\theta dx=\int\int\frac{M\ dx^2}{EI}.$$

Thus it may be supposed that the cantilever or beam carries a uniformly distributed load of intensity $\frac{1}{\rho}$, and by the process of integration the curves of slope and deflection can be obtained.

Let A be the effective B.M. area between any two points P and Q of the deflected neutral axis.

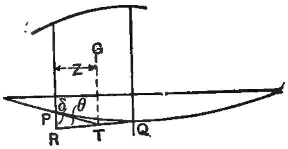


FIG. 518.

Let the tangents at P and Q meet in the point T and produce QT to meet the vertical through P in R.

The point T is vertically below the C. of G. of the area A. Let z be its horizontal distance from PR, and take $PR=\delta$. If θ is the change of curvature between P and Q, i.e., the angle PTR, then

$$\theta=\int\frac{\mathrm{M dx}}{\mathrm{EI}}=\frac{\mathrm{A}}{\mathrm{EI}},$$

assuming that E and I are constant.

Also, $$\delta=PR=\theta \mathrm{z}=\frac{\mathrm{Az}}{\mathrm{EI}}.$$

Ex. 45. *A cantilever OA of length l with a weight W at A.*

The B.M. area is the triangle OBC (Fig. 519), and $OC=Wl$.

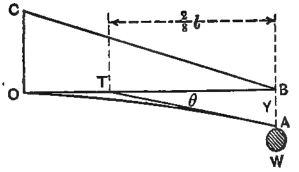


FIG. 519.

Therefore $$A=\tfrac{1}{2}Wl^2,$$

and the slope at A $$=\theta=\frac{1}{2}\frac{Wl^2}{EI}.$$

Also, $$Az=\tfrac{1}{2}Wl^2\cdot\tfrac{2}{3}l=\tfrac{1}{3}Wl^3$$

and the deflection of A $$=\mathrm{Y}=\frac{1}{3}\frac{\mathrm{Wl^3}}{\mathrm{EI}}.$$

Ex. 46. *A cantilever OA of length l, carrying a uniformly distributed load of intensity w.*

The B.M. area is OBC, the curve BC being a parabola with its vertex at B.

Also, $$OC=\frac{wl^2}{2}.$$

Therefore $$A=\frac{1}{3}\frac{wl^2}{2}l=\frac{1}{6}wl^3,$$

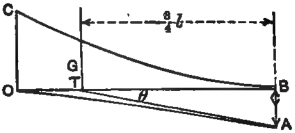


FIG. 520.

and the slope at A $$=\theta=\frac{1}{6}\frac{wl^3}{EI}.$$

Again, $$Az=\frac{1}{6}wl^3\frac{3}{4}l=\frac{1}{8}wl^4,$$

and the deflection of A $$=\mathrm{Y}=\frac{1}{8}\frac{\mathrm{wl}^4}{\mathrm{EI}}.$$

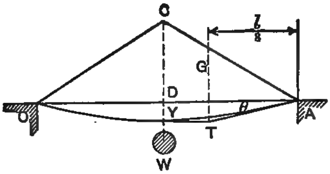


FIG. 521.

Ex. 47. *A horizontal beam OA of length l, resting upon supports at O and A and carrying a weight W at its middle point B.*

The B.M. area is the triangle OCA, the vertical distance of C above D, the middle point of the horizontal line OA being $\frac{Wl}{4}$.

Considering one half of the beam,

$$A=\frac{1}{2}\frac{Wl}{4}\frac{l}{2}=\frac{Wl^2}{16},$$

and the slope at A $$=\theta=\frac{1}{16}\frac{Wl^2}{EI}.$$

Also, $$Az=\frac{1}{16}Wl^2\frac{2}{3}\frac{l}{2}=\frac{1}{48}Wl^3,$$

and the maximum deflection

$$=\mathrm{Y}=\frac{1}{48}\frac{\mathrm{wl}^3}{\mathrm{EI}}.$$

Ex. 48. *A horizontal beam OA of length l, resting upon supports at O and A, and carrying a weight W at a point distant a from each end.*

Let b be the length of the intermediate segment.

The B.M. area is the trapezoid $OCDA$, the vertical distance CE $(=DF)$ being Wa.

Considering one half of the beam,

$$A=\tfrac{1}{2}Wa^2+\tfrac{1}{2}Wab=\frac{Wa}{2}(a+b),$$

and $$\text{the slope at } O=\theta=\frac{Wa}{2EI}(a+b).$$

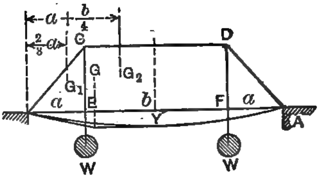


FIG. 522.

Also $$Az=\tfrac{1}{2}Wa^2\tfrac{2}{3}a+\tfrac{1}{2}Wab\left(a+\frac{b}{4}\right)$$

$$=\frac{Wa}{24}(8a^2+12ab+3b^2),$$

and the max. deflection

$$=Y=\frac{1}{24}\frac{Wa}{EI}(8a^2+12ab+3b^2).$$

If the two weights are concentrated at the points of trisection,

$$a=b \quad \text{and the max. deflection}=\frac{23}{24}\frac{Wa^3}{EI}$$

$$=\frac{Wa^3}{EI}, \text{ approximately.}$$

Ex. 49. *A horizontal beam OA of length l, resting upon supports at O and A, and carrying a weight W concentrated at a point B distant a from O.*

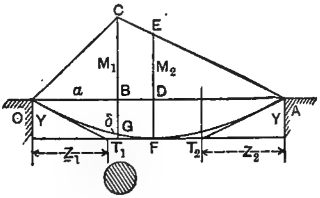


FIG. 523.

The B.M. area is the triangle OCA, the vertical distance CB being $\frac{Wa(l-a)}{2}=M_1$, suppose. The most deflected point F is at some point between B and A.

Let the horizontal tangent at F meet the tangents at O and A in T_1 and T_2 respectively, and let z_1, z_2 be the horizontal distances of T_1 and T_2 from O and A respectively.

Let M_2 be the length of the B.M. ordinate ED, vertically above F. The length of AD is evidently $\frac{3}{2}z_2$.

Consider first the portion of the beam between O and F, and let A_1 be the B.M. area $ACED$. Then, since the horizontal distance of the centre of gravity of the triangle OCA from O is $\frac{l+a}{3}$,

$$A_1z_1=\frac{1}{2}M_1l\frac{l+a}{3}-\tfrac{3}{2}M_2\tfrac{1}{6}z_2(l-\tfrac{3}{2}z_2+\tfrac{1}{2}z_2)$$

$$=\tfrac{1}{6}M_1l(l+a)-\tfrac{3}{4}M_2z_2(l-z_2)=EIY.$$

Consider in the second place the portion of the beam between A and D. Then

$$\tfrac{1}{2}M_2\tfrac{3}{2}z_2\cdot z_2=\tfrac{3}{4}M_2z_2^2=EIY.$$

Therefore $$\tfrac{1}{6}M_1l(l+a)-\tfrac{3}{4}M_2z_2(l-z_2)=\tfrac{3}{4}M_2z_2^2$$

and
$$\frac{9}{2}\frac{z_1}{l+a}=\frac{M}{M_2}=\frac{l-a}{\frac{3}{2}z_2},$$

or
$$z_2{}^2=\tfrac{4}{27}(l^2-a^2).$$

Also,
$$M_2=\frac{3}{2}M_1\frac{z_2}{l-a}=\frac{3}{2}\;\frac{Wa(l-a)}{l}\;\frac{\sqrt{\frac{4}{27}(l^2-a^2)}}{l-a}=\frac{Wa}{l}\sqrt{\frac{l^2-a^2}{3}}.$$

Hence
$$\text{the max. deflection}=Y=\frac{3}{4}\frac{M_2z_2{}^2}{EI}=\frac{1}{9\sqrt{3}}\frac{Wa}{l}(l^2-a^2)^{\frac{3}{2}}.$$

Again let δ be the vertical distance between G and F, and let z' be the horizontal distance of the centre of gravity of the area $BCED(=A')$ from G.

Then
$$EI\delta=A'z'=\tfrac{1}{2}M_1(l-a)\tfrac{1}{3}(l-a)-\tfrac{1}{2}M_2\tfrac{3}{2}z_2(\tfrac{1}{2}z_2+l-a-\tfrac{3}{2}z_2)$$
$$=\frac{M_1}{6}(l-a)^2-\tfrac{3}{4}M_2z_2(l-a-z_2)$$
$$=\frac{M_1}{6}(l-a)^2-\frac{M_1}{6}(l^2-a^2)+EIY.$$

Therefore
$$EI(Y-\delta)=\frac{M_1}{3}a(l-a)=\frac{1}{3}\;\frac{W}{l}a^2(l-a)^2,$$

which gives the deflection of the point at which W is concentrated.

Again,
$$A_2=\frac{3}{4}M_2z_2=\frac{9}{8}M_1\frac{z_2{}^2}{l-a}=\frac{1}{6}W\frac{a}{l}(l^2-a^2),$$
$$A_1=\frac{1}{2}W\frac{a(l-a)}{l}-A_2=\frac{1}{6}W\frac{a}{l}(l-a)(2l-a),$$

and
$$A'=\frac{1}{2}M_1\overline{l-a}-A_2=\frac{Wa}{l}(l-a)(l-2a).$$

Hence
$$\text{the slope at } O=\frac{1}{6}\frac{W}{EI}\frac{a}{l}(l-a)(2l-a)$$
$$\text{'' '' '' } G=\frac{W}{EI}\frac{a}{l}(l-a)(l-2a)$$
$$\text{'' '' '' } A=\frac{1}{6}\frac{W}{EI}\frac{a}{l}(l^2-a^2).$$

Ex. 50. *A beam OA of length l, supported at O and at A, and carrying a uniformly distributed load of intensity w.*

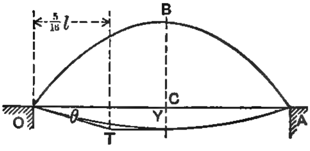


FIG. 524.

The B.M. area is OBA, the curve OBA being a parabola with its vertex at a distance $\frac{wl^2}{8}$ vertically above C, the middle point of OA. Considering one half of the beam,

$$A=\frac{2}{3}\frac{wl^2}{8}\frac{l}{2}=\frac{1}{24}wl^3 \quad\text{and}\quad z=\frac{5}{16}l.$$

Therefore $$\text{the slope at } O \text{ (or at } A)=\frac{1}{24}\frac{wl^3}{EI}$$

and $$\text{the max. deflection } Y=\frac{\frac{1}{24}\cdot wl^3\cdot\frac{5}{16}l}{EI}=\frac{5}{384}\frac{wl^4}{EI}.$$

Ex. 51. *A horizontal beam OA of length l fixed at the supports O and A and carrying a weight W concentrated at the middle point B.*

The fixture of the ends introduces negative bending moments at O and at A, and the B.M. diagram consists of the two lines BC, DC, the point C being vertically above the middle point G of OA. At the points E and F the B.M. is *nil*, and the beam may be supposed to be made up of two cantilevers EO, FA, and of an intermediate span EF resting upon the ends E and F.

Thus if $OE=x$,

$$OB=AD=\tfrac{1}{2}Wx.$$

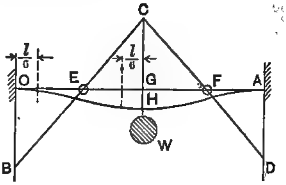


FIG. 525.

Again, since the neutral axis is horizontal at O and at A, the total change of curvature between O and A is *algebraically nil*, and therefore the total effective B.M. area must also be *algebraically nil*.

Hence the B.M. area above OA = B.M. area below OA,

or the triangle CEG = the triangle OBE,

and E is the middle point of OG; so that

$$x=\frac{l}{4} \quad\text{and}\quad OB(=AD=CG)=\tfrac{1}{8}Wl.$$

The B.M. area $OBE=\frac{1}{64}Wl^2$ and the corresponding $z=\frac{l}{6}$. Therefore

$$\text{the slope at } E=\frac{1}{64}\cdot\frac{Wl^2}{EI},$$

and
$$\text{the deflection of } E=\frac{1}{384}\cdot\frac{Wl^3}{EI}.$$

The B.M. area $ECG=\frac{1}{64}Wl^2$ and the corresponding $z=\frac{l}{6}$. Therefore

$$\text{the deflection } GH=\frac{1}{384}\frac{Wl^3}{EI}$$

and
$$\text{the \textit{total} maximum deflection}=\text{deflection of } E+GH=\frac{1}{192}\frac{Wl^3}{EI}.$$

The points E and F at which the B.M. is *nil* and at which the curvature necessarily changes from positive to negative are called *points of inflection.*

Ex. 52. *A horizontal beam OA of length l, fixed at the supports O and A and carrying a uniformly distributed load of intensity w.*

The B.M. diagram is now the parabola $BECFD$, with its vertex at C vertically above the middle point G.

The points E and F are evidently *points of inflection.*

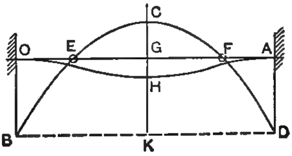


FIG. 526.

Let $EF=2x$.

Then
$$OE=\frac{l}{2}-x,$$

$$\text{the B.M. } CG=\frac{wx^2}{2},$$

and
$$\text{the B.M. } OB(=AD)=wx\left(\frac{l}{2}-x\right)+\frac{w}{2}\left(\frac{l}{2}-x\right)^2=\frac{w}{2}\left(\frac{l^2}{4}-x^2\right).$$

Also, as in the preceding example,

$$\text{the area } ECG=\text{the area } OBE.$$

and therefore
$$\text{the area } BECK=\text{the area } OGKB,$$

or
$$\frac{2}{3}\left\{\frac{wx^2}{2}+\frac{w}{2}\left(\frac{l^2}{4}-x^2\right)\right\}\frac{l}{2}=\frac{w}{2}\left(\frac{l^2}{4}-x^2\right)\frac{l}{2},$$

$$\text{so that } x=\frac{\sqrt{3}}{6}l=l\times.289.$$

The total max. deflection = deflection of E due to $w\left(\frac{l}{2}-x\right)$ uniformly distributed

$+$ “ “ “ wx at end of OE

$+GH$

$$=\frac{1}{8}\frac{w\left(\frac{l}{2}-x\right)^4}{EI}+\frac{1}{3}\frac{wx\left(\frac{l}{2}-x\right)^3}{EI}+\frac{5}{384}\frac{w(2x)^4}{EI}$$

$$=\frac{wl}{384EI}(3l^3-8l^2x-24lx^2+96x^3)$$

$$=\frac{1}{384}\frac{wl^4}{EI}.$$

22. Beams Supported at More Points Than Two.—Consider the rth span OX of a girder resting upon a number of supports in the same horizontal plane.

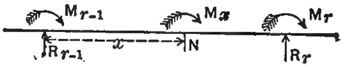


FIG. 527.

Let R_{r-1}, R_r be the reactions at O and at X;
M_{r-1}, M_r be the bending moments at O and at X;
M_x be the B.M. at any point N (i.e., x, y) of the neutral axis *due to the load on ON*;
M “ “ B.M. at X *due to the load on OX.*

Taking moments about X,

$$R_{r-1}l_r=M+M_r-M_{r-1},$$

and therefore
$$R_{r-1}=\frac{M}{l_r}+\frac{M_r}{l_r}-\frac{M_{r-1}}{l_r}.$$

Hence the shear at the $(r-1)$th support for the rth span
= the reaction at the same support, supposing the span an independent girder, i.e., cut at its supports,
+ the difference of the forces, or reactions, equivalent to the moments at the supports.

Again, the B.M. at N

$$=R_{r-1}x-M_x+M_{r-1}$$

$$=\left(\frac{M}{l_r}x-M_x\right)+\frac{M_{r-1}}{l_r}(l_r-x)+\frac{M_r}{l_r}x$$

= the moment at the same point supposing the span an independent girder.

+ the reactions equivalent to the moments M_{r-1}, M_r, multiplied respectively by the segments l_r-x and x.

Let OX, Fig. 528, be the rth span, and let OBX be the curve of bending moments, *supposing OX an independent girder,* i.e., cut at O and X. On the same scale as this curve is drawn, take the verticals OE and XF to represent M_{r-1} and M_r respectively in magnitude, and join EF. The curve OBX corresponds to the portion $\left(\frac{M}{l_r}x-M_x\right)$ of the above equation, and the line EF to the remainder, i.e., $\frac{M_{r-1}}{l_r}(l_r-x)+\frac{M_r}{l_r}x$.

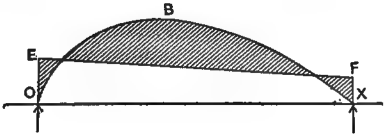


FIG. 528.

The actual bending moment at any point of OX is represented by the *algebraic* sum of the ordinates of the curve and line at the same point, which will be the intercept between them, since they represent bending moments of *opposite* kinds.

Let A be the *effective* moment area, or the algebraic sum of the areas for the load and for the moments at O and X, shown shaded in the figure, and let $\bar{x}$ be the horizontal distance of its centre of gravity from O.

Let A_r be the area for the load alone, i.e., the area of the curve OBX, and let z_r be the horizontal distance of its centre of gravity from O. Then

$$A\bar{x}=A_rz_r+M_{r-1}\frac{l_r^2}{2}+\frac{1}{3}(M_r-M_{r-1})l_r^2$$

$$=A_rz_r+\tfrac{1}{6}M_{r-1}l_r^2+\tfrac{1}{3}M_rl_r^2.$$

Ex. 53. *Let the load upon the rth span OX be a weight P concentrated at a point C distant p from O.*

The B.M. diagram consists of the two straight lines OB, XB, if it is assumed that OX is an independent girder. Also the vertical distance $BC = \frac{Pp(l_r - p)}{l_r}$.

Therefore $\quad A_r = \text{area } OBX = \frac{l_r}{2}\,\frac{Pp(l_r - p)}{l_r} = \frac{1}{2}Pp(l_r - p).$

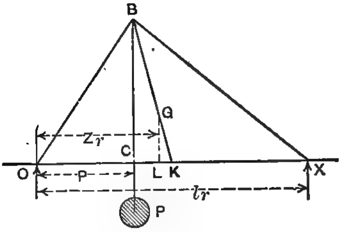


FIG. 529.

Again, join B to the middle point K of OX, and let GL be the vertical through the centre of gravity of the triangle. Then

$$z_r = OL = OC + CL = OC + \tfrac{2}{3}CK$$

$$= OC + \tfrac{2}{3}(OK - OC) = \frac{l_r + p}{3}.$$

Hence $\quad A_r z_r = \frac{1}{6}Pp(l_r^2 - p^2).$

Similarly, for the $(r+1)$th span,

$$A_{r+1}z_{r+1} = \tfrac{1}{6}Qq(l_{r+1}^2 - q^2),$$

z_{r+1} being the horizontal distance of the centre of gravity of A_{r+1}, the moment area of a load Q concentrated on the $(r+1)$th span at a distance q from the $(r+1)$th support, the span being considered an independent girder.

Ex. 54. *Let the load upon the rth span OX be a uniformly distributed load of intensity* w_r.

The B.M. diagram is now the parabola OBX, with its vertex B, vertically above the middle point C, Fig. 530. Then

$BC = \frac{w_r l_r^2}{8}.$

Therefore

$$A_r = \text{area } OBX = \frac{2}{3}\,\frac{w_r l_r^2}{8} l_r = \frac{1}{12} w_r l_r^3.$$

Also, $\quad z_r = \frac{l_r}{2}.$

Hence $\quad A_r z_r = \frac{1}{24} w_r l_r^4.$

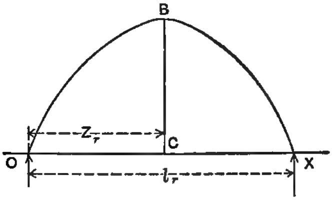


FIG. 530.

Similarly, for the $(r+1)$th span,

$$A_{r+1}z_{r+1} = \tfrac{1}{24} w_{r+1} l_{r+1}.$$

Ex. 55. *Let a uniformly distributed load of intensity* w_r *cover a length* $2a(<l_r)$ *of the rth span, and let* z *be the distance of its centre of gravity from the* $(r-1)$*th support.*

The load may be supposed to consist of a number of indefinitely small elements, and its effect may be obtained by superposing the several effects of these elements.

Let dp be the length of such an element at p from the $(r-1)$th support. Then, by Ex. 53,

$$A_r z_r = \int_{z-a}^{z+a} \frac{w_r dp \cdot p}{l_r}(l_r^2 - p^2) = 2aw_r \frac{z}{l_r}(l_r^2 - z^2 - a^2).$$

23. Continuous Girders.—When a girder overhangs its bearings, or is supported at more than two points, it assumes a wavy form and is said to be *continuous*. The convex portions are in the same condition as a loaded girder resting upon a single support, the upper layers of the girder being extended and the lower compressed. The concave portions are in the same condition as a loaded girder supported at two points, the upper layers being compressed and the lower extended. At certain points, called *points of contrary flexure*, or *points of inflexion*, the curvature changes sign and the flange stresses are necessarily zero. Hence, apart from other practical considerations, the flanges might be wholly severed at these points without endangering the stability of the girder.

24. Theorem of Three Moments.—Let O, X, V, the $(r-1)$th, rth, and $(r+1)$th supports of a continuous girder of several spans, be depressed below their true horizontal position $O_1O_2O_3$, through the vertical distances $d_{r-1}(=O_1O)$, $d_r(=O_2X)$, and $d_{r+1}(=O_3V)$, d_{r-1}, d_r, and d_{r+1} being necessarily *very small quantities*.

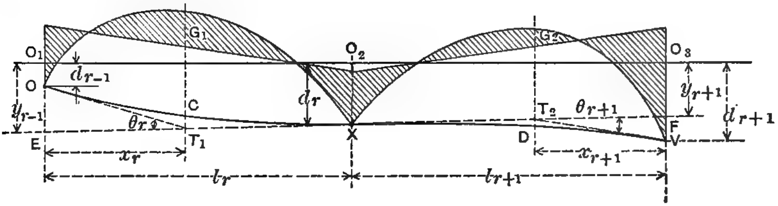


FIG. 531.

Let $OCXDV$ be the deflection curve, and let the tangent at X meet the verticals O_1O and O_3V in E and F, and the tangents at O and V in T_1 and T_2.

Take $O_1E=y_{r-1}$, $O_3F=y_{r+1}$ and let $\theta_r(=OT_1E)$, $\theta_{r+1}(=FT_2V)$ be the changes of curvature from O to X and from V to X respectively.

Let A_r, A_{r+1} be the *effective moment areas* for the spans OX, XV respectively.

Let x_r, x_{r+1} be the horizontal distances from O and V respectively of the points T_1 and T_2, which, as already proved, are vertically below the centres of gravity of the corresponding effective moment areas. Then

$$\frac{y_{r-1}-d_r}{l_r}=\frac{d_r-y_{r+1}}{l_{r+1}} \quad\text{or}\quad \frac{y_{r-1}}{l_r}+\frac{y_{r+1}}{l_{r+1}}=\frac{d_r}{l_r}+\frac{d_r}{l_{r+1}}.$$

Again, as already shown,

$$y_{r-1}-d_{r-1}=OE=x_r\theta_r=\frac{A_r}{EI}x_r$$

and $$d_{r+1}-y_{r+1}=FV=x_{r+1}\theta_{r+1}=-\frac{A_{r+1}}{EI}x_{r+1}.$$

Therefore

$$\frac{1}{EI}\left(\frac{A_rx_r}{l_r}+\frac{A_{r+1}x_{r+1}}{l_{r+1}}\right)=\frac{y_{r-1}}{l_r}+\frac{y_{r+1}}{l_{r+1}}-\frac{d_{r-1}}{l_r}-\frac{d_{r+1}}{l_{r+1}}$$

$$=\frac{d_r-d_{r-1}}{l_r}+\frac{d_r-d_{r+1}}{l_{r+1}}.$$

Hence by Art. 22

$$EI\left(\frac{d_r-d_{r-1}}{l_r}+\frac{d_r-d_{r+1}}{l_{r+1}}\right)=A_r\frac{z_r}{l_r}+\tfrac{1}{6}M_{r-1}l_r+\tfrac{1}{3}M_rl_r$$

$$+A_{r+1}\frac{z_{r+1}}{l_{r+1}}+\tfrac{1}{6}M_{r+1}l_{r+1}+\tfrac{1}{3}M_rl_{r+1},$$

and therefore

$$\mathbf{M}_{r-1}\mathbf{l}_r+2\mathbf{M}_r(\mathbf{l}_r+\mathbf{l}_{r+1})+\mathbf{M}_{r+1}\mathbf{l}_{r+1}$$

$$=-6\mathbf{A}_r\frac{\mathbf{z}_r}{\mathbf{l}_r}-6\mathbf{A}_{r+1}\frac{\mathbf{z}_{r+1}}{\mathbf{l}_{r+1}}+6\mathbf{EI}\left(\frac{\mathbf{d}_r-\mathbf{d}_{r+1}}{\mathbf{l}_r}+\frac{\mathbf{d}_r-\mathbf{d}_{r+1}}{\mathbf{l}_{r+1}}\right),$$

which is the analytical expression of the theorem of three moments in its most general form.

Exs. 53 and 54 give the values of A_rz_r and $A_{r+1}z_{r+1}$ for concentrated and for uniformly distributed loads.

If either of the supports, e.g. X, should lie *above* O_1O_3, then the sign of d_r is *negative.*

A similar relation holds true for every three consecutive supports, so that for a continuous girder of n spans there are $n-1$ equations connecting the $n+1$ bending moments M_1, M_2, M_3 . . . M_n, M_{n+1} at the several supports.

There must be *two* further conditions before these equations can be solved, and they are usually provided by the method adopted for carrying the ends of the girder.

If the ends rest freely on the supports,

$$\mathbf{M}_1 = 0 \quad \text{and} \quad \mathbf{M}_{n+1} = 0;$$

If the girder is fixed at 1 *and rests freely at* $n+1$,

$$2\mathbf{M}_1 + \mathbf{M}_2 = -\frac{w_1 l_1^2}{4} \quad \text{or} \quad = -\Sigma P\frac{p}{l_1^2}(l_1 - p)(2l_1 - p);$$

according as the load upon the first span (l_1) is uniformly distributed and of intensity w_1, or consists of a number of weights P_1, P_2, P_3 . . . concentrated at points distant p_1, p_2, p_3 . . . , respectively, *from the fixed end,* the symbol Σ denoting algebraic sum.

Also, $$\mathbf{M}_{n+1} = 0.$$

If both ends are fixed,

$$2\mathbf{M}_1 + \mathbf{M}_2 = -\frac{w_1 l_1^2}{4} \quad \text{or} \quad = -\Sigma P\frac{p}{l_1^2}(l_1 - p)(2l_1 - p),$$

and $$2\mathbf{M}_{n+1} + \mathbf{M}_n = -\frac{w_n l_n^2}{4} \quad \text{or} \quad = -\Sigma Q\frac{q}{l_n^2}(l_n - q)(2l_n - q),$$

w_n being the intensity of the uniformly distributed load on the nth span, and q the distance of the concentrated load Q from the $(n+1)$th support. These *fixture conditions* can easily be proved as follows:

It is assumed that the supports at 1 and 2 are in the same horizontal plane.

First. Let w_1 be the intensity of the load uniformly distributed over the first span (l_1), and let R_1 be the vertical reaction at 1.

At any point (x, y),

$$-EI\frac{d^2y}{dx^2} = R_1 x - \frac{w_1 x^2}{2} + M_1.$$

Integrating twice, and remembering that $\frac{dy}{dx}$ and y are each zero when $x=0$,

$$-EI\frac{dy}{dx}=R_1\frac{x^2}{2}-\frac{w_1x^3}{6}+M_1x$$

$$-EIy=R_1\frac{x^3}{6}-\frac{w_1x^4}{24}+M_1\frac{x^2}{2}.$$

Also, $y=0$ when $x=l_1$. Therefore

$$0=R_1\frac{l_1^3}{6}-\frac{w_1l_1^4}{24}+M_1\frac{l_1^2}{2}.$$

or

$$R_1l_1+3M_1=\frac{w_1l_1^2}{4}.$$

Taking moments about 2,

$$R_1l_1+M_1-M_2=\frac{w_1l_1^2}{2}.$$

Hence

$$\mathbf{2M_1+M_2=-\frac{w_1l_1^2}{4}}.$$

Second. Let a weight P be concentrated at B distant p from 1.
From 1 *to B.* At any point (x, y) in $1B$,

$$-EI\frac{d^2y}{dx^2}=R_1x+M_1.$$

Integrating twice and remembering that $\frac{dy}{dx}$ and y are each zero when $x=0$,

$$-EI\frac{dy}{dx}=R_1\frac{x^2}{2}+M_1x$$

$$-EIy=R_1\frac{x^3}{6}+M_1\frac{x^2}{2}.$$

If θ_B and y_B are the values of $\frac{dy}{dx}$ and y at B, i.e., when $x=p$,

$$-EI\theta_B=R_1\frac{p^2}{2}+M_1p$$

$$-EIy_B=R_1\frac{p^3}{6}+M_1\frac{p^2}{2}.$$

From B to 2. At any point (x, y) in $B2$,

$$-EI\frac{d^2y}{dx^2}=R_1x-P(x-p)+M_1.$$

Integrating, $$-EI\frac{dy}{dx}=R_1\frac{x^2}{2}-\frac{P}{2}(x-p)^2+M_1x+c_1.$$

At B, $$-EI\theta_B=R_1\frac{p^2}{2}+M_1p+c_1,$$

and therefore $c_1=0$, so that the slope equation becomes

$$-EI\frac{dy}{dx}=R_1\frac{x^2}{2}-\frac{P}{2}(x-p)^2+M_1x.$$

Integrating, $$-EIy=R_1\frac{x^3}{6}-\frac{P}{6}(x-p)^3+M_1\frac{x^2}{2}+c_2.$$

At B, $$-EIy_B=R_1\frac{p^3}{6}+M_1\frac{p^2}{2}+c_2,$$

and therefore $c_2=0$, so that the deflection equation becomes

$$-EIy=R_1\frac{x^3}{6}-\frac{P}{6}(x-p)^3+M_1\frac{x^2}{2}.$$

But y is also at zero at 2, i.e., when $x=l_1$. Therefore

$$0=R_1\frac{l_1^3}{6}-\frac{P}{6}(l_1-p)^3+M_1\frac{l_1^2}{2},$$

or $$R_1l_1+3M_1=\frac{P}{l_1^2}(l_1-p)^3.$$

Taking moments about 2,

$$R_1l_1+M_1-M_2=P(l_1-p).$$

Hence $$2M_1+M_2=-P\frac{p}{l_1^2}(l_1-p)(2l_1-p).$$

Ex. 56. (*a*) *The bridge over the Garonne at Langon carries a double track, is about* 695 *ft. in length, and consists of three spans, AB, BC, CD. The two main girders are continuous and rest upon the abutments at A and D and upon piers at B and C. The effective length of each of the spans AB, CD is* 208 *ft.* 6 *ins., and of the centre span BC* 243 *ft. The permanent load upon a main girder is* 1277 *lbs. per lineal foot, and the proof load is* 2688 *lbs. per lineal foot. Find the reactions at the supports* (1) *when the proof load covers the span AB;* (2) *when* *the proof load covers the span BC;* (3) *when the proof load covers the spans AB and BC;* (4) *when the proof load covers the whole girder.*

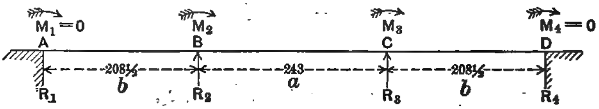


FIG. 532.

Take $BC=a=243'$, and $AB=208\frac{1}{2}'=b=CD$. Assume the B.Ms. at B and at C to be *right-handed*. The B.Ms. at A and at D are *nil*.

First. Let M_2', M_3' be the B.Ms. at B and C due to the dead load of 1277 lbs./lin. ft. Then

$$2M_2'(b+a)+M_3'a=-\frac{1277}{4}(a^3+b^3)$$

and

$$M_2'a+2M_3'(a+b)=-\frac{1277}{4}(a^3+b^3).$$

Therefore

$$M_2'=-\frac{1277}{4}\frac{a^3+b^3}{3a+2b}=-6522303 \text{ ft.-lbs.}=M_3'$$

$$=R_1'208\tfrac{1}{2}-\frac{1277}{2}(208\tfrac{1}{2}).$$

Hence

$$R_1'=101845 \text{ lbs.}=R_4'$$

and

$$R_2'=319565 \text{ ''} =R_3'.$$

Points of inflexion For AB (or CD) the point of inflection is at x feet from the end support, x being given by

$$\text{B.M.}=0=R_1'x-\frac{1277}{2}x^2,$$

or

$$x=\frac{2R_1'}{1277}=159'.5.$$

The *maximum positive B.M.* in AB (or CD) is at $79\frac{3}{4}$ ft. from the end support and its value is $\frac{1}{2}\frac{(R_1')^2}{1277}=4{,}061{,}240$ ft.-lbs.

For BC, measuring x from B, the

$$\text{B.M.}=0=R_1'(208\tfrac{1}{2}+x)+R_2'x-\frac{1277}{2}(208\tfrac{1}{2}+x)^2,$$

or

$$x^2-242.984x+10215.15=0 \quad \text{and} \quad x=54'.15 \quad \text{or} \quad 188'.83.$$

The maximum positive B.M. in BC is at $121\frac{1}{2}$ ft. from B (or C) and its value is 2,903,347 ft.-lbs.

If the *proof load of 2688 lbs. covers the whole girder* the B.Ms. and reactions are $\frac{3965}{1277}\left(=\frac{1277+2688}{1277}\right)$ times the corresponding reactions first obtained. Then,

$$R_1=\frac{3965}{1277}\times 101845=316223 \text{ lbs.}=R_4,$$

$$R_2=\frac{3965}{1277}\times 319565=992227 \text{ lbs.}=R_3,$$

$$M_2=\frac{3965}{1277}\times -6522303=-20251320 \text{ ft.-lbs.}=M_3.$$

The maximum *positive B.M.* in AB (or CD) $=\frac{3965}{1277}\times 4{,}061{,}240$ ft.-lbs.

$=12{,}609{,}900$ ft.-lbs.

The maximum *positive B.M.* on $BC=\frac{3965}{1277}\times 2903347=9{,}014{,}700$ ft.-lbs.

Fig. 533 shows the S.F. and B.M. diagrams for the two cases, the dotted lines being the diagrams when the proof load covers the whole girder.

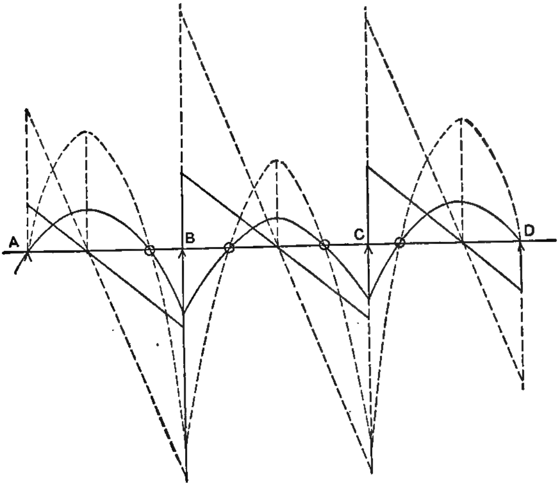


FIG. 533.

Second. Let M_2'', M_3''' be the bending moments due to the proof load of 2688 lbs./lin. ft. on the span AB. Then

$$2M_2''(a+b)+M_3'''a=-\frac{2688}{4}b^3$$

and $$M''_2 a + 2M_3''(a+b) = 0.$$

Therefore $$M_2'' = -1344\frac{b^3(a+b)}{3a^2+8ab+4b^2} = -7,271,879 \text{ ft.-lbs.}$$
$$= R_1'' \times 208.5 - 2688\frac{(208.5)^2}{2}.$$

and $$M_3'' = -M_2''\frac{a}{2(a+b)} = +1,956,885 \text{ ft.-lbs.}$$
$$= R_4'' \times 208.5.$$

Hence $R_1'' = 296340$ lbs. and $R_4'' = 9386$ lbs.

Again, $$R_4''(208.5+243) + R_3''243 = M_2'' = -7,271,879.$$

Therefore $$R_3'' = -47364 \text{ lbs.}$$

But $$R_1'' + R_2'' + R_3'' + R_4'' = 2688 \times 208.5 = 560,448 \text{ lbs.}$$

Hence $$R_2'' = 302,086 \text{ lbs.}$$

Superposing the results for the dead load and for the proof load on AB,

$R_1 = R_1' + R_1'' = 398,185$ lbs.; $R_3 = R_3' + R_3'' = 272,201$ lbs.;
$R_2 = R_2' + R_2'' = 621,651$ lbs.; $R_4 = R_4' + R_4'' = 111,231$ lbs.;
$M_2 = M_2' + M_2'' = -13,794,182$ ft.-lbs.; $M_3 = M_3' + M_3'' = -4,565,418$ ft.-lbs.

Points of inflexion. At such points the B.M. = 0.

For AB. $$\text{B.M.} = 0 = R_1 x - \frac{3965}{2}x^2$$

and $$x = 0 \quad \text{or} = \frac{2R_1}{3965} = 200'.85.$$

The maximum positive B.M. is at $\frac{R_1}{3965} = 100.42$ ft. from A and its value is

$$\frac{1}{2}\frac{R_1^2}{3965} = 19,993,810 \text{ ft.-lbs.}$$

For CD. Measuring x from D, the

$$\text{B.M.} = 0 = R_4 x - \frac{1277}{2}x^2$$

and $$x = 0 \quad \text{or} \quad = \frac{2R_4}{1277} = 174'.21.$$

The *maximum positive B.M.* in CD is at 87′.105 from D and its value is

$$\frac{1}{2}\frac{R_4^2}{1277}=4{,}844{,}300 \text{ ft.-lbs.}$$

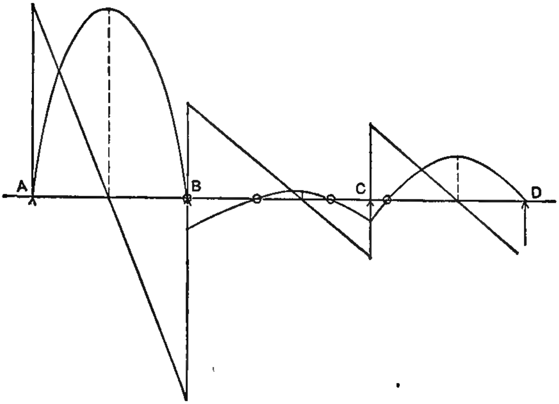


FIG. 534.

For BC. Measuring x from C, the

$$\text{B.M.}=0=R_4(208\tfrac{1}{2}+x)+R_3x-\frac{1277}{2}(x+208\tfrac{1}{2})^2,$$

or $$x^2-183.52x+7150.15=0,$$

and $$x=56'.13 \quad \text{or} \quad 127'.39.$$

The *maximum positive B.M.* is 91′.76 from C, and its value is

$$R_4\times 300.26+R_3\times 91.76-\frac{1277}{2}(300.26)^2=810{,}735 \text{ ft.-lbs.}$$

Fig. 534 shows the S.F. and B.M. diagrams for this case.

Third. Let M_2''', M_3''' be the B.Ms. due to the proof load of 2688 lbs./lin. ft. on the span BC. Then

$$R_1'''=R_4''', \quad R_2'''=R_3''', \quad M_2'''=M_3'''.$$

Therefore

$$M_2'''=-672\frac{a^3}{3a+2b}=M_3'''=-8414019 \text{ ft.-lbs.}=R_1'''\times 208\tfrac{1}{2}$$

and $$R_1'''=-40355 \text{ lbs.}=R_4'''.$$

Also, $$R_2'''=366947 \text{ lbs.}=R_3'''.$$

Superposing the results for the dead and proof loads,

$$R_1 = R_1' + R_1''' = 61490 \text{ lbs.} = R_4,$$

$$R_2 = R_2' + R_2''' = 686512 \text{ lbs.} = R_3.$$

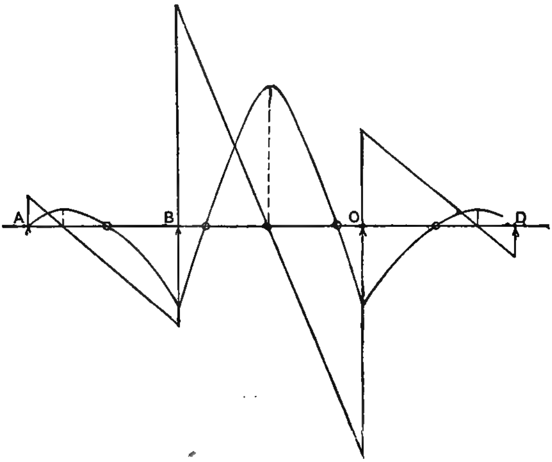


FIG. 535.

Points of inflexion. For AB (or CD) the point of inflection is at x feet from the end support, x being given by

$$\text{B.M.} = 0 = R_1 x - \frac{1277}{2} x^2, \quad \text{or} \quad x = \frac{2R_1}{1277} = 96.3 \text{ ft.}$$

The maximum positive B.M. in AB (or CD) is 48.15 ft. from the end support and its value is $\frac{1}{2}\frac{R_1^2}{1277} = 1,480,430$ ft.-lbs.

For BC. Measuring x from B, the

$$\text{B.M.} = 0 = R_1(208\tfrac{1}{2} + x) + R_2 x - 1277 \cdot 208\tfrac{1}{2}(104\tfrac{1}{4} + x) - \frac{3965}{2}x^2,$$

or $$x^2 - 243.05x + 7534.1 = 0,$$

and $$x = 38'.3 \quad \text{or} \quad 204'.7.$$

The *maximum positive B.M.* is 121½ ft. from B (or C), and its value is

$$R_1 \cdot 330 + R_2 \cdot 121\tfrac{1}{2} - 1277 \cdot 208\tfrac{1}{2} \cdot 225\tfrac{3}{4} - \frac{3965}{2}(121\tfrac{1}{2})^2 = 14,330,098 \text{ ft.-lbs.}$$

Fig. 535 shows the S.F. and B.M. diagrams for this case.

Fourth. Let M_2'''', M_3'''' be the B.Ms. due to the proof load of 2688 lbs./ lin. ft. on the two spans AB, BC.

$$M_2'''' = M_2' + M_2'' + M_2''' = -22{,}208{,}201 \text{ ft.-lbs.}$$

$$M_3'''' = M_3' + M_3'' + M_3''' = -12{,}979{,}437 \quad \text{"}$$

$R_1 = R_1' + R_1'' + R_1''' = 357{,}830$ lbs.; $R_3 = R_3' + R_3'' + R_3''' = 639{,}148$ lbs.

$R_2 = R_2' + R_2'' + R_2''' = 988{,}598$ " ; $R_4 = R_4' + R_4'' + R_4''' = 70{,}876$ "

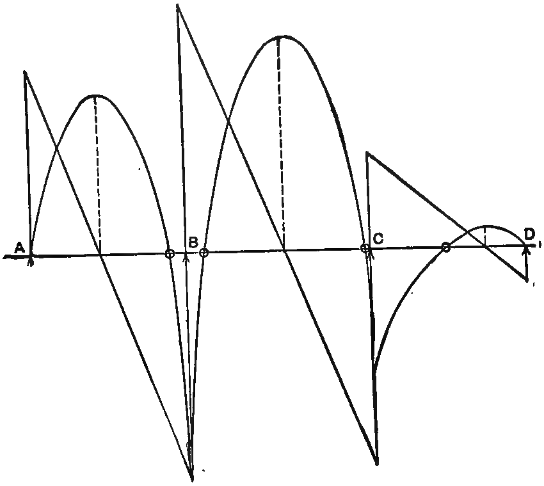


FIG. 536.

Points of inflexion. *For AB* the point of inflection is at x feet from A, x being given by

$$BM = 0 = R_1x - \frac{3965}{2}x^2, \quad \text{or} \quad x = \frac{2R_1}{3965} = 180.5 \text{ ft.}$$

The maximum positive B.M. is 90.25 ft. from A and its value is

$$\frac{1}{2}\frac{R_1^2}{3965} = 16{,}146{,}600 \text{ ft.-lbs.}$$

For CD the point of inflection is x feet from D, x being given by

$$\text{B.M.} = 0 = R_4x - \frac{1277}{2}x^2, \quad \text{or} \quad x = \frac{2R_4}{1277} = 111 \text{ ft.}$$

The maximum positive B.M. is 55.5 ft. from D and its value is

$$\frac{1}{2}\frac{R_4^2}{1277} = 1{,}966{,}880 \text{ ft.-lbs.}$$

For BC the point of inflexion is x feet from B, x being given by

$$\text{B.M.} = 0 = R_1(208.5+x) + R_2x - \frac{3965}{2}(208.5+x)^2,$$

or
$$x^2 - 262.156x = -5839.15,$$

and
$$x = 24.7 \text{ ft.} \quad \text{or} \quad = 237.65 \text{ ft.}$$

The maximum positive B.M. is at 131.15 ft. from B, and its value is

22,184,000 ft.-lbs.

Fig. 536 shows the S.F. and B.M. diagrams in this case.

Ex. 57. *The weights on five wheels passing over a continuous girder of two spans, each of* 50 *ft., taken in order, are as follows:* 15,000 *lbs.,* 24,000 *lbs.,* 24,000 *lbs.,* 24,000 *lbs.,* 24,000 *lbs. The distances of the wheels, centre to centre, taken in the same order, are* 90 *ins.,* 56 *ins.,* 56 *ins.,* 856 *ins. Let it be required to place the wheels in such a position as to give the maximum bending moment at the centre pier.*

The pier must evidently lie between the third and fourth wheels.

Let x be the distance in inches of the weight of 15,000 lbs. from the nearest abutment. The remaining two weights on the span are respectively $x+90$ ins. and $x+146$ ins. from the same abutment.

The two weights on the other span are $142-x$ ins. and $198-x$ ins. respectively from the nearest abutment.

Hence, if M is the bending moment at the centre pier,

$$\begin{aligned}-4M\times 600 = \frac{15000}{600}x(600^2-x^2) &+ \frac{24000}{600}(x+90)\{600^2-(x+90)^2\}\\ &+\frac{24000}{600}(x+146)\{600^2-(x+146)^2\}\\ &+\frac{24000}{600}\times(142-x)\{600^2-(142-x)^2\}\\ &+\frac{24000}{600}\times(198-x)\{600^2-(198-x)^2\},\end{aligned}$$

which reduces to

$$-M = \frac{x(600^2-x^2)}{96} + 3456000 - \frac{1}{60}\{(x+90)^3+(x+146)^3+(142-x)^3+198-x)^3\},$$

But M is to be a maximum, and therefore

$$-\frac{dM}{dx} = 0 = \frac{600^2-3x^2}{96} - \frac{1}{20}\{(x+90)^2+(x+146)^2+(142-x)^2+\overline{198-x}^2\},$$

which reduces to $x^2+1843.2x=167923.2$

and $x=87$ ins. $=7\frac{1}{4}$ ft.

Therefore $\text{B.M.}_{\text{max.}}=-3446581$ in.-lbs.

Ex. 58. *A swing-bridge consists of the tail end AB, and of a span BC, of length l ft., the pivot being at B. The ballast-box of weight W extends over a length AD (=2c ft.), and the weight of the bridge from D to B is w tons per lineal foot. If DB=x, if p is the cost per ton of the bridge, and if q is the cost per ton of the ballast, show that the total cost is a* minimum *when* $x+c=\left(\frac{q(l^2-c^2)}{2p-q}\right)^{\frac{1}{2}}$ *and that the corresponding weight of the ballast is* $w\frac{p}{q}(x+c)$. *Draw the S.F. and B.M. diagrams for the bridge when open and when closed, taking* $W=10wc$ *and* $l=4x=8c$.

For weight of ballast-box, take moments about B when bridge is open. Then

$$W(c+x)+\frac{wx^2}{2}=\frac{wl^2}{2},$$

or

$$W=\frac{w}{2}\frac{l^2-x^2}{c+x}.$$

Also, differentiating, $$dW=-\frac{w}{2}\frac{x^2+2xc+l^2}{(x+c)^2}dx.$$

Again, total cost $=Wq+w(l+x)p=$ a min.

Therefore $$q\cdot dW+wp\cdot dx=0,$$

or

$$p=-\frac{q}{w}\frac{dW}{dx}=+\frac{q}{2}\frac{x^2+2xc+l^2}{(x+c)^2},$$

from which

$$2p-q=\frac{q(l^2-c^2)}{(x+c)^2}$$

and

$$(x+c)^2=q\left(\frac{l^2-c^2}{2p-q}\right).$$

Hence, too, $$W=\frac{w}{2}\frac{l^2-x^2}{c+x}=w\frac{p}{q}(x+c).$$

S.F. and B.M. diagrams. Bridge open.
A to D at z from A,

$$S_z=-5wz,$$

being 0 at A and $-10wc$ at D.

D to B at z from A,

$$S_z = -6wc - 2wz,$$

being $-10wc$ at D and $-14wc$ at B.

B to C at z from A, since R_2 at $B = 20wc$,

$$S_z = -12wc + 20wc - w(z - 4c) = 12wc - wz,$$

being $8wc$ at B and 0 at C.

Again, A to D at z from A,

$$M_z = -\frac{5w}{2}z^2,$$

a parabola with its vertex at A.

At D the B.M. $= -10wc^2$.

D to B at z from A,

$$M_z = -10wc(z - c) - \frac{w}{2}(z - 2c)^2,$$

a parabola whose vertex is on the *left* of A at a point measured $8c$ horizontally from and $40wc^2$ vertically above A.

At D the B.M. $= -10wc^2$, and at B the B.M. $= -32wc^2$.

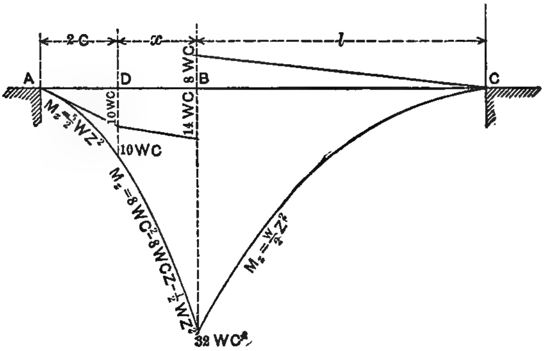


FIG. 537.—S.F. and B.M. diagrams with bridge open.

C to B at z from C,

$$M_z = -\frac{w}{2}z^2,$$

a parabola with its vertex at C.

At B the B.M. $= -32wc^2$.

Bridge closed. The bridge is now *continuous* over three supports,

$$-2M_z(8c + 4c) = +\frac{w}{4}(8c)^3 + \int_0^{2c} \frac{5w \cdot dz}{4c} z(16c^2 - z^2) + \int_{2c}^{4c} \frac{w \cdot dz}{4c} z(16c^2 - z^2),$$

or $$-24M_sc=172wc^3.$$

Therefore $$M_s=-\tfrac{43}{6}wc^2=R_1\cdot 4c-10wc\cdot 3c-2wc\cdot c$$
$$=R_3\cdot 8c-\ 8wc\cdot 4c.$$

Thus $$R_1=6\tfrac{5}{24}wc,\quad R_3=3\tfrac{5}{48}wc,\quad \text{and}\quad R_2=10\tfrac{11}{16}wc.$$

A to *D* at z from *A*,

$$S_z=R_1-5wz=6\tfrac{5}{24}wc-5wz,$$

being $6\tfrac{5}{24}\ wc$ at *A* and $-3\tfrac{19}{24}wc$ at *D*.

D to *B* at z from *A*,

$$S_z=R_1-8wc-wz=-1\tfrac{19}{24}wc-wz,$$

being $-3\tfrac{19}{24}wc$ at *D* and $-5\tfrac{19}{24}wc$ at *B*.

B to *C* at z from *A*,

$$S_z=R_1+R_2-12wc-w(z-4c)=8\tfrac{43}{48}wc-wz,$$

being $4\tfrac{43}{48}wc$ at *B* and $-3\tfrac{5}{48}wc$ at *C*.

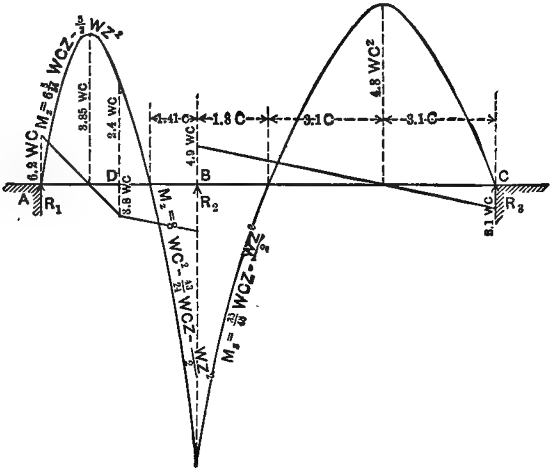

$M_z=6\frac{5}{24}WCZ-\frac{5}{2}WZ^2$
6.2 WC
3.85 WC
2.4 WC
3.8 WC
1.41 C
1.8 C
3.1 C
3.1 C
4.9 WC
4.8 WC²
8.1 WC
A
R₁
D
B
R₂
C
R₃
$M_z=8WC^2-\frac{43}{24}WCZ-\frac{WZ^2}{2}$
$M_z=\frac{23}{48}WCZ-\frac{WZ^2}{2}$

FIG. 538.—S.F. and B.M. diagrams with bridge closed.

Again, *A* to *D* at z from *A*,

$$M_z=R_1z-\frac{5wz^2}{2}=6\tfrac{5}{24}wcz-\tfrac{5}{2}wz^2,$$

a parabola with its vertex $1\tfrac{29}{120}c$ measured horizontally and $\tfrac{22201}{5760}wc^2$, measured vertically from *A*.

At D the B.M. $=2\frac{5}{12}wc^2$. Thus there is no point of inflexion in AD.
D to B at z from A,

$$M_z=R_1z-10wc(z-c)-\frac{w}{2}(z-2c)^2=-\frac{43}{24}wcz+8wc^2-\frac{w}{2}z^2,$$

which may be written

$$M_z-\frac{11065}{1152}wc^2=-\frac{w}{2}\left(\frac{43}{24}c+z\right)^2,$$

a parabola with its vertex on the left of A, at $\frac{43}{24}c$ measured horizontally and $\frac{11065}{1152}wc^2$ measured vertically upwards from A.

At B the B.M. $=-\frac{43}{6}wc^2$.

The B.M. is *nil* when

$$M_z=0=-\frac{43}{24}wcz+8wc^2-\frac{w}{2}z^2,$$

or

$$z^2+\frac{43}{12}zc=16c^2, \quad \text{i.e., when} \quad z=2.59c.$$

C to B at z from C,

$$M_z=R_3z-\frac{wz^2}{2}=3\frac{5}{48}wcz-\frac{wz^2}{2},$$

a parabola with its vertex at $3\frac{5}{48}c$ measured horizontally and $\frac{22201}{4608}wc^2$ measured vertically from C.

The B.M. at $B=3\frac{5}{48}wc\cdot 8c-\frac{w}{2}(8c)^2=-\frac{43}{6}wc^2$.

The B.M. is *nil* at point given by

$$M_z=0=3\frac{5}{48}wcz-\frac{wz^2}{2}, \quad \text{or} \quad z=6\frac{5}{24}c.$$

Ex. 59. *A continuous girder of two spans AB, BC, carrying a load of uniform intensity w, has one end A fixed, and the other end rests upon the support at C. If the bending moments at A and B are equal, show that the spans are in the ratio of $\sqrt{3}$ to $\sqrt{2}$, and find the reactions at the supports, W_1 being the load upon AB and W_2 that upon BC.*

$$M_1l_1+2M_2(l_1+l_2)=-\frac{w}{4}(l_1^3+l_2^3)=-\frac{1}{4}(W_1l_1^2+W_2l_2^2).$$

Therefore

$$M_1=M_2=-\frac{1}{4}\,\frac{W_1l_1^2+W_2l_2^2}{3l_1+2l_2}.$$

Also, $2M_1 + M_2 = -\frac{W_1 l_1^2}{4}$, or $M_1 = M_2 = -\frac{1}{12} w_1 l_1^2 = -\frac{W_1}{12} l_1.$

Hence
$$\frac{1}{4}\frac{W_1 l_1^2 + W_2 l_2^2}{3l_1 + 2l_2} = \frac{W_1 l_1}{12}, \quad \text{or} \quad 3W_2 l_2 = 2W_1 l_1,$$

and
$$\frac{l_1}{l_2} = \frac{3W_2}{2W_1} = \frac{3wl_2}{2wl_1} = \frac{3l_2}{2l_1}, \quad \text{or} \quad \frac{l_1}{l_2} = \sqrt{\frac{3}{2}} = \frac{W_1}{W_2}.$$

Taking moments about B,
$$R_1 l_1 - W_1 \frac{l_1}{2} + M_1 = M_2.$$

Therefore
$$R_1 = \frac{W_1}{2}.$$

Also,
$$R_3 l_2 - W_2 \frac{l_2}{2} = M_2 = -\frac{1}{4}\frac{W_1 l_1^2 + W_2 l_2^2}{3l_1 + 2l_2}$$

and
$$R_3 = \frac{1}{2} W_2 - \frac{1}{4}\frac{W_2}{l_2^2}\left(\frac{l_1^3 + l_2^3}{3l_1 + 2l_2}\right)$$
$$= \frac{W_2}{4}\left(\frac{6l_1 l_2^2 + 3l_2^3 - l_1^3}{(3l_1 + 2l_2)l_2^2}\right) = \frac{W_2}{4}\left(\frac{6l_1 + 3l_2 - \frac{3}{2}l_1}{3l_1 + 2l_2}\right) = \frac{3}{8} W_2.$$

Hence, too,
$$R_2 = W_1 + W_2 - R_1 - R_3 = \frac{W_1}{2} + \frac{5}{8} W_2.$$

Ex. 60. *A continuous girder of four equal spans, and with one end fixed at the first support, carries a uniformly distributed load. How much must the third support be raised to make the reactions equal (a) at the third and fourth supports; (b) at the fourth and fifth supports?*

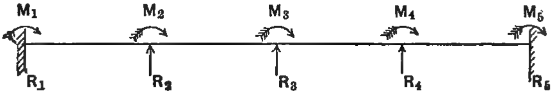


FIG. 539.

Take each span $= l (= 100$ ft.) and let w be the intensity of the load, so that $wl = 194$ tons.

From condition of fixture
$$2M_1 + M_2 = -\frac{wl^2}{4}.$$

Let d be the amount by which the third support must be raised. Then, by the general equation, and taking $N = \frac{EId}{l^2}$,

$$M_1+4M_2+M_3=-\frac{wl^2}{2}+6N,$$

$$M_2+4M_3+M_4=-\frac{wl^2}{2}-12N,$$

$$M_3+4M_4=-\frac{wl^2}{2}+6N.$$

Hence
$$M_1=-\frac{8}{97}wl^2-\frac{144}{97}N,\quad M_2=-\frac{33}{388}wl^2+\frac{288}{97}N,$$

$$M_3=-\frac{30}{388}wl^2-\frac{426}{97}N,\quad M_4=-\frac{41}{388}wl^2+\frac{252}{97}N.$$

Taking moments

$$R_5l=\frac{wl^2}{2}+M_4=\frac{153}{388}wl^2+\frac{252}{97}N. \quad . \quad . \quad . \quad . \quad (1)$$

$$R_52l+R_4l=2wl^2+M_3=\frac{746}{388}wl^2-\frac{426}{97}N. \quad . \quad . \quad . \quad . \quad (2)$$

$$R_53l+R_42l+R_3l=\frac{9wl^2}{2}+M_2=\frac{1713}{388}wl^2+\frac{288}{97}N. \quad . \quad . \quad . \quad . \quad (3)$$

(*a*) If $R_3=R_4$, these three equations give

$$\frac{22}{388}wl^2=\frac{774}{97}N=\frac{774}{97}\frac{EId}{l^3} \quad \text{and} \quad d=\frac{11}{1548}\frac{wl^4}{EI}.$$

(*b*) If $R_4=R_5$, eqs. (1) and (2) give

$$\frac{287}{388}wl^2=\frac{1182}{97}N=\frac{1182}{97}\frac{EId}{l^3} \quad \text{and} \quad d=\frac{287}{4728}\frac{wl^4}{EI}.$$

The corresponding reactions and B.Ms. can be easily deduced.

Ex. 61. *A swing-bridge ABCD, 440 ft. in length, has equal arms AB and CD, and rests upon rollers at B and C which run in a circular path of 22 ft. diameter. Each arm is a truss of 9 panels.*

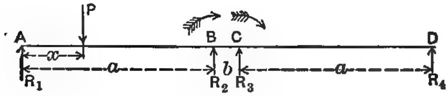


FIG 540

Take $AB=209'=a=CD$ and $BC=22'=b$.

Consider the effect of a load P tons concentrated at x ft. from A. Then

$$2M_2(a+b)+M_3b=-P\frac{x}{a}(a^2-x^2)$$

and $$(M_2b+2M_3(b+a)=0.$$

Therefore

$$M_2=-2\frac{P(a+b)(x)(a^2-x^2)}{a(4a^2+8ab+3b^2)}=-\frac{21P}{2023120}x(a^2-x^2)$$

and $$M_3=-\frac{M_2b}{2(a+b)}=-\frac{M_2}{21}.$$

Putting $A=\frac{361P}{80\times729}$, the values of M_2 and M_3 in foot-tons, corresponding to the concentration of P at the 1st, 2d, 3d . . . and 8th panel-points, may be tabulated as follows:

Value of $\frac{x}{a}$.	M_2.	M_3.
$\frac{1}{9}$	$-1680A=-10.399P$	$+80A=+.4952P$
$\frac{2}{9}$	$-3234A=-20.019P$	$+154A=+.9533P$
$\frac{3}{9}$	$-4536A=-28.078P$	$+216A=+1.3371P$
$\frac{4}{9}$	$-5460A=-33.798P$	$+260A=+1.6094P$
$\frac{5}{9}$	$-5880A=-36.398P$	$+280A=+1.7332P$
$\frac{6}{9}$	$-5670A=-35.095P$	$+270A=+1.6711P$
$\frac{7}{9}$	$-4704A=-29.118P$	$+224A=+1.3866P$
$\frac{8}{9}$	$-2856A=-17.679P$	$+136A=+.8419P$

Again, taking moments and simplifying,

$$R_1=\frac{M_2}{209}+P\left(1-\frac{x}{a}\right),\quad R_2=-\frac{230}{4389}M_2+P\frac{x}{a},$$

$$R_3=\frac{10}{209}M_2,\qquad R_4=-\frac{M_2}{4389}.$$

The values of the reactions, corresponding to the concentrations of P at the 1st, 2d, 3d . . . and 8th panel-points, are given in the following table:

Value of $\frac{x}{a}$.	R_1 in Tons.	R_2 in Tons.	R_3 in Tons.	R_4 in Tons.
$\frac{1}{9}$	$.8391P$	$.6561P$	$-.4976P$	$.0024P$
$\frac{2}{9}$	$.6820P$	$1.2713P$	$-.9578P$	$.0046P$
$\frac{3}{9}$	$.5323P$	$1.8047P$	$-1.3434P$	$.0064P$
$\frac{4}{9}$	$.3938P$	$2.2156P$	$-1.6171P$	$.0077P$
$\frac{5}{9}$	$.2703P$	$2.4629P$	$-1.7415P$	$.0083P$
$\frac{6}{9}$	$.1654P$	$2.5057P$	$-1.6791P$	$.0080P$
$\frac{7}{9}$	$.0829P$	$2.3037P$	$-1.3932P$	$.0066P$
$\frac{8}{9}$	$.0265P$	$1.8153P$	$-.8459P$	$.0040P$

Adding all these reactions together, the result is $8P$ tons, which verifies the calculations.

Similar values for the B.Ms. and reactions are obtained for the arm *CD*, and by means of the preceding tables the values may be found for every distribution of the panel load. Let 1, 2, 3, . . . 16 denote the 1st, 2d, . . . 16th panel-points from the point of support *A*. The following table of B.Ms. is now easily prepared:

Load P at	M_2 in Ft.-tons.	M_3 in Ft.-tons.
1	$-10.399P$	$+.4952P$
1 and 2	$-30.418P$	$1.4485P$
1 to 3	$-58.496P$	$2.7856P$
1 " 4	$-92.294P$	$4.3950P$
1 " 5	$-128.692P$	$6.1282P$
1 " 6	$-163.787P$	$7.7993P$
1 " 7	$-192.905P$	$9.1859P$
1 " 8	$-210.584P$	$10.0278P$
1 " 9	$-209.7421P$	$-7.6512P$
1 " 10	$-208.3555P$	$-36.7692P$
1 " 11	$-206.6844P$	$-71.8642P$
1 " 12	$-204.9512P$	$-108.2622P$
1 " 13	$-203.3418P$	$-142.0602P$
1 " 14	$-202.0047P$	$-170.1382P$
1 " 15	$-201.0514P$	$-190.1572P$
1 " 16	$-200.5562P$	$-200.5562P$

The B.Ms. in the last line are equal, and this must necessarily be the case, as *all* the panel-points are loaded.

A similar table of reactions in tons may also be prepared as follows:

Load P at	R_1.	R_2.	R_3.	R_4.
1	$.8391P$	$.6561P$	$-.4976P$	$.0024P$
1 and 2	$1.5211P$	$1.9274P$	$-1.4554P$	$.0070P$
1 to 3	$2.0534P$	$3.7321P$	$-2.7988P$	$.0134P$
1 " 4	$2.4472P$	$5.9477P$	$-4.4169P$	$.0211P$
1 " 5	$2.7175P$	$8.4106P$	$-6.1574P$	$.0294P$
1 " 6	$2.8829P$	$10.9163P$	$-7.8365P$	$.0374P$
1 " 7	$2.9658P$	$13.2200P$	$-9.2297P$	$0440P$
1 " 8	$2.9923P$	$15.0353P$	$-10.0756P$	$.0480P$
1 " 9	$2.9963P$	$14.1894P$	$-8.2603P$	$.0745P$
1 " 10	$3.0029P$	$12.7962P$	$-5.9566P$	$.1574P$
1 " 11	$3.0109P$	$11.1171P$	$-3.4509P$	$.3228P$
1 " 12	$3.0192P$	$9.3756P$	$-0.9880P$	$.5931P$
1 " 13	$3.0269P$	$7.7585P$	$+1.2276P$	$.9869P$
1 " 14	$3.0333P$	$6.4151P$	$3.0323P$	$1.5192P$
1 " 15	$3.0379P$	$5.4573P$	$4.3036P$	$2.2012P$
1 " 16	$3.0403P$	$4.9597P$	$4.9597P$	$3.0403P$

Adding the last line together it is found that the total sum of the reactions is $16P$, which verifies the calculations

The B.M. and S.F. diagrams for all distributions of the panel load can now be easily drawn.

25. Advantages and Disadvantages of Continuous Girders.—The advantages claimed for continuous girders are facility of erection, a smaller average B.M. and therefore a saving in the flange material; the concentration of the maximum B.M. over the piers, and the removal of a portion of the weight from the centre of a span towards the piers. Circumstances, however, may modify these advantages, and even render them completely valueless. The flange stresses are governed by the position of the points of inflexion, which under a moving load will fluctuate through a distance dependent upon the number of intermediate supports and upon the nature of the loading. In bridges in which the ratio of the dead load to the live load is small the fluctuation is considerable, so that for a sensible length of the main girders a passing train will subject local members to stresses which are alternately positive and negative. This necessitates a local increase of material, as each member must be designed to bear a much higher stress than if it were strained in one way only.

Again, the web of a continuous girder, even under a uniformly distributed dead load, is theoretically heavier than if each span were independent, and its weight is still further increased when it has to resist the complex stresses induced by a moving load.

Hence in such bridges the slight saving, if there be any, cannot be said to counterbalance the extra labor of calculation and workmanship.

In girders subjected to a dead load only, and in bridges in which the ratio of the dead load to the live load is large, the saving becomes more marked and increases with the number of intermediate supports, being theoretically a maximum when the number is infinite. This maximuum economy may be approximated to in practice by making the end spans about four fifths the intermediate spans.

In the calculations relating to the Theorem of Three Moments, it has been assumed that the quantity EI is constant, while in reality E, even for mild steel, may vary 10 to 15 per cent from a mean value, and I may vary still more. It does not appear, however, that this variation has any appreciable effect if the depth of the girder or truss changes *gradually,* but the effect may become very marked with a *rapid* change of depth, as, e.g., in the case of swing-bridges of the triangular type.

The graphical method of treatment may still be employed by substituting a *reduced* curve for the actual curve of moments, formed by changing the lengths of the ordinates in the ratio of the value of EI at a datum section to EI.

It is often found economical to increase the depth of the girder over the piers, which introduces a local stiffness and moves the points of inflexion farther from the supports. A point of inflexion may be made to travel a short distance by raising or depressing one of the supports.

In order to insure the full advantage of continuity the utmost care and skill are required both in design and workmanship. Allowance has to be made for the excessive expansion and contraction due to changes of temperature, and the piers and abutments must be of the strongest and best description, so that there shall be no settlement. Indeed, the difficulties and uncertainties to be dealt with in the construction of continuous girders are of such a serious if not insurmountable character that American engineers have almost entirely discarded their use except for draw-spans.

Much, in fact, is mere guesswork, and it is usual in practice to be guided by experience, which confines the points of inflexion within certain safe limits.

Under these circumstances it may prove desirable *to fix the points of inflexion absolutely,* and the advantages of doing so are (*a*) that the calculation of the web stresses becomes easy and definite instead of being complicated and even indeterminate; (*b*) that reversed stresses (for which pin-trusses are less adapted than riveted trusses) are almost entirely avoided; (*c*) that the stresses are not sensibly affected by slight inequalities in the levels of the supports; (*d*) that the straining due to a change of temperature takes place under more favorable conditions.

The *fixing* may be effected in the following manner:

(*a*) A *hinge* may be introduced at the selected point.

The benefit of doing so is very obvious when circumstances require a wide centre span and two short side spans.

(*b*) If the web is open, i.e., lattice-work, the point of inflexion in the upper flange may be fixed by cutting the flange at the selected point and lowering one of the supports so as to produce a slight opening between the severed parts. The position of the point of inflexion in the lower flange is then defined by the condition that the algebraic sum of the horizontal components of the stresses in the diagonals intersected by a line joining the two points of inflexion is zero.

It must be remembered, however, that this *fixing* of the points of inflexion, or the *cutting* of the chords, destroys the property of continuity, and, indeed, is the essential distinction between a continuous girder and a cantilever. A combined cantilever and girder possesses all the advantages and none of the disadvantages of a continuous girder.

Four methods may be followed in the erection of a continuous girder, viz.:

1. It may be built on the ground and *lifted* into place.

2. It may be built on the ground and rolled endwise over the piers. As the bridge is pushed forward, the forward end acts as a cantilever for the whole length of a span until the next pier is reached. This method of erection is common in France.

3. It may be built in position on a scaffold.

4. Each span may be erected separately and continuity produced by securely jointing consecutive ends, having drawn together the upper flanges. A more effective distribution of the material is often made by leaving a little space between the flanges and forming a wedge-shaped joint.

26. Maximum Bending Moments at the Points of Support of Continuous Girders of n Euqal Spans.

Let the figure represent a continuous girder of n spans, 1, 2, 3, . . . $n-1$ being the $n-1$ intermediate supports.

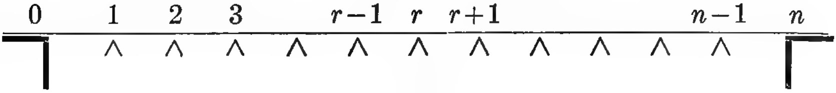


FIG. 541.

CASE I. Assume all the spans to be of the same length l, and let w_1, w_2, . . . w_{n-1}, w_n be the intensities of loads uniformly distributed over the 1st, 2d, . . . $(n-1)$th and nth spans, respectively.

By the Theorem of Three Moments,

$$4m_1+m_2=-\frac{l^2}{4}(w_1+w_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$m_1+4m_2+m_3=-\frac{l^2}{4}(w_2+w_3), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$m_2+4m_3+m_4=-\frac{l^2}{4}(w_3+w_4), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$m_3+4m_4+m_5=-\frac{l^2}{4}(w_4+w_5), \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$m_4+4m_5+m_6=-\frac{l^2}{4}(w_5+w_6). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

. . . .

$$m_{n-3}+4m_{n-2}+m_{n-1}=-\frac{l^2}{4}(w_{n-2}+w_{n-1}), \quad . \quad (n-2)$$

$$m_{n-2}+4m_{n-1} \qquad =-\frac{l^2}{4}(w_{n-1}+w_n). \quad . \quad . \quad (n-1)$$

m_1 and m_n are both zero, as the girder is supposed to be resting upon the abutments at 0 and n.

From these $(n-1)$ equations, the bending moments m_1, m_2, . . . m_{n-1} may be found in terms of the distributed loads.

Eliminating m_2 from 2 and 3,

$$m_1-15m_3-4m_4=-\frac{l^2}{4}\{(w_2+w_3)-4(w_3+w_4)\}. \quad . \quad . \quad (x_1)$$

Eliminating m_3 from 4 and x_1,

$$m_1+56m_4+15m_5=-\frac{l^2}{4}\{(w_2+w_3)-4(w_3+w_4)+15(w_4+w_5)\}. \quad (x_2)$$

Eliminating m_4 from 5 and x_2,

$$m_1-209m_5-56m_6$$

$$=-\frac{l_2}{4}\{(w_2+w_3)-4(w_3+w_4)+15(w_4+w_5)-56(w_5+w_6)\}. \quad . \quad (x_3)$$

Finally, by successively eliminating m_5, m_6, . . . m_{n-2},

$$m_1 \pm a_{n-1}m_{n-1} = -\frac{l^2}{4}\{(w_2+w_3)-4(w_3+w_4)+15(w_4+w_5)-. . .$$

$$\pm a_{n-4}(w_{n-3}+w_{n-2}) \mp a_{n-3}(w_{n-2}+w_{n-1}) \pm a_{n-2}(w_{n-1}+w_n)\}, \quad (y)$$

the upper or lower sign being taken for the terms within the brackets according as n is odd or even, and the coefficients a_{n-1}, a_{n-2}, a_{n-3}, . . . being given by the law,

$$\begin{aligned}
a_{n-1} &= 4a_{n-2}-a_{n-3},\\
a_{n-2} &= 4a_{n-3}-a_{n-4},\\
&. \quad . \quad . \quad . \quad . \quad . \quad .\\
a_5 &= 4a_4-a_3 = 209,\\
a_4 &= 4a_3-a_2 = \ \ 56,\\
a_3 &= 4a_2-a_1 = \ \ 15,\\
a_2 &= 4a_1 \qquad\ = \ \ \ 4,\\
a_1 &= 1.
\end{aligned}$$

Commencing with equations $n-3$ and $n-2$, and proceeding as before,

$$a_{n-1}m_1 \pm m_{n-1} = -\frac{l^2}{4}\{a_{n-2}(w_1+w_2)-a_{n-3}(w_2+w_3)+a_{n-4}(w_3+w_4)$$

$$-. . . \pm 15(w_{n-4}+w_{n-3}) \mp 4(w_{n-3}+w_{n-2}) \pm (w_{n-2}+w_{n-1})\}, \quad (z)$$

the upper or lower sign being taken for the terms within the brackets according as n is odd or even.

Solving the two equations y and z,

$$m_1(a^2{}_{n-1}-1) = -\frac{l^2}{4}\{a_{n-1}a_{n-2}w_1+(a_{n-1}a_{n-2}-a_{n-1}a_{n-3}-1w)_2$$

$$-(a_{n-1}a_{n-3}-a_{n-1}a_{n-4}-3)w_3+. . . \pm(3a_{n-1}+a_{n-4}-a_{n-3})w_{n-2}$$

$$\pm(a_{n-1}+a_{n-3}-a_{n-2}w_{n-1} \mp a_{n-2})w_n\},$$

and $$\pm m_{n-1}(a^2{}_{n-1}-1) = -\frac{l^2}{4}\{-a_{n-2}w_1+(a_{n-1}+a_{n-3}-a_{n-2})w_2$$

$$-(3a_{n-1}+a_{n-4}-a_{n-3})w_3 \mp (a_{n-1}a_{n-3}-a_{n-1}a_{n-4}-3)w_{n-2}$$

$$\pm(a_{n-1}a_{n-2}-a_{n-1}a_{n-3}-1)w_{n-1} \pm a_{n-1}a_{n-2}w_n\}.$$

Hence, since $w_1, w_2, \dots w_n$ are positive integers, the value of m_n will be *greatest* when $w_1, w_2, w_4, w_6, w_8, \dots$ are greatest and $w_3, w_5, w_7, \dots$ are least; and the value of m_{n-1} will be *greatest* when $w_n, w_{n-1}, w_{n-3}, w_{n-4}, \dots$ are greatest and $w_{n-2}, w_{n-4}, w_{n-6}, \dots$ are least. In other words, the bending moments at the 1st and $(n-1)$th intermediate supports have their maximum values when the two spans adjacent to the support in question, and then every alternate span, are loaded, and the remaining spans unloaded.

$m_2, m_3, \dots m_{n-2}$ may now be easily determined.

Thus, by eq. (1),

$$m_2 = -\frac{l^2}{4}(w_1+w_2) - 4m_1$$

$$= -\frac{l^2}{4}\left\{(w_1+w_2) - \frac{4}{a^2_{n-1}-1}a_{n-1}a_{n-2}w_1 \right.$$

$$\left. + (a_{n-1}a_{n-2} - a_{n-1}a_{n-3} - 1)w_2 - \dots\right\}$$

$$= -\frac{l^2}{4(a^2_{n-1}-1)}\{(a^2_{n-1}-1-4a_{n-1}a_{n-2})w_1$$

$$+ (a^2_{n-1}-1-4a_{n-1}a_{n-2}+4a_{n-1}a_{n-3}+4)w_2+\dots\}$$

But $a_{n-1} = 4a_{n-2} - a_{n-3}$. Therefore

$$m_2 = -\frac{l^2}{4(a^2_{n-1}-1)}\{-(a_{n-1}a_{n-3}+1)w_1 + (3a_{n-1}a_{n-3}+3)w_2+\dots\},$$

and is *greatest* when $w_2, w_3, w_5, w_7, \dots$ are greatest and $w_1, w_4, w_6, w_8, \dots$ are least.

Similarly, by eqs. (1) and (2),

$$m_3 = -\frac{l^2}{4}(w_2+w_3) + \frac{l^2}{4}\cdot 4(w_1+w_2) + 15m_1$$

$$= -\frac{l^2}{a^2_{n-1}-1}\{(a_{n-4}a_{n-1}+4)w_1 - (3a_{n-1}a_{n-4}+12)w_2$$

$$+ (11a_{n-1}a_{n-4}+44)w_3 + \dots\},$$

and is *greatest* when w_1, w_3, w_4, w_6, w_8, . . . are greatest and w_2, w_5, w_7, w_9, . . . are least.

Thus the general principle may be enunciated, that "*in a horizontal continuous girder of n equal spans, with its ends resting upon two abutments, the bending moment at an intermediate support is greatest when the two spans adjacent to such support, and the alternate spans counting in both directions, carry uniformly distributed loads, the remainder of the spans being unloaded.*"

CASE II. The principle deduced in Case I also holds true when the loads are distributed in any arbitrary manner.

Consider the effect of a weight w in the rth span concentrated at a point distant p from the $(r-1)$th support.

By the Theorem of Three Moments,

$$4m_1+m_2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$m_1+4m_2+m_3=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$m_2+4m_3+m_4=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$. \quad . \quad . \quad . \quad . \quad . \quad .$$

$$m_{r-2}+4m_{r-1}+m_r=-w\frac{p}{l}(l^2-p^2)=-A, \text{ suppose} \quad . \quad . \quad . \quad (r-1)$$

$$m_{r-1}+4m_r+m_{r+1}=-w\frac{l-p}{l}\{l^2-(l-p)^2\}$$

$$=-w\frac{p}{l}(l-p)(2l-p)=-B, \text{ suppose}, \quad . \quad . \quad (r)$$

$$m_r+4m_{r+1}+m_{r+2}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (r+1)$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$m_{n-3}+4m_{n-2}+m_{n-1}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (n-2)$$

$$m_{n-1}+4m_{n-1}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (n-1)$$

By equations (1), (2), (3), . . . $(r-2)$,

$$m_1=-\frac{1}{4}m_2=\frac{1}{15}m_3=-\frac{1}{56}m_4=.\;.\;.=\mp\frac{1}{a_{n-2}}m_{r-2}=\pm\frac{m_{r-1}}{a_{r-1}},$$

the upper or lower sign being taken according as r is even or odd.

By equations $(n-1)$, $(n-2)$, $(n-3)$, . . . $(r+1)$,

$$m_{n-1} = -\tfrac{1}{4}m_{n-2} = \tfrac{1}{15}m_{n-3} = -\tfrac{1}{56}m_{n-4} = \ldots$$

$$= \mp\frac{m_{r+2}}{a_{n-r-1}} = \pm\frac{m_{r+1}}{a_{n-r}} = \mp\frac{m_r}{a_{n-r-1}}.$$

The coefficients a are given by the same law as for the coefficients a in Case I. Thus

$$m_{r-2} = -\frac{a_{r-2}}{a_{r-1}}m_{r-1} \quad \text{and} \quad m_{r+1} = -\frac{a_{n-r}}{a_{n-r+1}}m_r.$$

Substituting these values of m_{r-2} and m_{r+1} in the $(r-1)$th and rth equations,

$$m_{r-1}\left(4-\frac{a_{r-2}}{a_{r-1}}\right)+m_r = -A = m_{r-1}b + m_r$$

and $$m_{r-1}+m_r\left(4-\frac{a_{n-r}}{a_{n-r+2}}\right) = -B = m_{r-1}+m_r c,$$

where $$b = 4-\frac{a_{r-2}}{a_{r-1}} \quad \text{and} \quad c = 4-\frac{a_{n-r}}{a_{n-r+1}}.$$

Hence, solving the last two equations,

$$m_{r-1} = -\frac{Ac-B}{bc-1} \quad \text{and} \quad m_r = -\frac{Bb-A}{bc-1}.$$

The ratios $\frac{a_{r-2}}{a_{r-1}}$ and $\frac{a_{n-r}}{a_{n-r+1}}$ are each less than unity, and hence b and c are each <4 and >3.

It may now easily be shown that $Ac-B$ and $Bb-A$ are each positive. Hence m_{r-1} and m_r are both of the same sign.

The bending moment m_q at any intermediate support on the left of $r-1$ is given by

$$m_q = +\frac{a_q}{a_{r-1}}m_{r-1} \text{ if } q \text{ and } r \text{ are the one even and the other odd,}$$

or

$$m_q = -\frac{a_q}{a_{r-1}} m_{r-1} \text{ if } q \text{ and } r \text{ are both even or both odd.}$$

Thus the bending moment at the qth support is increased in the former case and diminished in the latter.

If q is on the right of r,

$$m_q = +\frac{a_{n-q+1}}{a_{n-r+1}} m_r \text{ if } q \text{ and } r \text{ are both even or both odd,}$$

or

$$m_q = -\frac{a_{n-q+1}}{a_{n-r+2}} m_r \text{ if } q \text{ and } r \text{ are the one even and the other odd,}$$

and the bending moment on the qth support is increased in the former case and diminished in the latter.

Thus the general principle may be enunciated, that, "*in a horizontal continuous girder of n equal spans, with its ends resting upon two abutments, the bending moment at an intermediate support is greatest when the two spans adjacent to such support, and the alternate spans counting in both directions, are loaded, the remainder of the spans being unloaded.*

CASE III. The same general principle still holds true when the two end spans are of different lengths.

E.g., let the length of the first span be kl, k being a numerical coefficient, and let $2(1+k)=x$.

Eq. (1) now becomes

$$m_1 x + m_2 = 0.$$

Proceeding as before,

$$\frac{m_1}{b_1} = -\frac{m_2}{b_2} = +\frac{m_3}{b_3} = -\frac{m_4}{b_4} = + \ldots,$$

the coefficients b_1, b_2, b_3, . . . being given by the same law as before, viz.,

$$
\begin{aligned}
b_1 &= 1,\\
b_2 &= x,\\
b_3 &= 4b_2 - b_1 = 4x - 1,\\
b_4 &= 4b_3 - b_2 = 15x - 4,\\
b_5 &= 4b_3 - b_2 = 56x - 15,\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{aligned}
$$

The two sets of coefficients (a) and (b) are identical when $x=4$, and when $x>4$, all the coefficients b except the first ($b_1=1$) are numerically increased.

Hence the same general results will follow.

N.B.—The equations giving m_q are simple and easily applicable in practice. They may be written

$$m_q = \pm \frac{a_q}{a_{r-1}} \frac{B-Ac}{bc-1} \text{ if } q \text{ is on the left of } r,$$

and
$$m_q = \pm \frac{a_{n-q+1}}{a_{n-r+1}} \frac{A-Bc}{bc-1} \text{ if } q \text{ is on the right of } r.$$

If there are several weights on the rth span,

$$A = \Sigma \frac{wp}{l}(l^2-p^2) \quad \text{and} \quad B = \Sigma w \frac{p}{l}(l-p)(2l-p).$$

Ex. 62. *The viaduct over the Osse consists of two end spans, each of* 94 *ft., and five intermediate spans, each of* 126 *ft. The platform is carried by two main girders which are continuous from end to end. The total dead load upon the girders may be taken at one ton (of* 2000 *lbs.) per lineal foot.*

Denote the supports, taken in order, by the letters a, b, c, d, e, f, g, h, and let it be required to find the maximum bending moment at d when the bridge is subjected to an additional proof load of 1½ tons per lineal foot.

The spans ab, cd, de, fg of each girder carry 1¼ tons per lineal foot.

The spans bc, ef, gh of each girder carry ½ ton per lineal foot.

Denoting the bending moments at a, b, c, d, e, f, g, h, respectively, by m_1, m_2, . . . m_8, the intermediate spans by l, the end spans by kl, and remembering that $m_1=0=m_8$, we have

$$2m_2(k+1)+m_3 = -\frac{l^2}{4}(k1\tfrac{1}{4}+\tfrac{1}{2}),$$

$$m_2+4m_3+m_4 = -\frac{l^2}{4}(\tfrac{1}{2}+1\tfrac{1}{4}),$$

$$m_3+4m_4+m_5 = -\frac{l^2}{4}(1\tfrac{1}{4}+1\tfrac{1}{4}),$$

$$m_4+4m_5+m_6 = -\frac{l^2}{4}(1\tfrac{1}{4}+\tfrac{1}{2}),$$

$$m_5+4m_6+m_7 = -\frac{l^2}{4}(\tfrac{1}{2}+1\tfrac{1}{4}),$$

$$m_6+2m_7(k+1) = -\frac{l^2}{4}(1\tfrac{1}{4}+k\tfrac{1}{2}).$$

But $k = \frac{94}{126} = \frac{3}{4}$, very nearly.

Therefore

$$7m_2 + 2m_3 = -\frac{l^2}{4}\,\frac{23}{8}, \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$m_2 + 4m_3 + m_4 = -\frac{l^2}{4}\,\frac{7}{2}, \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$m_3 + 4m_4 + m_5 = -\frac{l^2}{4}\,\frac{5}{2}, \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$m_4 + 4m_5 + m_6 = -\frac{l^2}{4}\,\frac{7}{2}, \quad \ldots\ldots\ldots\ldots \quad (4)$$

$$m_5 + 4m_6 + m_7 = -\frac{l^2}{4}\,\frac{7}{2}, \quad \ldots\ldots\ldots\ldots \quad (5)$$

$$2m_6 + 7m_7 = -\frac{l^2}{4}\,\frac{13}{4}. \quad \ldots\ldots\ldots\ldots \quad (6)$$

From eqs. (1), (2), (3),

$$97m_4 + 26m_5 = -\frac{l^2}{4}\,\frac{347}{8}.$$

From eqs. (4), (5), (6),

$$26m_4 + 97m_5 = -\frac{l^2}{4}\,\frac{279}{4}.$$

Hence m_4, the maximum required,

$$= -\frac{l^2}{4}\cdot\frac{19151}{8\times 8733} = -605.5 \text{ ft.-tons}.$$

TABLE OF RESILIENCES OF SPRING MATERIALS.*

	Maximum Tensile or Compressive Stress in Thousands of Pounds which will not Produce Set.	E in Millions of Pounds per Square Inch.	$\frac{f^2}{2E}$.	Maximum Shear Stress in Thousands of Pounds which will not Produce Set.	G in Millions of Pounds per Square Inch.	$\frac{q^2}{2G}$.
Brass.	6.95	9.2	2.62	5.2	3.4	4
Cast steel, unhardened . .	80	30	107	64	11	186
Cast steel, hardened.	190	36	501	145	13	809
Copper.	4.3	15	.62	2.9	5.6	.75
Glass.	4.5	8	1.26			
Gun-metal.	6.2	9.9	2	4.15	3.7	2.33
Mild steel.	35	30	20	26.5	11	32
Mild steel, hardened.	70.5	30	83	53	11	128
Phosphor-bronze.	19.7	14	13.85	14.5	5.25	20
Wrought iron.	24	29	10	20	10.5	19

* Perry's "Applied Mechanics."

EXAMPLES.

1. A flat spiral spring .2 in. wide and .03 in. thick is subjected to a bending moment of 10 in.-lbs. Find its radius of curvature, E being 36,000,000 lbs. *Ans.* 1.62 ins.

2. A straight strip of tempered steel .7 in. broad, .1 in. thick (representing the *depth* of a beam), is subjected to a bending moment of 100 in.-lbs. Find its radius of curvature. *Ans.* 21 ins.

3. What is the greatest stress in a bar which is subject to a bending moment of 4000 in.-lbs (1) if the section is a circle of $\frac{3}{4}$ in. radius; (2) if of I form, 2 ins. deep and 1 in. wide, the web and flanges each being $\frac{3}{8}$ in. thick? *Ans.* (1) 5.4 tons; (2) 3.1 tons.

4. A $3'' \times \frac{3}{8}''$ steel bar is bent into a circle of 50 ft. radius. Find the maximum stress induced in the material, the coefficient of elasticity being 28,000,000 lbs./sq. in. *Ans.* 8750 lbs.

5. A square bar $\frac{3}{4}'' \times \frac{3}{4}''$ is subjected to a bending moment of 350 in.-lbs. What is the greatest stress in the bar, and the radius of the circle into which it is bent, E being taken at 2,000,000 lbs./sq. in.? *Ans.* 4978 lbs.; 12.5 ft.

6. A $2'' \times 1''$ bar is bent into the arc of a circle of 1000 ft. radius. Find the moment of resistance and the maximum stress developed, E being 30,000,000 lbs./sq. in. *Ans.* 1250 lbs./sq. in.; $408\frac{1}{3}$ in.-lbs.

7. A round steel rod $\frac{1}{2}$ in. in diameter, resting upon supports A and B, 4 ft. apart, projects 1 ft. beyond A and 9 ins. beyond B. The extremity beyond A is loaded with a weight of 12 lbs., and that beyond B with a weight of 16 lbs. Neglecting the weight of the rod, investigate the curvature of the rod between the supports, and calculate the greatest deflection between A and B. Find also the greatest intensity of stress in the rod due to the two applied forces. ($E = 30,000,000$ lbs.)

Ans. Neutral axis between A and B is arc of a circle; .45 in.; 11,750 lbs./sq. in.

8. One side of a plate of metal is at $\theta°_1C$ and the other is at $\theta_2°C$. The plate when cold is plane; what is now its curvature? What is the greatest stress in the material if curvature be prevented? *Ans.* $E\alpha(\theta_1 - \theta_2)$.

9. At a certain point a bar is subjected to a bending moment of 4000 in.-lbs., the greatest stress in the material being 10,000 lbs./sq. in. If the bar is of circular section, find its diameter; if it is square in section, find the side. If its coefficient of elasticity is 28,000,000 lbs., find the curvature.

Ans. 1.59 ins.; 1.48 ins.; *radius of curvature*, 186.2 ft. and 172.8 ft.

10. A horizontal beam of depth d, breadth b, and length 12 ft. rests upon supports at the ends. A weight W_1 at the centre deflects the beam .1 in. when the side of length b is vertical. An additional weight of 1250 lbs. is required to produce the same deflection when the side of length d is vertical. If $d = 2b$ and if $E = 1,200,000$ lbs., find the sectional area of the beam and the maximum skin stress. *Ans.* 72 sq. ins.; $416\frac{2}{3}$ lbs./sq. in.

11. A band-saw is ½ in. wide, .02 in. thick, and passes over two pulleys each 12 ins. in diameter. If the tight tension is 100 lbs. find the maximum intensity of stress in the band. ($E=30{,}000{,}000$ lbs.) *Ans.* 15,000 lbs./sq. in.

12. A steel strip 2″×.1″ has an initial curvature of .0025. Find the B.M. and the corresponding maximum stress developed in the material which will (*a*) straighten the strip; (*b*) give it a curvature in the opposite direction equal to the initial curvature, E being 36,000,000 lbs./sq. in.

Ans. (*a*) 15 in.-lbs.; 4500 lbs./sq. in.; (*b*) 30 in.-lbs., 9000 lbs./sq. in.

13. A laminated spring of 3 ft. span has 20 plates, each 375 in. thick and 2.95 ins. wide. Calculate the deflection when centrally loaded with 5 tons. ($E=11{,}600$ tons/sq. in.) *Ans.* 2.45 ins.

14. A laminated spring of 40 ins. span has twelve plates each .375 in. thick and 3.40 ins. wide. Calculate the deflection when centrally loaded with 4 tons. ($E=11{,}600$ tons/sq. in.) *Ans.* 3.9 ins.

15. A laminated spring of 75 ins. span has thirteen plates, each .39 inch thick and 3.5 ins. wide. Calculate the deflection when centrally loaded with 1 ton. ($E=11{,}600$ tons/sq. in.) *Ans.* 5 ins.

16. The section through the back of a hook is a trapezium with the wide side inwards. The narrow side is 1 in. and the wide side 2 ins.; the depth of the section is 2½ ins.; the line of pull is 1½ ins. from the wide side of the section. Calculate the load on the hook that will produce a tensile skin stress of 7 tons/sq. in. *Ans.* 3.87 tons.

17. A steel 3″×¾″ eye-bar has a 4-in. pin-hole and an 8-in. head. The stress along the bar is 30,000 lbs., but its line of action is ½ in. from the axis of the bar. Find the maximum and minimum stresses in the main body of the bar, and also in the metal at a diametral section through the pin-hole at right angles to the line of stress. *Ans.* 13,333⅓ lbs.; 0; 12,142$\frac{6}{7}$, 7857$\frac{4}{7}$ lbs.

18. The direction of the 30,000 lbs. pull on a 6″×1″ eye-bar is parallel to and 1 in. from the axis. Find the maximum and minimum stresses in the material. *Ans.* 10,000 lbs. and 0.

19. If the pin-holes for a bridge eye-bar were drilled out of truth sideways and the main body of the bar were 5 ins. wide and 2 ins. thick, what proportion would the maximum stress bear to the mean over any cross-section of the bar at which the mean line of force was ⅛ in. from the middle of the section?

Ans. 23 to 20.

20. Calculate the maximum and minimum stresses in a member of circular section 1½ ins. in diameter when subjected to a pull of 6000 lbs. and a bending moment of 4000 in.-lbs. *Ans.* 15,462 lbs. and 8674 lbs./sq. in.

21. A bar of steel 2½″×⅝″ in section is subjected to a pull of 15 tons. The mean line of the load passes ¼ in. from the centre of gravity of the section in the direction of the width of the bar. Find the greatest and least stress on the bar. *Ans.* 34.82 and 21 tons/sq. in.

22. The horizontal section of a crane-hook is a rectangle 3.5 ins. in width, the thickness of the hook being 1.3 ins. The stress in a horizontal section is not to exceed 6 tons/sq. in. Find the maximum load which can be raised, the horizontal distance between the centre of the section and the line of action of the load being 2.5 ins. *Ans.* 5.614 tons.

23. In a steel $8'' \times 1\frac{1}{4}''$ eye-bar it was found that the line of action of a stress of 160,000 lbs. was .25 in. from the centre. Calculate the extra stress produced by this. *Ans.* 3000 lbs./sq. in.

24. An iron bar is rectangular in section, its width being 3 ins. and its thickness 1 in. The tensile strength of the metal is 50,000 lbs./sq. in. Find the total tensile force which will break the bar, the line of action of the pull being $\frac{3}{8}$ in. distant from the axis of the bar. *Ans.* $85{,}714\frac{2}{7}$ lbs.

25. A tension bar 8 ins. wide and $1\frac{1}{4}$ ins. thick, is slightly curved in the plane of its width, so that the mean line of the stress passes 2 ins. from the axis at the middle of the bar Calculate the maximum and minimum stresses in the material. Total load on bar, 25 tons. *Ans.* $6\frac{1}{4}$ and $1\frac{1}{4}$ tons/sq. in.

26. A tension bar of T section has a $2'' \times \frac{1}{2}''$ flange and a $4'' \times \frac{1}{2}''$ web. The line of action of the pull P on the bar is 1 in. from the centre of gravity of the section. Find the maximum and minimum stresses developed in the metal. *Ans.* $\frac{229}{291}P$ and $\frac{85}{291}P$ or $\frac{181}{291}P$ and $\frac{11}{291}P$.

27. A horizontal beam of weight P rests at the ends in recesses. A weight W suspended from the centre of the beam by a string 3 ft. in length makes 60 revolutions per minute. Find the minimum value of P so that the beam may not rise out of the recesses. *Ans.* $\frac{98}{363}P$.

28. Find the skin stress due to bending in a connecting-rod from the following data: Radius of crank, 10 ins.; length of rod, 4 ft.; diameter of rod, 3 ins.; number of revolutions per minute, 120.

29. The coupling-rod of a locomotive is 10 ft. in length, the crank-radius is 1 ft. in length, and the driving-wheels are 6 ft. in diameter. What must be the depth of the rod if the stress in it is not to exceed 5 tons per square inch at 70 miles an hour? *Ans.* 11.28 ins.

30. A locomotive has two pairs of 6-ft. driving-wheels and 8-ft. coupling-rods 6 ins. deep, of wrought iron, capable of withstanding a working load of 20,000 lbs. per square inch. The cranks are 12 ins. in length. Find the speed corresponding to the greatest centrifugal force. *Ans.* 90.6 miles per hour.

31. A shaft $5\frac{1}{2}$ ins. deep × 5 ins. wide × 98 ins. long has one end absolutely fixed, while at the other a wheel turns at the rate of 270 revolutions per minute; a weight of 20 lbs. is concentrated in the rim, its centre of gravity being $2\frac{1}{2}$ ft. from the axis of the shaft. Find the maximum stress in the material of the shaft, and also find the maximum deviation of the shaft from the straight, E being 27,000,000 lbs. *Ans.* 4860 lbs. per/sq. in.; .31431 in.

32. An iron bar 12 ft. long and 6 ins. deep is held freely at the ends and rapidly rotated, every point in its axis describing a circle of 24 ins. radius. Find the number of revolutions per minute for which the maximum intensity of stress is 40,000 lbs./sq. in.

33. A 2-in. wrought-iron bar 10 ft. long is held at the ends and is whirled about a parallel axis at the rate of 50 revolutions per minute. If the distance between the axis of the bar and the axis of rotation is 10 ft., find the maximum stress to which the material is subjected. *Ans.* 17,148.5 lbs./sq. in.

34. A steel coupling-rod for a locomotive with 6-ft. drivers and a crank-

radius of 1 ft. is 10 ft. long and has an I cross-section with a $4'' \times \frac{1}{2}''$ web and equal $2'' \times \frac{1}{2}''$ flanges. Find the maximum intensity of stress for a speed of 80 miles per hour. *Ans.* 20,265 lbs./sq. in.

35. The coupling-rod of a locomotive is of uniform I section and of the following dimensions: Depth = 6 ins., width = 3 ins., thickness of web = $1\frac{1}{2}$ ins., thickness of flanges = 1 in., length between centres = 91 ins. The driving-wheels have a diameter of 82 ins. and the crank-throw is 14 ins. Assuming that the rod weighs .3 lb./cu. in., find the maximum stress due to bending when the engine runs at 70 miles per hour. *Ans.* 9493 lbs./sq. in.

36. Find the bending stress in the middle section of a coupling-rod of rectangular section from the following data: Radius of coupling-crank, 11 ins.; length of coupling-rod, 8 ft.; depth of coupling-rod, 4.5 ins.; width, 2 ins.; revolutions per minute, 200.

If the rod has an I section with a $3'' \times 1''$ web, what will the bending stress be?

37. Assuming that one half of the force exerted by the steam on one piston of a locomotive is transmitted through a coupling-rod, find the maximum stress in the two rods. Diameter of cylinder, 16 ins.; steam-pressure, 140 lbs./sq. in. *Ans.* 482.7 lbs./sq. in.

38. Determine the isosceles section of maximum strength which can be cut out of a circular section of given diameter, and compare the strengths of the two sections. *Ans.* $H = \frac{5}{6} \times$ diam.; $175\sqrt{5}$ to 1782.

39. Show that the moments of resistance of an elliptic section and of the strongest rectangular section that can be cut out of the same are in the ratio of $99\sqrt{3}$ to 112, and that the areas of the sections are in the ratio of 33 to $14\sqrt{2}$.

40. Show that the moments of resistance of an isosceles triangular section and of the strongest rectangular section that can be cut out of the same are in the ratio of 27 to 16, and that the areas of the two sections are in the ratio of 9 to 4.

41. Determine the dimensions of the strongest section in the form of (*a*) a rectangle with vertical sides, (*b*) an isosceles triangle with horizontal base, that can be cut out of an elliptical section having a vertical major axis of length $2p$ and a minor axis of length $2q$.

Ans. (*a*) depth of rect. $= p\sqrt{3}$; width of rect. $= q$.

(*b*) depth of triangle $= \frac{7}{4}p$; base of triangle $= \dfrac{q}{2}\sqrt{7}$.

42. Determine the isosceles section of maximum strength which can be cut out of a circular section of given diameter, and compare the strengths of the two sections. *Ans.* $H = \frac{5}{6} \times$ diam.; $175\sqrt{5} : 1782$.

43. A round and a square beam of equal length and equally loaded are to be of equal strength. Find the ratio of the diameter to the side of the square. *Ans.* $\sqrt[3]{56} : \sqrt[3]{33}$.

44. Compare the relative strengths of two rectangular beams of equal length, the breadth (*b*) and depth (*d*) of one being the depth (*b*) and breadth (*d*) of the other. *Ans.* $d^2 : b^2$

45. Compare the uniformly distributed loads which can be borne by two

beams of rectangular section, the several linear dimensions of the one being n times the corresponding dimensions of the other. Also compare the moments of resistance of corresponding sections. *Ans.* n^2; n^3.

46. Compare the moments of resistance to bending of a rectangular section and of the rhomboidal and isosceles sections which can be inscribed in the rectangle, the base of the triangle being the lower edge of the rectangle. *Ans.* 4:1:1 or 4:1:2.

47. Compare the relative strengths of two beams of the same length and material (a) when the sections are *similar* and have areas in the ratio of 1 to 4; (b) when one section is a circle and the other a square, a side of the latter being equal to the diameter of the former. *Ans.* (a) 1 to 8; (b) 56 to 33.

48. Compare the strength of a cylindrical beam with the strength of the strongest (a) rectangular and (b) square beam that can be cut from it.

Ans. (a) $112:99\sqrt{3}$; (b) $33:14\sqrt{2}$.

49. There are two beams of the same sectional area, the one A being circular and of radius r, the other B being square. A hollow square of side r is cut through the middle of A. What must be the radius of a hollow round through the middle of B, so that both sections may have the same moment of resistance? The sides of the squares are vertical. *Ans.* $r \times .626$.

50. A triangular knife-edge of a weighing-machine overhangs $1\frac{1}{2}$ inches and supports a load of 2 tons (assume evenly distributed). Taking the triangle to be equilateral, find the requisite size for a tensile stress at the apex of 10 tons per square inch. *Ans.* 1.53 ins.

51. A circular link of a chain having an internal diameter of 3 ins. is made of iron having a diameter of 1 in. and one side is cut through. Determine the maximum and minimum stresses in the link when the chain is loaded with 1000 lbs. *Ans.* 21,636 lbs./sq. in. and 19,091 lbs./sq. in.

52. A bar of larch 6 ft. long×2 ins. square, resting on supports at the two ends, fails under a load of 515 lbs. at the centre. Find the breadth of each of two cantilevers, of 4 ft. length by 10 ins. depth, made of this material, which are to carry a cistern of water weighing 2 tons, 5 being a factor of safety. *Ans.* 4.15 ins.

53. A rectangular beam of pine 8.9 ins. deep by 5 ins. wide, of uniform section throughout, is supported horizontally on two walls 15 ft. apart. What weight will the beam safely carry at 5 ft. from one of the walls, the breaking load being four times the safe load? How much must the depth be increased, the breadth remaining the same, if the load is shifted to the middle of the beam? (Assume that the breaking weight of a pine bar 15 ins. long by 1 in. by 1 in., supported at both ends, fails under a load of 360 lbs. at the centre.) *Ans.* 3342 lbs.; new depth = 9.44 ins.

54. What must be the thickness of a mild-steel tube of 10 ins. external diameter to carry a load of 10,000 lbs., placed centrally between two supports 10 ft. apart, the safe working unit stress being 10,000 lbs./sq. in.?

Ans. .38 in.

55. A block of ice 3 ins. wide and 4 ins. deep has its ends resting upon supports 30 ins. apart and carries a uniformly distributed load of 4800 lbs. An increase of pressure to the extent of 1125 lbs./sq. in. lowers the freezing-

point 1° F. Assuming that the ordinary theory of flexure holds good, find the temperature of the ice. *Ans.* 30° F.

56. A beam supported at the ends can just bear its own weight W together with a single weight $\frac{W}{2}$ at the centre. What load may be placed at the centre of a beam whose transverse section is similar but m^2 as great, its length being n times as great? If the beam could support only its own weight, what would be the relation between m and n? *Ans.* $W\left(\frac{m^3}{n}-\frac{nm^2}{2}\right)$; $m=\frac{n^2}{2}$.

57. A wooden beam 12 ins. deep, 6 ins. wide, and 12 ft. long is embedded in a wall at one end. What weight will the beam carry at the outer end if the breaking weight of a beam 1 ft. long, 1 in. broad, and 1 in. deep, supported at the ends and loaded at the centre, is 500 lbs? *Ans.* 9000 lbs.

58. A spruce-tree 60 ft. high and 14 ins. in diameter at the ground level is subjected to a horizontal wind pressure which may be assumed to be uniformly distributed over the upper 30 ft. of the tree. Find the intensity of this pressure if the maximum fibre stress at the base is 1080 lbs./sq. in. *Ans.* 18 lbs./lin. ft.

59. Find the stress in tons/sq. in. at the skin and also at a point 2 ins. from the neutral axis in a piece of 10″×8″ oak, (*a*) with the 10-in. side vertical, (*b*) with the 8-in. side vertical. The oak rests upon supports 3 ft. apart and carries a load of 4900 lbs. at its middle point. Also compare (*c*) the strength of the beam with its strength when a *diagonal* is horizontal.

Ans. (*a*) $330\frac{3}{4}, 132\frac{3}{10}$; (*b*) $413\frac{7}{16}, 206\frac{23}{32}$; (*c*) $4:\sqrt{41}$ or $5:\sqrt{41}$.

60. In the case of a tram-rail, the area of the modulus figure is 8.24 sq. ins., and the distance between the two centres of gravity is 5.55 ins.; the neutral axis is situated at a distance of 3.1 ins. from the skin of the bottom flange. Find the I and Z. *Ans.* 70.9; 22.87.

61. Show that the *modulus of rupture* of any material is 18 times the load which will break a beam 12 ins. long, 1 in. deep, and 1 in. wide when applied at the centre.

62. Find the limiting length of a wrought-iron cylindrical beam 4 ins. in diameter, the modulus of rupture being 42,000 lbs. What uniformly distributed load will break a cylindrical beam of the same material 20 ft. long and 4 ins. in diameter? *Ans.* 64.8 ft.; 8800 lbs.

63. A red-pine beam 18 ft. long has to support a weight of 10,000 lbs. at the centre. The section is rectangular and the depth is twice the breadth. Find the transverse dimensions, the modulus of rupture being 8500 lbs. and 10 being a factor of safety. (Neglect the weight of the beam.)

Ans. $b=9.84$ ins.; $d=19.68$ ins.

64. A round oak cantilever 10 ft. long is just broken by a load of 600 lbs. suspended from the free end. Find its diameter, the modulus of rupture being 10,000 lbs. (Neglect the weight of the beam.) *Ans.* 4.185 ins.

65. Determine the breaking weight at the centre of a cast-iron beam of 6 ft. span and 4 ins. square, the coefficient of rupture being 30,000 lbs.

Ans. $26,666\frac{2}{3}$ lbs.

66. The flooring of a corn warehouse is supported upon yellow-pine joists

20 ft. in the clear, 8 ins. wide, 10 ins. deep, and spaced 3 ft. centre to centre. Find the height to which corn weighing 48½ lbs./cu. ft. may be heaped upon the floor, 10 being a factor of safety and 3000 lbs. the coefficient of rupture. *Ans.* .68 ft.

67. A yellow-pine beam 14 ins. wide, 15 ins. deep, and resting upon supports 126 ins. apart broke down under a uniformly distributed load of 60.97 tons. Find the coefficient of rupture. *Ans.* 2731.456.

68. Find the breaking weight at the centre of a Canadian ash beam 2½ ins. wide, 3½ ins. deep, and of 45 ins. span, the coefficient of rupture being 7250. *Ans.* 4934$\frac{1}{36}$ lbs.

69. A timber beam 6 ins. deep, 3 ins. wide, 96 ins. between supports, and weighing 50 lbs./cu. ft. broke down under a weight of 10,000 lbs. at the centre. Find the coefficient of rupture. *Ans.* 8911$\frac{1}{9}$.

70. A wrought-iron bar 2 ins. wide, 4 ins. deep, and 144 ins. between supports carries a uniformly distributed load W in addition to its own weight. Find W, 4 being a factor of safety and 50,000 lbs. the coefficient of rupture. *Ans.* 5235$\frac{5}{9}$ lbs.

71. Find the length of a beam of Canadian ash 6 ins. square which would break under its own weight when supported at the ends. The coefficient of rupture = 7000 lbs., and the weight of the timber = 30 lbs./cu. ft. *Ans.* 274.95

72. The teeth of a cast-iron wheel are 3½ ins. long, 2¼ ins. deep, and 7 ins. wide. What is the breaking weight of a tooth, the coefficient of rupture being 5000 lbs? *Ans.* 50,625 lbs.

73. A wrought-iron bar 4 ins. deep, ¾ in. wide, and rigidly fixed at one end gave way at 32 ins. from the load when loaded with 1568 lbs. at the free end. Find the coefficient of rupture. *Ans.* 4181⅓.

74. A cast-iron beam is 12 ins. wide, rests upon supports 18 ft. apart, and carries a 12-in. brick wall which is 12½ ft. in height and weighs 112 lbs./cu. ft. Taking 63,000 as the modulus of rupture for a uniformly distributed load and 5 as a factor of safety, find the depth of the beam, (*a*) neglecting its weight, (*b*) taking its weight into account.

Also (*c*) determine the depth of a cedar beam which might be substituted for the cast-iron beam, taking 11,200 lbs. as the modulus of rupture for the cedar. *Ans.* (*a*) 6 ins.; (*b*) 6½ ins.; (*c*) 14.23 ins.

75. A cast-iron girder 27½ ins. deep rests upon supports 26 ft. apart. Its bottom flange has an area of 48 sq. ins. and is 3 ins. thick. Find the breaking weight at the centre, the ultimate tensile strength of the iron being 15,000 lbs./sq. in. (Neglect the effect of the web.) *Ans.* 253,846$\frac{2}{13}$ lbs.

76. Is it safe for a man weighing 160 lbs. to stand at the centre of a spruce plank 10 ft. long, 2 ins. wide, and 2 ins. thick, supported by vertical ropes at the ends? The safe working strength of the timber is 1200 lbs./sq. in.

Ans. No; the maximum safe weight at the centre is 53⅓ lbs.

77. A timber beam weighing 42 lbs./cu. ft. rests on piers 16 ft. apart; it is 8 ins. thick. What depth must it be made if the deflection in the centre under its own weight and a load of 1 cwt./ft. run is not to exceed ¼ of an inch? (Assume $E = 700$ tons/sq. in.)

78. A piece of timber 10 ft. long, 12 ins. deep, 8 ins. wide, and having a working strength of 1000 lbs./sq. in. carries a load, including its own weight, of w lbs./lin. ft. Find the value of w, (*a*) when the timber acts as a cantilever (*b*) when it acts as a beam supported at the ends. Find (*c*) stress in material 3 ins. from neutral axis at fixed end of cantilever and at middle of beam.

Ans. (*a*) 320 lbs.; (*b*) 1280 lbs.; (*c*) 500 lbs./sq. in.

79. A cast-iron beam of square cross-section 1 in. deep×1 in. wide is tested on a span of 36 ins. The breaking load in the centre is $7\frac{1}{2}$ cwt. Calculate (*a*) the maximum intensity of tensile stress, assuming the beam formula to hold up to the breaking-point, (*b*) the probable deflection in the centre You may assume $E=12{,}000{,}000$ lbs./sq. in.

Ans. (*a*) 22,680 lbs./sq. in.; (*b*) .8165 in.

80. A yellow-pine beam 14 ins. wide, 15 ins. deep, and resting upon supports 129 ins. apart was just able to bear a weight of 34 tons at the centre. What weight will a beam of the same material of 45 ins. span and 5 ins. square bear?

Ans. 4.133 tons.

81. A horizontal circular tube of steel is 7 ft. in diameter, $\frac{1}{8}$ in. thick, 100 ft. long supported at the ends, its total load distributed uniformly all over being 30 tons; what are the greatest stresses in the metal? The tube is filled with compressed air; what must its pressure be if there is just no compressive stress in the metal? State what is now the nature of the stress in the metal at the place where it is greatest. *Ans.* $6\frac{1}{2}$ tons/sq. in.; 38.6 lbs./sq. in.

82. An oak beam of circular section and 22 ft. long is strained to the elastic limit (2 tons/sq. in.) by a uniformly distributed load of $2\frac{1}{24}$ tons. Find the diameter of the beam. What load 2 ft. from one end would strain the material to the same limit? *Ans.* 7 ins.; 3.088 tons.

83. A uniform beam of weight W_1 crossing a given span can bear a uniformly distributed load W_2. What load may be placed upon the same beam if it crosses the span in n equal lengths supported at the joints by piers whose widths may be disregarded? *Ans.* $n^2(W_1+W_2)-W_1$.

84. A cast-iron water-main 30 ins. inside diameter and 32 ins. outside is unsupported for a length of 12 ft. Find the stress in the metal due to bending.

Ans. 180 lbs./sq. in.

85. A wrought-iron bar $1\frac{1}{2}$ ins. wide and 20 ft. long is fixed at one end and carries a load of 500 lbs. at the free end. Find the depth of the bar, so that the stress may nowhere exceed 10,000 lbs./sq. in.

Ans. 6.928 ins.; if weight of bar is included, the depth d is given by $d^2-4.8d-48=0$, and $d=9.73$ ins.

86. Determine the diameter of a solid round wrought-iron beam resting upon supports 60 ins. apart and about to give way under a load of 30 tons at 14 ins. from one end. Take 5 as a factor of safety and 8960 lbs./sq. in. as the safe working intensity of stress.

Ans. 5.47 ins.; if weight of beam is taken into account, d is given by $2019584+196\frac{7}{9}d-12320d^3=0$.

87. A yellow-pine beam 14 ins. wide and 15 ins. deep was placed upon supports 10 ft. 9 ins. apart and deflected $\frac{2}{5}$ in. under a load of 20 tons at the centre. Find E, neglecting the weight of the beam. What were the intensities of

the normal and tangential stresses at 2 ft. from a support and $2\frac{1}{2}$ ins. from neutral plane upon a plane inclined at 30° to the axis of the beam?

Ans. 1,272,112 lbs.; 132.83 and 218.91 lbs.

88. A beam of uniform section of depth d, with equal flanges and of span l, is built into walls so that its ends are horizontal and at the same level. One of the walls settles a distance δ without disturbing the horizontality of the ends of the beam. Show that, due to settling, the maximum stress induced is $\frac{3E\delta d}{l^2}$, E being Young's modulus.

89. A stress of 1 lb. /sq. in. produces a strain of $\frac{1}{2000000}$ in a beam 12 ins. square and 20 ft. between supports. Find the radius of curvature and the central deflection under a load of 2000 lbs. at the middle point.

Ans. 2400 ft.; $\frac{1}{4}$ in.

90. A bar of wrought iron 3 ins. broad and $1\frac{1}{4}$ ins. thick is supported in a horizontal position at two points $2\frac{1}{2}$ ft. apart What deflection at the middle will be caused by placing there a load of 15 cwts.? *Ans.* .065 in.

91. A $2'' \times 4''$ beam 20 ft. long has both ends fixed and is loaded at the centre with a weight of 400 lbs. Find the deflection at the centre. How much work is done in bending the beam? (Take $E = 28,000,000$ lbs./sq. in.)

Ans. .097 in.; $7405\frac{5}{7}$ in.-lbs.

92. A steel girder 20 ft. long between supports is loaded with 3000 lbs. per foot run. Calculate what its moment of inertia must be, so that the deflection at the centre is $\frac{1}{1200}$ of the span, the value of E in pounds and inches being taken as 30,000,000. Also calculate the slope at the ends of such a beam. *Ans.* 1800; 9.16 minutes.

93. A beam 8 ins. wide and weighing 50 lbs. /cu. ft. rests upon supports 30 ft. apart. Find its depth so that it may deflect $\frac{3}{4}$ in. under its own weight. ($E = 1,200,000$ lbs.) *Ans.* 9.185 ins.

94. A rectangular girder of given length l and breadth b rests upon two supports and carries a weight P at the centre. Find its depth so that the elongation of the lowest fibres may be $\frac{1}{1400}$ of the original length.

Ans. $\sqrt{\frac{2100Pl}{bE}}$.

95. A beam AB, 60 ft. long, is fixed horizontally at A and hinged at a point C 40 ft. from A, and the other end B is supported on a pier at the same level as A. The beam carries a uniformly distributed load of 2000 lbs./ft. run over the whole of its length. Determine the shearing force and bending moment at every point. Also find the deflection and slope at C.

Ans. Reaction at $A = 50$, at $B = 10$ tons; B.M. at $A = 1200$ in.-tons.

Slopes at C for $AC = \frac{2688000}{EI}$; for $BC = \frac{192000}{EI}$; deflection of $C = \frac{92160000}{EI}$.

96. Deduce expressions for the slope and deflection at the end of a cantilever of rectangular section having a length of 60 ins., a breadth of 2 ins., and a depth of 3 ins., when loaded uniformly with 100 lbs. /ft. run. ($E = 30,000,000$.)

Ans. $\frac{1}{450}$; $\frac{1}{10}$ in.

97. An angle-iron $3'' \times 3'' \times \frac{9}{16}''$ was placed upon supports 12 ft. 9 ins. apart and deflected $1\frac{1}{2}$ ins. under a load of 8 cwts. uniformly distributed and 2 cwts. at the centre. Find E and the position of the neutral axis.

Ans. $E = 16{,}079{,}611$ lbs.; neutral axis $\frac{895}{928}$ in. from upper face.

98. A yellow-pine beam 14 ins. wide, 15 ins. deep, and weighing 32 lbs./cu. ft. was placed upon supports 10 ft. 6 ins. apart. Under uniformly distributed loads of 59,734 lbs. and of 127,606 lbs. the central deflections were respectively .18 in. and .29 in. Find the mean value of E. Also determine the additional weight at the centre which will increase the first deflection by $\frac{1}{10}$ of an inch.

Ans. 2,552,980 lbs.; 24,121 lbs.

99. For the load of 59,734 lbs., find the maximum intensities of thrust, tension, and shear at a point half-way between the neutral axis and the outside skin in a transverse section at one of the points of trisection of the beam. Also find the inclinations of the planes of principal stress at the point.

Ans. 800.01, 3.55, 401.78 lbs.; $\theta = 3° \, 48\frac{3}{4}'$.

100. If the shape of the section and the load are given, and the linear dimensions of section are chosen so as to make the greatest stress a given quantity, show that the deflection at centre is proportional to $(\text{length})^{\frac{5}{3}}$.

101. A steel rectangular girder 2 ins. wide, 4 ins. deep is placed upon supports 20 ft. apart. If E is 35,000,000 lbs., find the weight which if placed at the centre will cause the beam to deflect 1 in. *Ans.* $1296\frac{8}{10}$ lbs.

102. A cylindrical beam of 2 ins. diameter, 60 ins. in length, and weighing $\frac{1}{4}$ lb./cu. in. deflects $\frac{8}{10}$ in. under a weight of 3000 lbs. at the centre. Find E.

Ans. $E = 21{,}688{,}210$ lbs.

103. A $3'' \times 3'' \times \frac{1}{2}''$ angle-iron, with both ends fixed and a clear span of 20 ft., carries a uniformly distributed load of 500 lbs., which causes it to deflect .03909 in. Find E. What single load at the centre will produce the same deflection? Find the work done due to bending in each case.

Ans. $E = 20{,}775{,}415$ lbs.; 250 lbs.

104. Show that the work done in deflecting, within the elastic limits, a uniform rectangular bar, supported at the ends and loaded in the middle, is equal to the volume of the bar multiplied by $\frac{1}{18}\frac{f^2}{E}$, f being the maximum stress in the bar.

105. A girder fixed at both ends carries $(2n+1)$ weights W concentrated at points dividing the length of the girder into $2n+2$ equal divisions. Find the total central deflection.

Ans. $\frac{n+1}{192}\frac{Wl^3}{EI}$.

106. The deflection of a uniformly loaded horizontal beam supported at the ends is not to exceed 1 in. in 50 ft. of span, and the stress in the material is not to exceed 400 lbs./sq. in. Find the ratio of span to depth, E being 1,200,000 lbs./sq. in., and the neutral axis being at half the depth of the beam.

Ans. 20.

107. A round wrought-iron bar l ft. long and d ins. in diameter can just carry its own weight. Find l in terms of d, (a) the allowable deflection being 1 in./per 100 ft. of span, E being 30,000,000 lbs.; (b) the allowable stress

being 8960 lbs./sq. in.; (*c*) the stiffness given by (*a*) and the strength given by (*b*) being of equal importance.

Ans. (*a*) $l=\sqrt[3]{250d^2}$; (*b*) $l=\sqrt{224d}$; (*c*) $l=\frac{25}{18}d$.

108. A square steel bar l ft. long and having a side of length d ins. can just carry its own weight; its stiffness is $\frac{1}{1200}$ and the maximum allowable working stress is 7 tons/sq. in. Find l in terms of d, E being 13,000 tons.

Ans. $\dfrac{l \text{ (in ft.)}}{d \text{ (in in.)}}=\dfrac{13}{21}$.

109. A beam is supported at the ends and bends under its own weight. Show that the upward force at the centre which will exactly neutralize the bending action is equal to $\frac{5}{8}$ or $\frac{1}{2}$ of the weight of the beam (w), according as the ends are *free* or *fixed*. Find the neutralizing forces at the quarter spans.

Ans. Ends free: $\frac{57}{128}w$ at *each* or $\frac{19}{56}w$ at *one* of the points of division.

Ends fixed: $\frac{9}{20}w$ at *each* or $\frac{2}{3}w$ at *one* of the points of division.

110. A horizontal girder AB of length l, and fixed at A and at B, carries two weights P and Q, concentrated at points C and D respectively. If $AC=X$ and $BD=Y$, show that the bending moment at C is greater or less than the bending moment at D according as $PX^2(3l-2X) \gtrless QY^2(3l-2Y)$.

111. Show that the work done in bending a horizontal beam is the same whether it has two ends fixed, or one end fixed and one resting upon its support, or two ends resting upon supports, if the load intensities in the several cases are in the ratio of $2\sqrt{6}$ to 3 to 2.

112. A horizontal girder AC is fixed at A, rests upon the support at C, carries a uniformly distributed load of intensity w, and is hinged at B, dividing the girder into segments $AB=a$ and $BC=b$. Find (1) the reactions at A and C, (2) the moment of fixture, (3) the deflection at B.

Ans. (1) $w\left(a+\dfrac{b}{2}\right)$; $w\dfrac{b}{2}$; (2) $\dfrac{wa}{2}(a+b)$; (3) $\dfrac{wa^3}{EI}\left(\dfrac{a}{8}+\dfrac{b}{6}\right)$.

113. A horizontal beam with both ends absolutely fixed is loaded with a weight W at a point dividing the span into two segments a and b. Show that the deflection at the point is $\dfrac{W}{3EI}\left(\dfrac{ab}{a+b}\right)^3$, and find the work done in bending the beam.

Ans. $\dfrac{W^2}{6EI}\left(\dfrac{ab}{a+b}\right)^3$.

114. A girder with both ends fixed carries two equal loads W at points dividing the girder into segments a, b, c. Determine the reactions and bending moments at the supports.

Ans.

$$R_1=W\frac{3ab^2+b^3+6abc+3b^2c+2c^3+6ac^2+6bc^2}{(a+b+c)^3};$$

$$R_2=W\frac{2a^3+6a^2b+3ab^2+b^3+6a^2c+6abc+3b^2c}{(a+b+c)^3};$$

$$M_1=W\frac{2a^2c+2abc+bc^2+ab^2}{(a+b+c)^2};$$

$$M_2=W\frac{2ac^2+2abc+a^2b+b^2c}{(a+b+c)^2}.$$

115. A bridge a ft. in the clear is formed of two cantilevers which meet in the centre of the span and are connected by a bolt capable of transmitting a vertical pressure from the one to the other. A weight W is placed at a distance b from one of the abutments. Find the pressure transmitted from one cantilever to the other, and draw the curve of bending moments for the loaded cantilever.

Ans. $R_1 = W\left(1 - 3\frac{b^2}{a^2} + 2\frac{b^3}{a^3}\right)$; $R_2 = W\left(3\frac{b^2}{a^2} - 2\frac{b^3}{a^3}\right)$.

116. Two weights P and Q ($<P$) are carried by a horizontal girder of length l resting upon supports at the ends, the distance between the weights being a. Place the weights so as to throw a maximum bending moment on the girder and find the value of this moment. Also find the corresponding work of flexure.

Ans. Distance of P from support $= \dfrac{Pl + Q(l-a)}{2(P+Q)}$.

Max. B.M. $= \dfrac{\{Pl + Q(l-a)\}^2}{4(P+Q)l}$.

117. The section of a cantilever of length l is an ellipse, the major axis (*vertical*) being twice the minor axis. Find the deflection at the end under a single weight W, f being the coefficient of working strength and E the coefficient of elasticity.

Ans. $\left(\dfrac{297}{3500}\dfrac{f^4 l^5}{E^3 W}\right)^{\frac{1}{3}}$.

118. A bar of wrought iron 4 ft. long, 4 ins. deep, and 1½ ins. thick is fixed at one end and loaded with 2000 lbs. at the free end. Find the maximum slope and the maximum deflection, E being 28,000,000 lbs.

119. A horizontal girder AB, of length l, and fixed at A and B, carries a load P at a point C. If $AC = a$, find the maximum deflection.

Ans. $\dfrac{2P(l-a)^3 a^2}{3EI(3l-2a)^2}$.

120. A horizontal girder AB, of length l, is fixed at A, and rests upon its support at B. It carries a weight P at a point C, and $AC = a$. Find the position of the most deflected point, and show that the bending moment at C is greatest when $a = l \times .634$.

Ans. Distance of most deflected point from point of fixture $= l\left(\dfrac{l-a}{3l-a}\right)^{\frac{1}{2}}$.

121. A piece of greenheart 142 ins. between supports, 9 ins. deep, and 5 ins. wide was tested by being loaded at two points, distant 23 ins. from the centre, with equal weights. Under weights at each point of 4480 lbs., 11,200 lbs., and 17,920 lbs. the central deflections were .13 in., .37 in., .67 in., respectively. Find the mean coefficient of elasticity. The beam broke under a load of 32,368 lbs. at each point. Find the coefficient of bending strength.

Ans. 8,508,000 lbs./sq. in.; 23,017 lbs./sq. in.

122. A beam of span l is uniformly loaded. Compare its strength and stiffness (a) when merely resting upon supports at the ends; (b) when fixed at one end and resting upon a support at the other; (c) when fixed at both ends. In case (c) two hinges are introduced at points distant x from the centre,

show that the *strength* of the beam is economized to the best effect when $x=\frac{l}{2\sqrt{2}}$, and that the *stiffness* is a maximum when $x=\frac{l}{4}$ very nearly.

Ans. 3:3:2; 5:2.08:1.

123. A steel-plate beam of uniform section and 30 ft. span has both ends fixed and is freely hinged at the points of trisection. Determine the neutral axis (*a*) for a uniformly distributed load of 6000 lbs.; (*b*) for a single load of 10,000 lbs. concentrated, *first*, $7\frac{1}{2}$ ft. and, *second*, 15 ft. from a support.

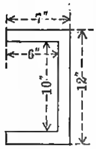


FIG. 542.

124. A channel of the dimensions shown by Fig. 542 rests upon supports 20 ft. apart. If the stress developed in the material is not to exceed 9000 lbs./sq. in., find the allowable uniformly distributed load when *AA* is (*a*) vertical, (*b*) horizontal. Also find (*c*) the central deflection in each case.

Ans. (*a*) 25,400 lbs.; (*b*) 6979 lbs.; (*c*) $\frac{9}{28}$ in., $\frac{19.89}{28}$ in.

125. Find the moment of resistance to bending of a steel I beam, each flange consisting of a pair of $3''\times3''\times\frac{1}{2}''$ angle-irons riveted to a $12''\times\frac{3}{8}''$ web, the coefficient of strength being 5 tons/sq. in. What load will the beam carry at 5 ft. from one end, its span being 20 ft.? Find the central deflection, and also the deflection at the loaded point, *E* being 15,000 tons.

Ans. 287.85 in.-tons; 6.4 tons, disregarding weight of beam, or 6.13 tons, taking weight of beam into account; .4 in. at centre and .3 in. at loaded point.

126. A plate girder of 64 ft. span and 8 ft. deep carries a dead load of 2 tons/lin. ft. At any section the two flanges are of equal area, and their joint area is equal to that of the web. Find the sectional area at the centre of girder, so that the intensity of stress in the metal may not exceed 3 tons/sq. in. The deflection of the girder is $\frac{2}{5}$ in. at the centre. Find *E* and the radius of curvature. *Ans.* 128 sq. in.; 15,360 ft.; 25,804,800 lbs.

127. Taking the coefficient of *direct* elasticity at 15,000 tons, the coefficient of *lateral* elasticity at 60,000 tons, and the limit of elasticity at 10 tons determine the greatest deviation from the straight line of a wrought-iron girder of breadth *b* and depth *d*.

Ans. $\frac{b^2}{24000d}$.

128. An I beam $7\frac{1}{2}$ ins. deep, with a flange $3\frac{1}{2}''\times\frac{3}{8}''$ and web $6\frac{3}{4}''\times\frac{1}{4}''$, was placed on centres 6 ft. apart and tested by loading at the centre. The following readings were obtained:

Load in Pounds.	Deflection in Inches.	Load in Pounds.	Deflection in Inches.	Load in Pounds	Deflection in Inches.
0	0	16,000	.110	28,000	0.460
4,000	.028	20,000	.138	30,000	1.10
8,000	.055	24,000	.167	32,000	2.25
12,000	.082	27,000	.310		

Determine the modulus of elasticity (*E*) before the yield-point is reached, and the maximum stress on the beam when the load is 16,000 lbs.

129. A 9″×9″ beam 168 ins. centre to centre of supports was tested by concentrating gradually increasing loads and the following observations were made:

Load in Pounds.	Deflection in Inches.	Load in Pounds.	Deflection in Inches.	Load in Pounds.	Deflection in Inches.
1,000	.11	9,000	1.32	19,000	Fracture
3,000	.41	12,000	1.79		
6,000	.86	15,000	2.28		

Disregarding weight of beam, find E and skin stress at fracture. Hence indicate the material of the beam.

130. A simplex steel beam with equal flange areas rests upon supports 60 ins. centre to centre, and is tested by being loaded in the centre. The following observations were made:

Load at Centre in Pounds.	Central Deflection in Inches.	Load at Centre in Pounds.	Central Deflection in Inches.	Load at Centre in Pounds.	Central Deflection in Inches.
1,000	.011	5,000	.059	8,000	.094
2,000	.023	6,000	.070	9,000	.107
3,000	.035	7,000	.082	10,000	.119
4,000	.047				

Determine the coefficient of transverse elasticity and also find the maximum tensile and fibre stresses under a load of 10,000 lbs. ($I=13.6$; depth of beam $=5$ ft.)

131. A gas-pipe resting upon supports 45 ins. apart has an external diameter of *one* inch and an internal diameter of *seven tenths* of an inch. The pipe was loaded at the centre and the following observations made:

Load in Pounds.	Deflection in Inches.	Load in Pounds.	Deflection in Inches.	Load in Pounds.	Deflection in Inches.
0	0	40	.062	80	.127
10	.015	50	.078	90	.142
20	.030	60	.094	100	.158
30	.046	70	.110		

Calculate the modulus of elasticity.

132. The following observations were made in the transverse test of a Douglas-fir beam 14.85 ins. deep×6 ins. wide×150 ins. between supports:

Load at Centre in Pounds.	Deflection in Inches.	Load at Centre in Pounds.	Deflection in Inches.	Load at Centre in Pounds.	Deflection in Inches.
1,000	1.525	16,000	2.160	32,000	2.850
2,000	1.570	18,000	2.250	34,000	2.970
4,000	1.650	20,000	2.330	36,000	3.065
6,000	1.740	22,000	2.420	38,000	3.180
8,000	1.820	24,000	2.510	40,000	3.290
10,000	1.900	26,000	2.590	41,700	Failed on tension side
12,000	1.990	28,000	2.680		
14,000	2.060	30,000	2.750	42,000	Sheared longitudinally

Calculate the modulus of elasticity, and the maximum stress developed at the point of failure.

133. From the transverse test of a white-pine beam 9″×9″×14′ between supports the following observations were made:

Load at Centre in Pounds.	Deflection in Inches.	Load at Centre in Pounds.	Deflection in Inches.	Load at Centre in Pounds.	Deflection in Inches.
0	1.51	8,000	2.67	16,000	3.97
2,000	1.76	10,000	2.98	18,000	4.48
4,000	2.06	12,000	3.30	19,000	4.69
6,000	2.37	14,000	3.62		

This beam commenced to fail on the compression side when the load was 17,000 lbs. Calculate the modulus of elasticity and the maximum stress developed before failure took place.

134. A reinforced concrete beam is to support at mid-length a concentrated load of 10,000 lbs., the distance centre to centre of supports being 12 ft. Assuming E to be 29,000,000 lbs. for the steel reinforcing bars and 3,000,000 lbs. for concrete in compression, also that the compressive and tensile strengths of the concrete are 2000 and 200 lbs./sq. in. respectively, determine the section of the beam, the elastic limit of the steel being 50,000 lbs./sq. in.

Ans. Sectional area in square inches of concrete = 99.2 and of steel = .635.

135. Two cantilever girders, the one 4 ft. and the other 6 ft. long, are placed at right angles to each other, with their ends meeting, and support at their intersection a common load. What must be the ratio of moments of inertia in order that the cantilevers may carry equal loads? *Ans.* 8:27.

136. In a transverse test of a slate beam 3.96″×3.97″×66″ between the supports, and with the bed horizontal, the following observations were made:

Central Load in Pounds.	Deflection in Inches.	Central Load in Pounds.	Deflection in Inches.	Central Load in Pounds.	Deflection in Inches.
0	0	3,000	.057	6,000	.110
1,000	.020	4,000	.075	7,000	Failed
2,000	.038	5,000	.095		

Find E and the coefficient of bending strength.

137. In a transverse test of a slate beam 3.99″×3.98″×66″ between supports, and with its bed vertical, the following observations were made:

Central Load in Pounds.	Deflection in Inches.	Central Load in Pounds.	Deflection in Inches.	Central Load in Pounds.	Deflection in Inches.
0	0	4,000	.070	7,000	.122
1,000	.017	5,000	.089	8,000	.140
2,000	.035	6,000	.106	8,500	Failed
3,000	.052				

Find E and the coefficient of bending strength.

138. In a transverse test of an oak beam 2.94 ins. wide×11.4 ins. deep×144 ins. between supports the following observations were made:

Central Load in Pounds.	Deflection in Inches.	Central Load in Pounds.	Deflection in Inches.	Central Load in Pounds.	Deflection in Inches.
2,000	.17	7,000	.64	12,000	1.120
3,000	.26	8,000	.73	13,000	1.230
4,000	.35	9,000	.82	14,000	1.350
5,000	.45	10,000	.92	15,000	1.500
6,000	.55	11,000	1.015	15,500	Failed on tension side

Determine E and the coefficient of bending strength.

Ans. 1,830,000 lbs./sq. in.; 8870 lbs./sq. in.

139. A wrought-iron beam of rectangular section and 20 ft. span is 16 ins. deep, 4 ins. wide, and is loaded with a proof load at the centre. If the proof strength is 7 tons/sq. in., find the proof deflection and the resilience, E being 12,000 tons (1 ton = 2240 lbs.). *Ans.* .029 ft.; 650 ft.-lbs.

140. In a rolled-steel beam (symmetrical about the neutral axis) the moment of inertia of the section is 72 in.-units. The beam is 8 ins. deep, is laid across an opening of 10 ft., and carries a distributed load of 9 tons. Find the maximum fibre stress, also the central deflection, taking E at 13,000 tons/sq. in. *Ans.* $7\frac{1}{2}$ tons/sq. in.; .216 in.

141. A girder 30 m. long has both ends fixed and carries a uniformly distributed load of 5800 k./lineal metre. Find the maximum deflection and the work of flexure. *Ans.* $\frac{4078125000}{EI}$ mm.; $\frac{567675000000}{EI}$ km.

142. A steel beam of circular section is to cross a span of 15 ft. and to carry a load of 10 tons at 5 ft. from one end. Find its diameter, the stiffness being such that the ratio of *maximum deflection* to span is .00125. (E = 13,000 tons.) *Ans.* 10.3 ins.

143. Determine the dimensions of a beam of rectangular section which might be substituted for the round beam in the preceding question, the stiffness remaining the same and the coefficient of working strength being $7\frac{1}{2}$ tons/sq. in. *Ans.* $bd^2=320$.

144. Two equal weights are placed symmetrically at the points of trisection of a beam of uniform section supported at the ends. These weights are then removed and other two equal weights are placed at the quarter spans. Find the ratio of the two sets of weights so that the maximum intensity of stress may be the same in each case. Also show that the *stiffness* of the beam is the same in each case. *Ans.* 3 to 4.

145. A cast-iron beam of an inverted T section rests upon supports 22 ft. apart; the web is 1 in. thick and 20 ins. deep; the flange is 1.2 ins. thick and 12 ins. wide; the beam carries a uniformly distributed load of 99,000 lbs. Find the maximum deflection, E being 17,920,000 lbs.

Ans. .822 in. (I = 1608.65).

146. Find the maximum deflection of a cast-iron cantilever 2 ins. wide×3 ins. deep×120 ins. long under its own weight, E being 17,920,000 lbs.

Ans. $\frac{15}{16}$ in.

147. Calculate the central deflection of a tram rail due to (*a*) bending, (*b*) shear, when centrally loaded on a span of 3 ft. 6 ins. with a load of 10 tons. ($E = 12{,}300$ tons/sq. in.; $I = 80.5$ in.-units; $A = 10.5$ sq. ins.; $G = 4900$ tons/sq. in.; $K = 4.03$.) *Ans.* (*a*) .016 in.; (*b*) .008 in.

148. A beam of 20-ft. span carries a load which varies uniformly in intensity from 0 at one end to 100 lbs. at the other end. Find the work done in bending the beam. *Ans.* $\frac{(2400)^5}{9450EI}$ in.-lbs.

149. A pitch-pine beam 14 ins. wide, 15 ins. deep, and weighing 45 lbs./cu. ft. is placed upon supports 10 ft. 9 ins. apart and carries a load of 20 tons at the centre. Find the deflection and radius of curvature, E being 1,270,000 lbs. What stiffness does this give? What amount of uniformly distributed load will produce the same deflection?

Ans. .361 in.; 3810 ins.; $\frac{1}{358}$; 32 tons.

150. A beam is supported horizontally on two posts, one under each end; C is a point of the beam one fourth of its length from one of the points of support. Compare the curvature at C, supposing the beam to be uniformly loaded, with what it would be if the beam were without weight and the load concentrated at the middle point, the total load in both cases being the same. *Ans.* 3 to 4.

151. A straight bar of wrought iron 1''×1'' section is loaded as a tie bar with 5 tons. It is found that the portion between two points on it 4 ft. apart elongates .019 in. What is the value of E? If the bar be subject to a bending moment of 1800 in.-lbs., what would be the radius of curvature? Find also the greatest stress and deflection if the bar be supported at points 4 ft. apart and centrally loaded with 120 lbs.

Ans. 12,632 tons; 97.466 ft.; 8640 lbs./sq. in.; .1313 in.

152. A cantilever of rectangular section and constant breadth is loaded at the free end. If the curvature is constant show that the greatest stress developed in any section is proportional to the depth of the section and also to the cube root of the distance of the section from the loaded end.

153. A carriage-spring is made up of six plates each ½ in. thick×3 ins. wide and the top plate is 36 ins. in length; determine the initial curvature and dip of this plate. Also find the overlap, the deflection, and the load which will produce a deflection of .64 in., taking $E = 30{,}000{,}000$ lbs./sq. in. and assuming a proof stress of 30,000 lbs./sq. in.

Ans. 250 ins.; .648 ins.; 3 ins.; .002592 ins.; 2470 lbs.

154. A weight of 2500 lbs. is to be the proof load of a carriage-spring made of ⅜''×3'' steel plates. If the proof stress is 30,000 lbs., how many plates will be required, the top plate being 30 ins. in length? *Ans.* 9.

155. Find the load which will deflect 2 ins. a carriage-spring made of ten strips each ⅜''×2½'', the length of the upper plate being 40 ins. and the overlap 2 ins. *Ans.* 3296 lbs.

156. A cantilever of length l, specific weight w, and square in section, a side of the section being $2b$ at the fixed and $2a$ at the free end, bends under

its own weight. Find the slope and deflection of the neutral axis at the free end. Hence, also, deduce corresponding results when the cantilever is a regular pyramid.

Ans. $\dfrac{(b+a)wl^3}{4Eb^3}$; $\dfrac{(b+2a)wl^4}{8Eb^3}$; $\dfrac{wl^3}{4Eb^2}$; $\dfrac{wl^4}{8Eb^2}$.

157. If the section of the cantilever in the preceding example, instead of being square, is a regular figure with *any* number of equal sides, show that the neutral axis is a parabola with its vertex at the point of fixture.

158. Find the slope and deflection at the free end of the following cantilevers when bending under their own weight, l being the length, $2b$ the depth at the fixed end, w the specific weight, and E the coefficient of elasticity:

(*a*) Of constant thickness t and with profile in the form of a trapezoid with the non-parallel sides equal and of depth $2a$ at the free end.

(*b*) Of circular section and with profile in the form of an isosceles triangle.

(*c*) Of constant thickness and with profile in the form of a parabola symmetrical with respect to the axis and having its vertex at the free end.

(*d*) When the depth $2a$ in (*a*) is *nil*, i.e., when the profile is an isosceles triangle.

(*e*) Due to a uniformly distributed load of intensity p over the cantilever (*c*).

(*f*) Due to a weight W at the free end of (*a*).

(*g*) Due to a uniformly distributed load of intensity p over the cantilever (*a*). Hence also deduce the deflection and slope when the depth $2a$ is *nil*.

Ans. (*a*) $\dfrac{wl^3}{2Eb^2}$; $\dfrac{3wl^4}{2E(b-a)^3}\left\{\dfrac{a^2}{b-a}\log_e\dfrac{a}{b}+\dfrac{b^3-ab^2+8a^2b-2a^3}{6b^2}\right\}$.

(*b*) $\dfrac{1}{3}\dfrac{wl^3}{Eb^2}$; $\dfrac{1}{6}\dfrac{wl^4}{b^2E}$. (*c*) $\dfrac{2wl^3}{5Eb^2}$; $\dfrac{4wl^4}{15Eb^2}$.

(*d*) $\dfrac{wl^3}{2Eb^2}$; $\dfrac{wl^4}{4Eb^2}$; (*e*) $\dfrac{p}{2}\dfrac{l^3}{Eb^3t}$; $\dfrac{3}{10}\dfrac{pl^4}{Eb^3t}$.

(*f*) $\dfrac{3}{4}\dfrac{Wl^2}{Eatb^2}$; $\dfrac{3}{2}\dfrac{Wl^3}{Et(b-a)^2}\left\{\dfrac{\log\dfrac{b}{a}}{b-a}-\dfrac{3b-a}{2b^2}\right\}$.

(*g*) $\dfrac{3}{4}\dfrac{pl^3}{Et(b-a)^3}\left\{\log\dfrac{b}{a}-\dfrac{(3b-a)(b-a)}{2b^2}\right\}$;

$\dfrac{3}{4}\dfrac{pl^3}{Et(b-a)^4}\left\{3a\log\dfrac{a}{b}+\dfrac{(2b^2+5ab-a^2)(b-a)}{2b^2}\right\}$;

$\dfrac{3}{4}\dfrac{pl^4}{Etb^3}$; 90°.

159. A vertical row of water-tight sheet-piling 12 ft. high is supported by a series of uprights placed 6 ft. centre to centre and securely fixed at the base. Find the greatest deviation of an upright from the vertical when the water rises to the top of the piling. What will the maximum deviation be when the water is 6 ft. from the top?

Ans. $\dfrac{wbh^5}{30EI}=\dfrac{3110400}{EI}$; $\dfrac{wb}{30EI}(h-c)^5+\dfrac{wbc}{24EI}(h-c)^4=\dfrac{218700}{EI}$.

160. Show that the curved profile of a cantilever of uniform strength designed to carry a load W at the free end is theoretically a *cubical* parabola. Also show that by taking the tangents to the profile at the fixed end as the boundaries of the cantilever a cantilever of *approximately* uniform strength is obtained having a depth at the free end equal to *two thirds* of the depth at the fixed end (breadth to be proportional to depth).

161. Design a wheel-spoke 33 ins. in length to be of *approximately* uniform strength, the intensity of stress being 1000 lbs./sq. in.; the load at the end of the spoke is a force of 1000 lbs. applied tangentially to the wheel's periphery, and the section of the spoke is to be (*a*) *circular*, (*b*) *elliptical*, the ratio of the depth to the breadth being $2\frac{1}{2}$.

Ans. (*a*) Depth at hub =6.952 ins., at periphery =4.634 ins.
(*b*) " " " =9.435 " " " =6.29 "
Breadth at hub =3.774 " " " =2.516 "

162. A beam of 17 ft. span is loaded with 7, 7, 11, and 11 tons at points 1, 6, 11, and 15 ft. from one end. Determine the depths at these points, the beam being of uniform breadth and of *approximately* uniform strength; the coefficient of working strength =2 tons/sq. in., the depth of the section of maximum resistance to bending =16 ins.

$$Ans.\ b=\frac{11358}{1088};\ d_1^2=\frac{277\times16^2}{1262};\ d_2^2=\frac{1067\times16^2}{1262};\ d_3^2=16^2;$$

$$\text{and } d_4^2=\frac{670\times16^2}{1262}.$$

163. Design a timber cantilever of *approximately* uniform strength from the following data: Length =12 ft.; square section; load at free end =1 ton; coefficient of working strength $=\frac{1}{4}$ ton/sq. in. What must be the dimensions at the fixed and free ends so that the cantilever might carry an additional uniformly distributed load of 2 tons?

Ans. Side =15.1 ins. at fixed end and =10 ins. at free end; side =19.1 ins. at fixed end and $=\frac{1}{2}$(19.1 ins.) at free end.

164. Design a cantilever 10 ft. long, of *approximately* uniform strength, to carry a load of 4000 lbs. at the free end, the coefficient of strength being 2000 lbs./sq. in., and the section (*a*) a rectangle of constant breadth and 12 ins. deep at the fixed end; (*b*) a square.

How will the results be modified if it is to carry an additional uniformly distributed load of 4800 lbs.?

Ans.—First. (*a*) b =10 ins., d at free end =6 ins.; (*b*) side $=\sqrt[3]{1440}$ at fixed end and $=\sqrt[3]{426\frac{2}{3}}$ at free end.

Second. (*a*) b =16 ins., d at free end =6 ins.; (*b*) side $=\sqrt[3]{2304}$ at fixed end and $=\sqrt[3]{85\frac{1}{3}}$ at free end.

165. Design a cantilever 10 ft. long, of constant breadth and of *approximately* uniform strength, to carry a uniformly distributed load of 5000 lbs. on the half of the length next the free end, the intensity of stress being 2000 less sq. in., and the section a rectangle 12 ins. deep at the fixed end. What

must the dimensions be if 1000 lbs. are concentrated at 30 ins. from fixed end?

Ans. $b = 9\frac{3}{8}$ ins.; d at centre $= 6.928$ ins.; at free end $= 0$.
$b = 10$ ins.; depth $= 9.48$ ins. at $7\frac{1}{2}$ ft. from free end, $= 6.708$ ins. at centre, and $= 0$ at free end.

166. A gallery 30 ft. long and 10 ft. wide is supported by four $9'' \times 5''$ cantilevers spaced so as to bear equal portions of the superincumbent weight. What load per square foot will the gallery bear, the coefficient of working strength being 700 lbs./sq. in.? Find the depth of cast-iron cantilevers 3 ins. wide which may be substituted for the above, the coefficient of working strength being 2000 lbs./sq. in. How should the depth vary if the cantilevers are to be of uniform strength?

Ans. $10\frac{1}{2}$ lbs.; $4d^2 = 18.9$; variation of depth for cast-iron cantilever is given by $6400d^2 = 121x^2$, x being distance from free end.

167. A span of 60 ft. is crossed by a beam hinged at the points of trisection and fixed at the ends; the beam has a constant breadth of 3 ins. and is to be of *uniform strength;* the intensity of stress is 3 tons/sq. in. Determine the dimensions of the beam when a load of $\frac{1}{20}$ ton per lineal foot covers (*a*) the whole span; (*b*) the centre span.

Ans. (*a*) Depth at support $= 4\sqrt{10}$ ins., at centre $= \sqrt{20}$.
(*b*) " " " $= \sqrt{80}$ " " " $= \sqrt{20}$.

168. The weight of 200 lbs./sq. ft. upon a platform 60 ft. long and 10 ft. wide is equally borne by six cast-iron girders of rectangular section, triangular in profile, 10 ft. long and 3 ins. wide. Find the depth at the fixed end, taking 2 tons/sq. in. as the coefficient of safety.

If $E = 17,000,000$ lbs., find the deflection at the free end.

Ans. 24 ins.; $\frac{8}{85}$ in.

169. Find the limiting length of a cedar cantilever of rectangular section in which the length $= 40 \times$ depth, $w = 36$ lbs./cu. ft., and $f = 1800$ lbs./sq. in.

Ans. 60 ft.

170. A steel cantilever 2 ins. square has an elastic strength of 15 tons/sq. in. What must its limiting length be so that there may be no *set?*

Ans. 23.4 ft.

171. Find the limiting length ($= 64 \times$ depth) of a wrought-iron beam of circular section, the elastic strength being 8 tons/sq. in. What will this length be if a beam of I section having equal flange areas and a web area equal to the joint area of the flanges is substituted for the circular section?

Ans. 84 ft.; 224 ft.

172. A rectangular cast-iron beam having its length, depth, and breadth in the ratio of 60 to 4 to 1 rests upon supports at the two ends. Find the dimensions of the beam so that the intensity of stress under its own weight may nowhere exceed 4500 lbs./sq. in. *Ans.* $l = 128$ ft.; $d = 8\frac{8}{15}$ ft.; $b = 2\frac{2}{15}$ ft.

173. The effective length of the Conway tubular bridge is 412 ft.; the effective depths of a tube at the centre and quarter spans are 23.7 ft. and 22.25 ft. respectively; the sectional areas of the top and bottom flanges are respectively 645 sq. ins. and 536 sq. ins. at the centre and 566 sq. ins. and 461 sq. ins. at the quarter spans; the corresponding sectional areas of the web are 257 sq. ins. and 241 sq. ins. Assume the total load upon a tube to be equiv-

alent to 3 tons/lin. ft., and that the continuity of the web compensates for the weakening of the tension flange by the rivet-holes. Find (*a*) the flange stresses and the deflection at the centre and quarter spans, E being 24,000,000 lbs. What (*b*) will be the increase in the central flange stresses under a uniformly distributed live load of $\frac{3}{4}$ ton/lin. ft.?

Ans. (*a*) 4.59 tons/sq. in., 3.94 tons/sq. in.; 8.56 ins.
1.367 " " 1.158 " " 6.72 "
(*b*) Stresses and deflections are increased in ratio of 3 to 4.

174. Design a wooden cantilever 12 ft. long, of circular section and uniform strength, to carry a uniformly distributed load of 2 tons, the coefficient of working strength being 1 ton/sq. in. Also, find the deflection of the free end.

Ans. Taking fixed end as origin and z being radius in inches at distance x ft. from origin, then $11z^3 = 14(12-x)^2$.

Deflection at end $= \frac{2737.9}{E}$ ins.

175. A beam AB of span l, carrying a uniformly distributed load of intensity w, rests upon a support at B and is imperfectly fixed at A, so that the neutral axis at A has a slope of $\frac{1}{48}\frac{wl^3}{EI}$. The end B is lower than A by an amount $\frac{1}{32}\frac{wl^4}{EI}$. Find the reactions. How much must B be lowered so that the whole of the weight may be borne at A? Find the work done in bending the beam.

Ans. $\frac{21}{32}wl$, $\frac{11}{32}wl$; $\frac{7}{48}\frac{wl^4}{EI}$; $\frac{53}{30720}\frac{w^2l^5}{EI}$.

176. Two angle-irons, each $2'' \times 2'' \times \frac{1}{4}''$, were placed upon supports 12 ft. 9 ins. apart, the transverse outside distance between the bars being $9\frac{1}{2}$ ins., and were prevented from turning inwards by a thin plate upon the upper faces. The bars were tested under uniformly distributed loads, and each was found to have deflected $2\frac{5}{16}$ ins. when the load over the two was 1008 lbs. Find E and the position of the neutral axis.

Both bars failed together when the total load consisted of $10\frac{1}{2}$ cwts. (cwt. = 112 lbs.) uniformly distributed and 3 cwts. at the centre. Find the maximum stress in the metal.

Ans. 17,226,139 lbs.; $\frac{19}{28}$ in. from upper face; 20,323 and 39,577 lbs./sq. in.

177. A horizontal girder of uniform strength, of rectangular section and of length l rests upon supports at its ends and carries a uniformly distributed load of intensity w, which develops the *same maximum stress* f at every cross-section of the girder. If the depth (d) is constant and the breadth variable, find the maximum deflection.

Ans. $\frac{1}{4}\frac{fl^2}{Ed}$.

178. A semi-girder of *uniform strength*, of length l, and of rectangular section carries a weight W at the free end which produces a maximum stress f at every cross-section of the beam. Prove that the maximum deflection is $\frac{4}{3}\frac{(fl)^{\frac{3}{2}}}{E}\left(\frac{b}{6W}\right)^{\frac{1}{2}}$ when the breadth (b) is constant and the depth variable, and

that it is twice as great as it would be if the section were uniform throughout and equal to that at the support. What would be the maximum deflection if the semi-girder were subjected to a uniformly distributed load of w lbs. per unit of length?

Ans. $\frac{2fl}{E}\sqrt{\frac{bf}{3w}}$.

179. The vertical sections of a cantilever ABC, of length l, and fixed at the end BC are circular. The profile is a cubical parabola with its vertex at A. Show that, under its own weight, the slope at the free end of the neutral axis is seven tenths of the maximum deflection divided by the length.

180. A horizontal beam of length l carries a load P concentrated at a distance a from each end. Find the maximum deflection due to P (a) if the two ends rest upon the supports, (b) if both ends are fixed.

Ans. (a) $\frac{Pa}{EI}\left(\frac{l^2}{8}-\frac{a^2}{6}\right)$; ($b$) $\frac{Pa}{EI}\left(\frac{l}{8}-\frac{a}{3}\right)$.

181. What should be the diameter of a screw-blade for a cast-iron pile of 24 ins. diameter, so that the stress at the root of the blade, which is 2 ins. thick may not exceed 1 ton/sq. in. under a uniform soil pressure of 2 tons/sq. in.?

Ans. 25.63 ins.

182. If a bar of cast iron 1 in. square and 1 in. long, when secured at one end breaks transversely with a load of 6000 lbs. suspended at the free end, what would be the safe working pressure, employing a factor of safety of 10, between the two teeth which are in contact in a pair of spur-wheels whose width of tooth is 6 ins., the depth of the tooth, measured from the point to the root, being 2 ins., and the thickness at the root of the tooth $1\frac{1}{2}$ ins. (Assume that one tooth takes the whole load.) *Ans.* 4050 lbs.

183. Show that the moment of resistance of the equal flanged rolled joist with the web vertical is to the moment of resistance when the web is horizontal in the ratio of $ah_1^3-2bh_2^3:(h_1-h_2)a^3+h_2(a-b)^3$. (*Ex.* $a=3$ ins., $b=1.2$ ins., $h_1=5$ ins., $h_2=4\frac{1}{3}$ ins.) *Ans.* 8.16 to 1.

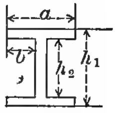


FIG. 543.

184. A piece of greenheart 140 ins. between supports, 9 ins. wide, and 8 ins. deep was successively subjected to loads of 4, 8, and 16 tons at the centre, the corresponding deflections being .32 in., .64 in., and 1.28 ins. Find E and the total work done in bending the beam.

What were the corresponding inch-stresses at three fourths of the depth of the beam?

Ans. $E=5,582,682$ lbs.; 13.44 in.-tons; $\frac{35}{48}$ tons, $\frac{70}{48}$ tons, $\frac{140}{48}$ tons.

185. The square of the radius of gyration of the equal-flanged section of a wrought-iron girder of depth d is $\frac{3}{14}d^2$; the area of the section $=\frac{1}{8}d^2$; the span $=50$ ft. In addition to its own weight it carries a uniformly distributed load of $1\frac{1}{48}$ lbs./lin. ft.; the maximum intensity of stress $=10,000$ lbs./sq. in. Find the depth. Also determine the *stiffness*, E being 25,000,000 lbs.

Ans. $3\frac{1}{2}$ ins.; $\frac{3}{175}$.

186. The section of a beam is of the form and dimensions shown by the figure, and the coefficient of strength is 10 tons/sq. in. Find the moment

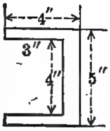


FIG. 544.

of resistance and the greatest load which may be placed at the centre of such a beam of 10 ft. span. *Ans.* $102\frac{2}{3}$ in.-tons; 3.42 tons.

187. The upper chord of a Howe truss is 24 ins. wide×12 ins. deep and is made up of four 12″×6″ timbers; the lower chord is 24 ins. wide×16 ins. deep and is made up of four 16″×6″ timbers; the distance between the inner faces of the chords is 24 ft. Find the moment of resistance to bending, taking 800 lbs./sq. in. as the coefficient of tensile strength and neglecting the effect of the web.

Ans. Neutral axis is $137\frac{3}{7}$ ins. from bottom face of lower chord; moment of resistance $=87,441,616$ in.-lbs.

188. A girder supported at the ends is 30 ft. in the clear and carries two stationary loads, viz., 7 tons concentrated at 6 ft. and 12 tons at 18 ft. from the left support. Find the position and amount of the maximum deflection, and also the work of flexure. The girder is built up of plates and angle-irons and is 24 ins. deep. If the moment of resistance due to the web is neglected, and if the intensity of the longitudinal stress is not to exceed 5 tons/sq. in., what should be the flange sectional area corresponding to the maximum bending moment? Determine the work of flexure and the necessary flange sectional area at the centre if the girder is subjected to a uniformly distributed load of 40 tons instead of the isolated loads.

$$\textit{Ans. Max. def.}=\left\{\frac{26}{15}x^3-\frac{7}{6}(x-6)^3-\frac{4536}{5}x\right\}\div EI\quad x\text{ being }15.22\text{ ft.}$$

$$\text{Work}=\frac{67161.5}{EI}\text{ ft.-tons, }10.32\text{ sq. ins.; work}=\frac{180000}{EI}\text{ ft.-tons,}$$

sectional area $=15$ sq. ins.

189. A beam AB of span l carrying a uniformly distributed load of intensity w is fixed at A and merely supported at B. The end B is lowered by an amount $\frac{wl^4}{16EI}$. Find the reactions. How much must B be lowered so that the whole of the weight may be borne at A?

Solve the example supposing the fixture at A to be imperfect, the neutral axis making with the horizontal an angle whose tangent is $\frac{1}{48}\frac{wl^3}{EI}$.

Ans. $\frac{13}{16}wl$ at A, $\frac{3}{16}wl$ at B; $\frac{1}{8}\frac{wl^4}{EI}$; $\frac{3}{4}wl$, $\frac{1}{4}wl$; $\frac{7}{48}\frac{wl^4}{EI}$.

190. The section of a girder of 24 ft. span supported at the two ends is shown by Fig. 545. The allowable working stress per square inch is 4 tons. Determine the load which may be uniformly distributed over such a beam, and find the maximum deflection.

Ans. 10.53 tons; .192 in.

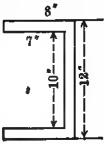


FIG. 545.

191. A steel bar of uniform rectangular section, 10 ft. long, has to support a load of 2 tons at the centre and to satisfy the condition that when the stress induced is 6 tons/sq. in. the deflection at the centre is .2 in. Determine the breadth and depth of the bar, being given that $E=13,500$ tons/sq. in. *Ans.* $2\frac{7}{64}$ ins.; $5\frac{1}{3}$ ins.

192. In a tram-rail the area of the modulus figure is 21 sq. ins. Also $I=80.5$, $\frac{f_y}{y}=4.03$, $E=12{,}300$ tons/sq. in.; $G=4900$ tons/sq. in. The rail is centrally loaded with 10 tons on a span of 42 ins. Find the central deflection (*a*) due to bending, (*b*) due to shear. *Ans.* (*a*) .016 in.; (*b*) .008 in.

193. A horizontal bracket of length a is attached to the upper end of a vertical pillar of length l which has its lower end built in. When carrying a load W at the extremity of the bracket the pillar bends slightly. Show that, due to flexure of the pillar, the bending moment at its base is increased, whatever the length of the bracket, in the ratio

$$\sec\sqrt{\frac{W}{EI}}\,l,$$

in which E is Young's modulus and I the moment of inertia of the cross-section of the pillar about line through its centre perpendicular to the plane of bending.

194. A rolled beam with equal flanges and a web whose section is equal to the joint section of the flanges has a span of 24 ft. and carries a weight of 8 tons at the centre. If the stiffness is .001 and if the coefficient of strength per square inch is 5 tons, find the depth of the beam and the web and flange sectional areas. ($E=15{,}000$ tons.) *Ans.* 16 ins.; 10.8 and 5.4 sq. ins.

195. In a rolled joist the sum of the two flange areas and the web area is a constant quantity. Find the proportion between them which will give a joist of maximum strength, the thickness of the web being fixed by practical considerations. *Ans.* Flange area $=\frac{2}{3}$ web area.

196. A wrought-iron beam of I section, 20 ft. between supports, carries a uniformly distributed load of 4000 lbs. and deflects .1 in.; the effective depth $=8$ ins.; $E=30{,}000{,}000$ lbs.; web area = joint area of the equal flanges. Find the total sectional area. Also find the width of a rectangular section 8 ins. deep which might be substituted for the above.

Ans. $I=288$; area $=27$ sq. ins.; width $=6\frac{3}{4}$ ins.

197. A girder has a moment of resistance of 550,000 ft.-lbs., a depth of 3.2 ft., and a web area of 15 sq. ins. Determine the area of the flanges, the coefficient of strength for compression and tension being 14,000 lbs./sq. in.

198. A plate girder 30 ft. long and 3.6 ft. deep carries a load of 6000 lbs./lin. ft. Calculate the area of the flange so that the flange stress shall not exceed 12,500 lbs./sq. in. If two $6''\times4''\times\frac{1}{2}''$ angles (area of section 4.5 sq. ins. each) and one cover-plate be used for each flange, find the length of the cover-plate. *Ans.* 15 sq. ins.; 18.96 ft.

199. A street-car weighing 40,000 lbs., the weight being concentrated on 2 axles 8 ft. apart, is to be carried across a span of 20 ft. by a number of timber stringers 16 ins. deep. Find the total width of the stringers so that the fibre stress may not exceed 1000 lbs./sq. in. *Ans.* 36 ins.

200. Determine the moment of inertia of a built-up beam composed of two equal flanges, each consisting of a $7\frac{1}{2}''\times\frac{1}{2}''$ plate connected with a $24''\times\frac{1}{2}''$ web, by four angle-irons, each $3''\times3''\times\frac{1}{2}''$. If the working stress

is 6 tons/sq. in., find the moment of resistance of the section. If the span is 20 ft., what uniformly distributed load will the beam safely carry?

Ans. 3058; 1529 in.-tons; 51 tons.

201. A cast-iron beam of rectangular section, 12 ins. deep, 6 ins. wide, and 16 ft. long, carries, in addition to its own weight, a single load P; the coefficient of working strength is 2000 lbs./sq. in. Find the value of P when it is placed (*a*) at the middle point; (*b*) at 2½ ft. from one end.

Ans. (*a*) 4200 lbs.; (*b*) $9577\frac{7}{9}$ lbs.

202. The top and bottom flanges of a rolled section of wrought iron are 8″×⅝″, and the web is of the same thickness. The height over all is 12 ins. What is the bending moment when the greatest tensile stress is 10,000 lbs./sq. in.?

Ans. 569,000 in.-lbs.

203. A rolled iron beam of I section is 1 in. thick throughout and rests upon supports 10 ft. apart. The flanges are 8 ins. wide and the depth over all is 8 ins. What weight will the joist carry at the centre if the safe working stress is 4 tons/sq. in.?

Ans. 7.18 tons.

204. Calculate approximately the safe central load for a single-web riveted girder 6 ft. deep; flanges 18 ins. wide, 2½ ins. thick. The flange is attached to the web by two 4″×½″ angles. Neglect the strength of the web, and assume that the section of each flange is reduced by two rivet-holes ⅞ in. in diameter passing through the flange and angles. (Span of girder, 50 ft.; stress in flanges, 5 tons/sq. in.)

Ans. 110 tons.

205. A trough section, Fig. 546, is used for the flooring of a bridge; each section has to support a uniformly distributed load of 150 lbs./sq. ft., and a concentrated central load of 4 tons. Find the span for which such a section may be safely used. Skin stress =5 tons/sq. in.; pitch of corrugation, 2 ft.; depth, 1 ft.; width of flange =8 ins.; thickness =⅝ in.

Ans. 19 ft.

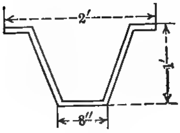


FIG. 546.

206. A lattice girder of 80 ft. span and 8 ft. deep is designed to carry a dead load of 60 tons and a live load of 120 tons uniformly distributed; at the centre the *net* sectional area of the bottom flange is 45 sq. ins. and the gross sectional area of the top flange 56½ sq. ins. Find the position of the neutral axis and the maximum flange intensities of stress. If the live load travels at 60 miles an hour, what will be the increased pressure due to centrifugal force? (1 ton =2240 lbs.)

Ans. 3.546 ft. from top; 11,200 lbs./sq. in.; 8920.35 lbs./sq. in.; $\frac{594000}{EI}$ lbs.

207. Two tracks 6 ft. apart cross the Torksey Bridge, and are supported by single-webbed plate cross-girders 25 ft. long and 14 ins. deep. If the whole of the weight upon a pair of drivers, viz., 10 tons, is directly transmitted to one of these cross-girders find the maximum deflection of the girder and the work of flexure when the ends (*a*) are fixed to the main girders, (*b*) merely rest on these girders.

Ans. (*a*) $\frac{499.32}{EI}$ at 14.205 ft. from end; $\frac{1416.365}{EI}$ ft.-tons.

208. Find the uniformly distributed load which can be borne by a rolled T-iron beam, $6'' \times 4'' \times \frac{1}{2}''$, 10 ft. long, fixed at one end and free at the other, the coefficient of strength being 10,000 lbs./sq. in. *Ans.* 438 lbs.

209. One of the tubes of the Britannia bridge has an effective length of 470 ft., depth of $27\frac{1}{2}$ ft., and deflects 12 ins. at the centre under a uniformly distributed load of 1587 tons. Find E and the central flange stresses, the sectional areas of the top flange, bottom flange, and web being 648 sq. ins., 585 sq. ins., and 302 sq. ins., respectively.

Ans. $E = 22,910,496$ lbs.; $f_t = 5.37$ tons/sq. in.; $f_c = 4.81$ tons/sq. in.

210. Compare the resistance to bending of a wrought-iron I section when the beam is placed like this, I, and like this, $\vdash\!\dashv$. The flanges of the beam are 6 ins. wide and 1 in. thick, and the web is $\frac{3}{4}$ in. thick and measures 8 ins. between the flanges. *Ans.* 4.57 to 1.

211. Compare the strength to resist bending of a rolled joist with an $8'' \times 1''$ web and two equal flanges each $4'' \times 1''$, when placed with its axis vertical and with its axis horizontal. *Ans.* Ratio of moments of resistance $= 308/17$.

212. The flanges of a rolled joist are each 4 ins. wide by $\frac{1}{2}$ in. thick; the web is 8 ins. deep by $\frac{1}{2}$ in. thick. Find the position of the neutral axis, the maximum intensities of stress per square inch being 10,000 lbs. in tension and 8000 lbs. in compression. *Ans.* $h_1 = 3\frac{5}{9}$; $h_2 = 4\frac{4}{9}$.

213. A railway girder is 101.2 ft. long, 22.25 ft. deep, and weighs 3764 lbs./lin. foot. Find the maximum shearing force and flange stresses at 25 ft. from one end when a live load of 2500 lbs./lin. ft. crosses the girder.

Ans. 168,078.3 lbs.; 268,155.5 lbs.

214. A floor with superimposed load weighs 160 lbs./sq. ft. and is carried by tubular girders 17 ft. centre to centre and 42 ft. between bearings. Find the depth of the girders (neglecting effect of web), the safe inch-stress in the metal being 9000 lbs. and the sectional area of the tension flange at the centre 32 sq. ins. *Ans.* 24.99 ins.

215. A hollow tube of wrought iron of 3 ins. outside and $2\frac{1}{2}$ ins. inside diameter is 20 ft. long; find its weight. What is its deflection with its own weight? What further weight on its middle will it carry safely if $f = 4\frac{1}{2}$ tons/sq. in.?

Ans. 21 lbs.; $144\frac{1}{2}$ lbs.; $193\frac{13}{59}$ lbs.

216. A horizontal beam of depth d, breadth b, and length 12 ft. rests upon supports at the ends. A weight W_1 at the centre deflects the beam .1 in. when the side of length b is vertical. An additional weight of 1250 lbs. is required to produce the same deflection when the side of length d is vertical. If $d = 2b$ and if $E = 1,200,000$ lbs., find the sectional area of the beam and the maximum skin stress. *Ans.* 72 sq. ins.; $277\frac{7}{9}$ lbs./sq. in.

217. A timber stringer $6'' \times 16''$ rests on two supports 20 ft. apart and carries two concentrated loads of 16,000 lbs. each at a distance of 2 ft. from each support. Find the greatest intensity of shearing stress along the fibre of the timber.

If the ultimate resistance to compression is 8000 lbs./sq. in. and the resistance to shear along the fibre 600 lbs./sq. in., indicate the probable mode of failure as the load is increased. *Ans.* Intensity of shear 250 lbs./sq. in.

218. A cast-iron beam has a cruciform section with equal ribs 2 ins. thick and 4 ins. long. If the intensity of longitudinal shear at the neutral axis is 1 ton/sq. in., find the total shear which the section can bear, and also find the moment of resistance, the least coefficient of working tensile and compressive stress being 1 ton/sq. in. *Ans.* 59.31 tons; 34.4 tons.

219. A cast-iron beam of an inverted T section has a uniform depth of 20 ins. and is 22 ft. between supports; the flange is 12 ins. wide and 1.2 ins. thick; the web is 1 in. thick; the load upon the beam is 4500 lbs./lin. ft.; $E=17{,}000{,}000$ lbs. Find the deflection at the centre, the moment of resistance to bending, the maximum tensile and compressive intensities of stress, and the position of the neutral axis. Why is the flange placed downwards?

Ans. .122 in.; 326,700 in.-lbs.; 1492 lbs. and 3274 lbs./sq. in.; 5.062 ins. from flange surface.

220. The section of a beam is in the form of an isosceles triangle with its base horizontal. Show that the moment of resistance to bending of the strongest trapezoidal beam that can be cut from it is very nearly $\frac{11}{240}fbd^2$, b being the width of the base and d the depth of the triangle.

221. Find the sectional area of a wrought-iron beam of T section which may be substituted for the cast-iron beam in the preceding question, the depth being the same and the coefficients of strength per square inch being 3 tons in compression and 5 tons in tension. Why should the flange be uppermost? What should the total sectional area be if the flange and web are of equal area? *Ans.* 10.35 sq. in.

222. A cast-iron beam of T section, with a 1.62″×1.62″ upper flange and a 5.67″×.57″ web, rests upon supports 102 ins. apart and carries a load of 8300 lbs. at the middle point. Find the tensile fibre stress.

Ans. 33,213 lbs./sq. in.

223. A beam of 15 ft. span carries loads of 2, 4, 1, and 3 tons at distances of 2, 5, 7, and 11 ft., respectively, from the left support. Determine the deflection graphically, taking $I=242$ and $E=12{,}000$ tons per sq. in.

Ans. 0.32 in.

224. Find the moment of resistance of the section (Fig. 547) (*a*) about the axis *AA*, (*b*) about the axis *BB*, each passing through the centre of gravity *C* of the section, the allowable extreme fibre stress being 16,000 lbs./sq. in. The section consists of *one* 19″×$\frac{3}{8}$″ cover, *two* 15-in.×35-lb. channels, and *two* 4″×$\frac{7}{8}$″ bars.

Ans. (*a*) 2,872,320 in.-lbs.; (*b*) 2,476,586 in.-lbs.

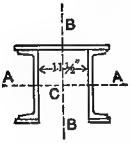


FIG. 547.

225. Fig. 548 shows a chord section designed for a 560-ft. double-track span. The unsupported length is 35 ft. Find (*a*) the moment of resistance about a vertical axis; (*b*) the deflection and maximum fibre stress due to its own weight. The section consists of *one* 48″×$\frac{1}{2}$″ cover, *four* top angles each 4″×4″×$\frac{1}{2}$″, *eight* 42″×$\frac{11}{16}$″ web plates in pairs, and *four* bottom angles each 7″×3$\frac{1}{2}$″×$\frac{13}{16}$″.

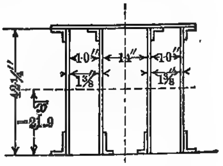


FIG. 548.

Ans. (*a*) 43,121,340 in.-lbs.; (*b*) fibre stress = 512 lbs./sq. in.; deflection = .014 in., E being 30,000,000 lbs./sq. in.

226. Taking f_t, f_c as the tensile and compressive intensities of stress, find the moment of resistance to bending of a section consisting of a $20t'' \times 7t''$ top flange, an $80t'' \times 10t''$ bottom flange, and a trapezoidal web $4t$ ins. thick at the top, $8t$ ins. thick at the bottom, and $120t$ ins. deep. Also, compare the *maximum* and *average* intensities of shear.

Ans. $84{,}864 f_t t^3$ or $36{,}419 f_c t^3$.

227. A cast-iron channel-beam having a web 12 ins. wide and two sides 7 ins. deep, the metal being everywhere 1 in. thick, crosses a span of 14 ft. If the tensile intensity of stress is 1 ton/sq. in., what uniformly distributed load will the beam carry (*a*) with the web at the bottom; (*b*) with the web at the top? Find (*c*) the maximum compressive intensity of stress to which the metal is subjected, and (*d*) compare the maximum and average intensities of shear. Also, (*e*) what should be the area of a rectangular section to bear the same total shear? *Ans.* $I = 110\frac{1}{2}$; (*a*) $\frac{442}{189}$ tons; (*b*) $\frac{442}{39}$ tons; (*e*) $\frac{5304}{361}$ sq. ins.

228. A trough-shaped steel beam of the dimensions shown by Fig. 549 has to carry a uniformly distributed load of 45 lbs./cu. ft. If the length of the beam between the supports is 16 ft., find to what height the load may be raised so that the stress developed may not exceed 10,000 lbs./sq. in. —*first*, assuming both ends free; *second*, assuming both ends fixed.

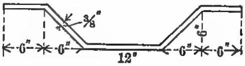


FIG. 549.

229. In the followng cases determine the position of the neutral axis, the moment of resistance, the shearing strength, the ratio of the maixmum to the average intensity of shear, the coefficients of strength per square inch being $4\frac{1}{2}$ tons for tension and compression and $3\frac{1}{2}$ tons for shear:

(*a*) A rectangle 2 ins. wide and 6 ins. deep.

Ans. At centre; 54 in.-tons; 28 tons; 3 to 2.

(*b*) A square with a diagonal vertical, the length of a side being 4 in.-tons.

Ans. At centre; $24\sqrt{2}$ in.-tons; 112 tons; 2 to 1.

(*c*) A circular section 4 ins. in diameter.

Ans. At centre; 28.2 in.-tons; 33 tons; 4 to 3.

(*d*) A regular hexagonal section with a diameter (*a*) vertical, (*b*) horizontal, *a* being a side of the hexagon.

Ans. (*a*) At centre; $\frac{45}{32}a^3\sqrt{3}$; $\frac{15}{4}a^2\sqrt{3}$; 7 to 5.

(*b*) At centre; $\frac{45}{16}a^3$; $\frac{2625\sqrt{3}}{629}a^2$; 1.258.

(*e*) A triangular section 6 ins. deep, with a base 6 ins. wide, the sides being equal. *Ans.* 4 ins. from vertex; 40.5 in.-tons; 42 tons; 3 to 2.

(*f*) A double-tee section composed of a $30'' \times \frac{3}{8}''$ web and four angle-irons each $5'' \times 3\frac{1}{2}'' \times \frac{5}{8}''$.

Ans. At centre; 1424.6 in.-tons; 34.534 tons; 3.1355 to 1.

(*g*) A section having a semicircular top flange of 8 ins. external diameter and 1 in. thick, a web 14 ins. deep and 1 in. thick, and a bottom flange 8 ins. wide and 1 in. thick. *Ans.* 609.57 in.-tons; 44.27 tons; .3833.

(*h*) A section having a semi-elliptic top flange 2 ins. thick, the internal major and minor axes being 8 ins. (*horizontally*) and 4 ins. (*vertically*), respectively, a bottom flange 8 ins. wide and 2 ins. thick, and a web 10 ins. deep and 2 ins. thick. *Ans.* 845 in.-tons; 73.26 tons; 34.23.

(*i*) A section having a semi-elliptic top flange 2 ins. thick, the external major and minor axes being 10 ins. (*horizontally*) and 6 ins. (*vertically*), respectively, a trapezoidal web 8 ins. deep having a width of 3 ins. at the top and 6 ins. at the bottom, and a bottom semicircular flange of 10 ins. diameter.

Ans. $1141\frac{1}{2}$ in.-tons; 12.6 tons; .382.

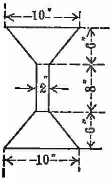


FIG. 550.

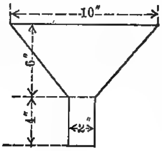


FIG. 551.

(*j*) The sections shown by Figs. 550 and 551. What is the ratio of the maximum tensile and compressive stresses in each section?

(*k*) A trapezoidal section, the top side, bottom side, and depth h (inches) being in the ratio of 1 to 2 to 4. *Ans.* $\frac{5}{9}h$ from top side; $\frac{39}{160}h^3$ in.-tons.

(*l*) A section in the form of a rhombus of depth $2c$ and with a horizontal diagonal of length $2b$. *Ans.* $\frac{3}{2}bc^2$; $\frac{56}{9}bc$; 9 to 8.

(*m*) An angle-iron $2'' \times 2'' \times \frac{1}{2}''$.

Ans. Neutral axis divides depth into segments of $\frac{19}{28}$ in. and $\frac{37}{28}$ in.; $\frac{2379}{1184}$ in.-tons; 2.3 tons; 4107 to 1586.

(*n*) A hollow circular section of external radius C and internal radius C'.

$$Ans.\ \frac{99}{28}\frac{C^4-C'^4}{C};\ \frac{33}{4}\frac{C^4-C'^4}{C^2+CC'+C'^2};\ \frac{4}{3}\frac{C^2+CC'+C'^2}{C^2+C'^2}.$$

(*o*) A cruciform section made up of a flat steel bar 10 ins. by $\frac{1}{2}$ in. and four steel angles, each $4'' \times 4'' \times \frac{1}{2}''$, all riveted together. (Neglect weakening effect of rivet-holes.) *Ans.* 76.425 in.-tons; 29.43 tons; 2.208.

230. A beam of triangular section 12 ins. deep and with its base horizontal can bear a total shear of 100 tons. If the safe maximum intensity of shear is 4 tons/sq. in., find the width of the base. *Ans.* $6\frac{1}{4}$ ins.

231. In a rolled beam with equal flanges, the area of the web is proportional to the nth power of the depth. Find the most economical distribution of metal between the flanges and web, and the moment of resistance to bending of the section thus designed. Also find the ratio of the average to the maximum intensity of shear.

Ans. Area of each flange: web area::$2n-1:6$; *moment of resistance* $=\frac{1}{2}\frac{n}{n+1}fAh$, f being the coefficient of strength, A the total sectional area, and h the depth of the web; *ratio of shears* $=\frac{(n+1)(4n+1)}{6n}$.

232. Assuming that the web and flanges of a rolled beam are rectangular in section, determine the ratio of the maximum to the average intensity of

shear in a section from the following data: the total *depth* is $\frac{n}{2}$ times the *breadth* of each flange, n times the *thickness* of each flange, and $2n$ times the *thickness* of the web. Show also that this ratio is $\frac{156}{77}$ or $\frac{18}{7}$, according as the area of the web is equal to the joint area of the two flanges or is equal to the area of each flange. How much of the shearing force is borne by the web? How much by the flange?

Ans. ratio $=\dfrac{3(n^3+12n-12)(n+6)}{2(n^2+18n^2-36n+24)}$; 70%; 85%.

233. A built-up beam is composed of two equal flanges, each consisting of a $6\frac{1}{2}'' \times \frac{1}{2}''$ plate connected to a $24'' \times \frac{1}{2}''$ web, open 6 ins. in the middle, by means of four equal angle-irons, each $3'' \times 3'' \times \frac{1}{2}''$. Determine the moment of resistance, the maximum shearing strength, the ratio of the maximum to the average intensity of longitudinal shear, and the intensity of longitudinal shear 12 ins. from the neutral axis, 6 tons/sq. in. being coefficient of strength.

Ans. $1391\frac{1}{2}$ in.-tons; 72 tons; .452; .16 ton/sq. in.

234. Find the moment of resistance to bending, the resistance to shear, and the ratio of maximum to the average intensity of a shear in the case of a section consisting of two equal flanges, each composed of a pair of $5'' \times 3\frac{1}{2}'' \times \frac{5}{8}''$ angle-irons riveted to a $31\frac{1}{4}'' \times \frac{3}{8}''$ web, the 5-in. sides of the angles being horizontal and $4\frac{1}{2}$ tons/sq. in. being the coefficient of strength.

Ans. 1501.06 in.-tons; 46.21 tons; 3.058.

235. The floor-beam for a single-track bridge is 15 ft. between bearings, and each of its flanges is composed of a pair of $2\frac{3}{4}'' \times 2\frac{3}{4}'' \times \frac{3}{8}''$ angle-irons riveted to a $30'' \times \frac{3}{8}''$ web. The uniformly distributed load (including weight of beam) upon the beam is 4200 lbs., and a weight of 1600 lbs. is concentrated at each of the rail-crossings, $2\frac{1}{2}$ ft. from the centre. Find (*a*) the maximum flange stress, (*b*) the ratio of the *maximum* and average intensities of shear; (*c*) the stiffness, E being 27,000,000 lbs.

Ans. (*a*) 1193.7 lbs.; (*b*) 2.039; (*c*) .0000663.

236. A lattice-girder of 100 ft. span carries 80 tons uniformly distributed; the girder is 10 ft. deep and the safe working stress is 4 tons/sq. in. If the width of the flange must be 20 ins. to carry the load exclusive of the weight of the girder, what must be the width of the flange when the weight of the girder is taken into account?

Ans. 23 ins.

237. A plate girder of double-tee section and of 80 ft. span is 8 ft. deep and carries a uniformly distributed load of 80 tons. If the width of the flange must be 12 ins. to carry the load exclusive of the weight of the girder, what must the width be when this weight is taken into account?

238. Design the central section of a plate girder of 45 ft. span and 5 ft. deep to carry a dead load of 500 lbs./ft. run, a live load of 3200 lbs./ft. run and an impact load of 2400 lbs./ft. run, and also determine the lengths of the flange plates.

239. A tubular girder rests upon supports 36 ft. apart. At 6 ft. from one end the flanges are each 27 ins. wide and $2\frac{3}{4}$ ins. thick, the net area of the tension flange being 60 ins., while the web consists of two $\frac{7}{16}$-in. plates, 36 ins. deep and 18 ins. apart. Neglecting the effect of the angle-irons uniting the web plates to the flanges, determine the moment of resistance. The girder has to carry a uniformly distributed dead load of 56 tons, a uniformly distributed live load of 54 tons, and a local load at the given section of 100 tons. What are the corresponding flange stresses per square inch? How many $\frac{7}{8}$-in. rivets are required at a given section to unite the angle-irons to the flanges?

Ans. 238.13 × coefficient of strength; 3.3186 tons; 3.896 tons; *six*, f_s being 4 tons/sq. in.

240. A beam of rectangular section, of breadth b and depth d, is acted upon by a couple in a plane inclined at 45° to the axis of the section. Compare the moment of resistance to bending with that about either axis.

$$\textit{Ans.}\ \frac{b\sqrt{2}}{b+d};\ \frac{d\sqrt{2}}{b+d}.$$

241. If the plane of bending does not coincide with the plane of symmetry of a beam, show that the neutral axis is parallel to a line joining the centres of two circles into which the beam would be bent by two component couples whose axes are the principal axes of inertia of the section, each couple being supposed to act alone.

242. If a spiral spring is fastened to the barrel so that there is no change of direction relatively to the barrel, show that the tendency to unwind is directly proportional to the amount of winding up. (Condition of perfect isochronism.)

243. A metal beam is subjected to the action of a bending moment steadily applied *beyond* the elastic limit. Assuming that the metal acts as if it were perfectly plastic, i.e., so that the stress throughout a transverse section is *uniform*, compare the moment of resistance to bending of a section of the beam with the moment on the assumption that the metal continued to fulfil the ordinary laws of elasticity, (a) the section being a rectangle; (b) the section being a circle.

244. The neutral axis of a symmetrically loaded girder whose moment of inertia is constant assumes the form of an elliptic or circular arc. Show that the bending moment at any point of the deflected girder is inversely proportional to the cube of the vertical distance between the point and the centre of the ellipse or circle.

245. The flange of a girder consists of a pair of angle-irons and of a plate which extends over the middle portion of the girder for a certain required distance. Show that the greatest economy of material is secured when the length of the plate is two thirds of the span and the sectional areas of the plate and angle-irons are as 4 to 5, the girder being uniformly loaded.

246. The flange of a uniformly loaded girder is to consist of two plates, each of which extends over the middle portion of the girder for a certain required distance, and of a pair of angle-irons. Show that the greatest economy of material is realized when the lengths of the plates and angle-irons are in the ratio of 12:18:23, and when the areas of the plates are in the ratio of 4:5. What should be the relative lengths of the plates if they are of equal area?

Ans. $1:\sqrt{2}:\frac{3}{2}(\sqrt{2}+1)$.

247. Determine the moment of resistance to bending of a section of a beam in which the top flange is composed of *two* 340-mm.×12-mm. plates and *one* 340-mm.×10-mm. plate, and the bottom flange of *one* 340-mm.×10-mm. plate and *one* 340-mm.×8-mm. plate, the flanges being riveted to a 1.4-m.×7-mm. web plate by means of four 100-mm.×100-mm.×8-mm. angle-irons. The coefficient of strength $=6$ k./mm.^{2}. What will the moment of resistance be if *three* 400-mm.×15-mm. plates are substituted for the top flange and *one* 400-mm.×15-mm. plate is substituted for the bottom flange?

Ans. 84.408 k.m.; 95.478 k.m.

248. Floor-beams 4.4 m. between bearings and spaced 2.548 m. centre to centre have a section composed of two equal flanges, each consisting of two 85-mm.×85-mm.×12-mm. angle-irons riveted to a 490-mm.×7-mm. web. A weight of 150 k. (due to longitudinals) and a weight of 150 k. (due to rails, etc.), i.e., 300 k. in all, are concentrated at the rail-crossings, and the ties have also to carry a uniformly distributed load of 400 k. due to weight of floor-beam, 4000 k. due to weight of platform, and 4000 k./sq. m. of platform due to *proof*-load. Find the moment of resistance to bending and the maximum flange intensities of stress. *Ans.* $I=.000438584615$.

249. Each of the flanges of a girder is a 350-mm.×10-mm. plate and is riveted to a 1.8-m.×8-mm. web by means of two 100-mm.×100-mm.×12-mm. angle-irons. Determine the moment of resistance to bending, the coefficient of strength being 6 k./sq. mm., (*a*) disregarding the weakening effect of riveting; (*b*) assuming that the flange plates are riveted to the angles by 20-mm. rivets. *Ans.* (*a*) 108.66104 k.m.; (*b*) 100.8933.

250. The cross-tie for a single-track bridge is 4.1 m. between bearings, the gauge of the rails being 1.51 m.; each of the flanges is composed of a 148-mm.×8-mm. plate riveted to a 550-mm.×8-mm. web by means of two 70-mm.×70-mm.×9-mm. angle-irons; a load of 296 k. (weight of rails, etc.) is concentrated at each rail-crossing. What uniformly distributed load will the tie safely bear, the metal's coefficient of strength being 6 k./sq. mm.? The load actually distributed over the tie is 19,782 k. Find the maximum intensity of stress. *Ans.* 24,162 k.; 4.94 k./sq. mm.

251. Design a longitudinal of .45 m. depth which is to be supported at intervals of 3.3 m. and to carry at its middle point a weight of 7000 k., the coefficient of strength being 5 k./sq. mm.

Ans. $I=259.875$, and the I of a section with two equal flanges, each composed of two 70-mm.×70-mm.×9-mm. angle-irons riveted to a 450-mm.×8-mm. web is 259.536.

252. A longitudinal 2.548 m. between bearings consists of two equal flanges, each composed of two 70-mm.×70-mm.×9-mm. angle-irons riveted to a 350-mm.×7-mm. web plate. Find the flange intensity of stress under a maximum load of 7000 k. at the centre.

Ans. $I=.000139284508$; stress $=5.6$ k./mm.^{2}.

253. A cross-tie resting upon supports at the ends and 2.26 m. between bearings is composed of two equal flanges, consisting of two 70-mm.×70-mm.×9-mm. angle-irons riveted at the top to a 450-mm.×7-mm. web plate, and

at the bottom to a 300-mm.×7-mm. web plate, the interval between the web plates, which is open, being 2.55 m.; the tie is designed to carry a uniformly distributed load of 676 k./lineal metre of its length, and also a load of 11,644.8 k. at each of the points distant .375 m. from the bearing. Find the position of the neutral axis and the *maximum* flange stresses.

Ans. 1.516 m. from top flange; $I = .02394564198$; maximum B.M. $= 4815.8161$ k.m.; maximum tensile stress $= .37$ k./mm.^{2}; maximum compressive stress $= .314$ k./mm.^{2}.

254. Find the maximum concentrated load on a cross-tie for a single track due to a six-wheel locomotive, the wheels being 2.3 m. centre to centre, the ties being 3.2 m. centre to centre and the weight on each wheel being 7000 k.

Ans. 10,937.5k.

255. The floor-beams for a double-track bridge are 8.3 m. between bearings and are spaced 2.58 m. centre to centre. The distance, centre to centre, between track rails is 1.5 m., and between inside rails is 2 m.; the tie has equal flanges, each consisting of two 90-mm.×90-mm.×9-mm. angle-irons riveted to a 900-mm.×7-mm. web; the maximum live load upon the tie is that due to a weight of 3000 k. upon each of the six wheels of two locomotives, the wheels being 2.4 m. centre to centre. If the coefficient of working strength is $5\frac{1}{2}$ k./sq. mm., what uniformly distributed load will the tie carry?

Ans. 3730 k.

256. Determine the safe value of the moment of inertia (I) of a cross-tie for a double-track bridge, the length of the tie between bearings being 7.624 m., its depth .6 m. the gauge of the rails 1.5 m., the distance between inside rails 2 m. The uniformly distributed load upon a tie consists of 850 k./sq. m. due to platform, etc., and of 1800 k. due to weight of tie; the ties are 3.584 m. centre to centre, the live load is that due to a weight of 7000 k. upon each of the centre wheels of a six-wheel locomotive and a weight of 6000 k. upon each of the front and rear wheels, the wheels being 2.4 m. centre to centre; the safe coefficient of strength $=6$ k./sq. mm.

Ans. 7,179,104,318.

257. The cross-ties of a single-track bridge consist of two equal flanges, each composed of two 70-mm.×70-mm.×9-mm. angle-irons riveted to a 650-mm.×7-mm. web; the ties are 4.1 m. long, and each carries 19,146 k. (viz., 384 k. *for ties*, 2762 k. *for platform, and* 16,000 k. *for proof load*) uniformly distributed and 635 k. (*due to longitudinals, rails, etc.*) concentrated at each rail-crossing, i.e., at 755 mm. from the middle point. Assuming that the cross-ties are merely supported at the ends, find the maximum intensity of stress.

Ans. 5.7552 k./mm.^{2}; $\frac{I}{c} = .0018423$. The fixture of the ends approximately doubles the strength.

258. The longitudinals of the bridge in the last example consist of two pairs of 70-mm.×70-mm.×9-mm. angle-irons riveted to a 4-m.×7-mm. web; the cross-ties are 3.2 m. centre to centre. Determine the maximum intensity of stress due to a load of 7000 k. concentrated on the longitudinal half-way

between the cross-ties, assuming that it is an independent girder. What would the stress be if the ties were 3 m. centre to centre?

Ans. $\frac{I}{c}=.00095458$; 5.866 k./mm.^{2}; 5.4994 k./mm.^{2}.

259. The section for the Estressol bridge cross-ties is the same as that for the Grande Baise (Ex. 257) bridge ties; the load at each rail-crossing is 335 k., and the uniformly distributed load is 18,062 k. Find the maximum intensity of stress in the flanges, assuming that the ties are merely supported at the ends. *Ans.* 5.26 k./mm.^{2}.

260. The area of the compression flange of a cast-iron beam is 17 sq. ins.; the thickness of the web is a certain fraction of the depth; the unit stresses are in the ratio of 2 to 5. Find the areas of web and tension flange which will give a section of maximum strength. *Ans.* 12 sq. ins.; 3.2 sq. ins.

261. Determine suitable dimensions for a cast-iron beam 20 ins. deep at a section subjected to a bending moment of 1200 in.-tons, the coefficients of strength per square inch being 2 tons for tension and 8 tons for compression. Take thickness of web $=\frac{9}{10}$ in.

Ans. Sectional area of tension flange $=36$ sq. ins.; of compression flange $=2\frac{1}{4}$ sq. ins.

262. In a cast-iron beam the area of the web is one half the area of the tension flange, the depth of the beam is 9 ins., and the unit stresses are 2 tons per square inch in tension and 4 tons per square inch in compression. The maximum moment of resistance is 162 in.-tons. Find (*a*) the flange and web areas, (*b*) the length of the beam so that its stiffness might not exceed .001, (*c*) the net weight of the beam, and (*d*) the work of flexure, E being 7500 tons.

Ans. (*a*) 9, $3\frac{3}{8}$, and $4\frac{1}{2}$ sq. ins.; (*b*) 9 ft.; (*c*) 11.775 tons; .020736 in.-tons.

263. A cast-iron girder of 20 ft. span has a top flange of $4'' \times 1\frac{1}{2}''$, a bottom flange of $12'' \times 1\frac{1}{2}''$, and a web of $16'' \times 1\frac{1}{2}''$. Find the position of the neutral axis. If the maximum tensile stress is 2000 lbs./sq. in., find the uniformly distributed load which the girder may carry and the maximum compressive stress.

264. A cast-iron girder with a $4'' \times 1''$ upper flange, a $25'' \times 1''$ lower flange, and a $20'' \times 1''$ web must not carry a greater load than will develope tensile and compressive stresses of 2000 and 5000 lbs /sq. in. respectively. Determine the moment of resistance of the section. *Ans.* 2,158,369 in.-lbs.

265. A cast-iron girder of 25 ft. span has a bottom flange of 36 sq. ins. sectional area. Find the most economic arrangement of material for the web and top flange which will enable the beam to carry a load of 18,900 lbs. at 10 ft. from one end, the tensile and compressive working strengths being 2000 and 5000 lbs./sq. in. respectively. Assume that the thickness of the web is a fraction of its depth.

Ans. Depth $=21\frac{1}{4}$ ins.; area of web $=28.8$ sq. ins.; area of top flange $=5.76$ sq. ins.

266. A double-flanged cast-iron girder has a sectional area of x sq. ins.; the web is 1 in. thick and 21 ins. deep; the moment of resistance of the section is 100,950 ft.-lbs.; the coefficients of strength are 2100 lbs./sq. in. in tension and 5250 lbs. in compression. Find x, the position of the neutral axis, and the areas of the two flanges. *Ans.* $a_1 = 29\frac{43}{196}$ sq. ins.; $a_2 = 5\frac{19}{49}$ sq. ins.

267. Find the safe distributed load for a cast-iron beam of the following dimensions: Top flange, 3″×1″; bottom flange, 8″×15″; web, 1.25 ins. thick; total depth, 10 ins.; with (*a*) the bottom flange in tension, (*b*) when inverted; span, 12 ft.; skin stress, 3000 lbs./sq. in. Also find the safe central loads for a stress of 3000 lbs./sq. in., including the stress due to the weight of the beam. The beam is of constant cross-section.

Ans. (*a*) 5.7 and 2.65 tons; (*b*) 3.2 and 1.4 tons.

268. Determine the thickness of the metal in a cast-iron beam of 12 ft. span and 8 ins. deep which has to carry a uniformly distributed load of 4000 lbs., the section being (*a*) a hollow square; (*b*) a circular annulus. The coefficient of working strength = 3000 lbs./sq. in. Also find the limiting safe span of the beam under its own weight.

Ans. Neglecting weight of beam, (*a*) .317 in.; (*b*) .3538 in. Taking weight of beam into account, (*a*) .599 in.; (*b*) .727 in. Limiting span = 39.527 ft. in (*a*) and = 32.69 ft. in (*b*).

269. The effective length and depth of a cast-iron girder which failed under a load of 18 tons at the centre were 57 ins. and 5⅛ ins. respectively; the top flange was 2.33 ins. by .31 in., the bottom flange 6.67 ins. by 66 ins., and the web was .266 in. thick. Assuming that the ordinary theory of flexure held good, what were the maximum intensities of stress in the flanges at the point of rupture? *Ans.* $f_t = 12.36$ tons/sq. in.; $f_c = 44.9$ tons/sq. in.

270. A double-flanged cast-iron girder 14 ins. deep and 20 ft. between supports carries a uniformly distributed load of 20 tons. Find suitable dimensions for the section, the tensile and compressive inch-stresses being 2 tons and 5 tons respectively. Also find the *stiffness* of the beam, E being 8000 tons.

Ans. Let thickness of web = 1 in.; $a_1 = 22\frac{25}{42}$ sq. ins.; $a_2 = 4\frac{88}{105}$ sq. ins.; stiffness = .001875.

271. A cast-iron beam with a 30″×1″ web is subjected to a bending moment of 2400 in.-tons, the coefficient of strength being 2 tons in tension and 8 tons in compression. Find the position of the neutral axis and the areas of the tension and compression flanges.

Ans. 6 ins. from foot of web; 1¼ sq. ins., 50 sq. ins.

272. The central section of a cast-iron girder is 10½ ins. deep; its web area is *five* times the area of the top flange, and the moment of resistance of the section is 360,000 in.-lbs.; the tensile and compressive intensities of stress are 3000 and 7500 lbs./sq. in. respectively. Find the span and load so that the girder may have a *stiffness* = .001, E being 17,000,000 lbs.

Ans. $a_1 = 12\frac{12}{49}$ sq. ins.; $a_2 = 1\frac{47}{49}$ sq. ins.; $a_3 + a_4 = 9\frac{39}{49}$ sq. ins.; span = 136 ins.; uniformly distributed load = $21{,}176\frac{8}{17}$ lbs.

273. Determine suitable dimensions for a cast-iron girder of 20 ft. span and 24 ins. deep, carrying a load of 30,000 lbs. at the centre, the coefficients of working strength in tension and compression being respectively 2000 and 5000 lbs./sq. in. *Ans.* $a_1 = \frac{1125}{28}$ sq. ins.; $A' = \frac{225}{7}$ sq. ins.; $a_2 = \frac{45}{7}$ sq. ins.

274. The dimensions of the section of a cast-iron girder are the following:

Top flange, 6″×2″; bottom flange, 12″×3″; web, 16″×2″. Determine the position of the neutral axis and the moment of resistance, the maximum tensile and compressive stresses being 2 tons and 6 tons respectively.

275. Determine the dimensions of a cast-iron beam at a section whose moment of resistance is 800 in.-tons and whose depth is 18 ins., taking 2 tons/sq. in. as the maximum tensile intensity of stress.

Ans. $a_1=\frac{500}{21}$ sq. ins.; $A'=\frac{400}{21}$ sq. ins.; $a_2=\frac{80}{21}$ sq. ins.

276. A 4″×4″×½″ inverted T section is used as a beam for a span of 12 ft., the uniformly distributed load on the flange being 0.6 ton. Find the maximum tensile and compressive skin stresses.

Ans. 4⅜ tons (comp.), 1⅞ tons (tens.).

277. A sample cast-iron girder for the Waterloo Corn Warehouses, Liverpool, 20 ft. 7⅓ ins. in length and 21 ins. in depth (total) at the centre, was placed upon supports 18 ft. 1⅓ ins. apart, and tested under a uniformly distributed load. The top flange was 5″×1¼″, the bottom flange was 18″×2″, and the web was 1¼ ins. thick. The girder deflected .15 in., .2 in., .25 in., and .28 in. under loads (including weight of girder) of 63,763 lbs., 88,571 lbs., 107,468 lbs., and 119,746 lbs., respectively, and broke during a sharp frost under a load of 390,282 lbs. Find the mean coefficient of elasticity and the central flange stresses at the moment of rupture.

Ans. $I=3309.122$; $E=17{,}279{,}567$ lbs.; 20,121 lbs., 47,168 lbs.

278. The effective length and central depth of a cast-iron girder resting upon two supports were respectively 11 ft. 7 ins. and 10 ins.; the bottom flange was 10 ins. wide and 1¼ ins. thick; the top flange was 2½ ins. wide and ⅞ in. thick; the thickness of the web was ¾ in. The girder was tested by being loaded at points 3¾ ft. from each end, and failed when the load at each point was 17½ tons. What were the total central flange stresses at the moment of rupture? What was the central deflection when the load at each point was 7½ tons? ($E=18{,}000{,}000$ lbs., and the weight of the girder $=3368$ lbs.)

Ans. 164,747.4 lbs.; .353 in.

279. The dimensions of the section of a cast-iron girder are the following: Top flange, 4″×1½″; bottom flange, 12″×1¾″; web, 16″×1½″. Determine the position of the neutral axis and calculate the moment of inertia of the section. Find, also, the moment of resistance, the greatest permissible tensile and compressive stresses being 2½ and 7½ tons/sq. in. respectively. If the girder is 20 ft. long, and is supoprted at its two ends, find also the greatest safe load which it will carry when uniformly distributed along its length.

Ans. 7⅛ ins. from top; 2280½ in.-units; 800 in.-tons; 26⅔ tons.

280. A continuous beam of four equal spans carries a uniformly distributed load of w intensity per unit of length. The second support is depressed a certain distance d below the horizontal, and the reaction at the second support is twice that at the first. Show that the reactions at the first, second, third, fourth, and fifth supports are in the ratio of the numbers 15, 30, 36, 34, and 13; find d. With this same value of d find the reactions when one end is *fixed*.

Ans. $d=\frac{1}{48}\frac{wl^4}{EI}$; $244rwl$, $285rwl$, $457\frac{1}{2}wl$, $407rw$, $158\frac{1}{2}rwl$, where $\frac{1}{r}=388$.

281. A continuous girder of three spans, the side spans being equal, carries a uniformly distributed load of intensity w. Find the error in the B.M. at the intermediate support, if an error Δ is made in the length (a) of the centre span, (b) of a side span. If the three spans are each of length l, show that this error is $.09w\Delta l$ in the first case and $.11w\Delta l$ in the second case.

282. A wrought-iron girder of I section, 2 ft. deep, with flanges of equal area and having their joint area equal to that of the web, viz., 48 sq. ins., carries $\frac{1}{2}$ ton per lineal foot, is 100 ft. long, consists of five equal spans, and is continuous over six supports. Find the reactions when the *third* support is lowered $\frac{1}{2}$ in. How much must this support be lowered so that the reaction may be *nil* at (a) the first support, (b) the third; (c) the fifth? How much must the support be raised so that the reaction may be *nil* at (d) the second, (e) the fourth, and (f) the sixth support? ($E=16{,}500$ tons.)

Ans. $R_1=2\frac{29}{38}$; $R_2=15\frac{51}{76}$; $R_3=3\frac{6}{19}$; $R_4=14\frac{43}{76}$; $R_5=9\frac{8}{19}$; $R_6=4\frac{5}{19}$ tons. (a) $1\frac{2}{3}$ ins.; (b) $\frac{185}{244}$ in.; (c) $2\frac{71}{72}$ ins.; (d) $1\frac{99}{331}$ ins.; (e) $1\frac{3}{367}$ ins.; (f) $6\frac{1}{4}$ ins.

283. Each of the main girders of the Torksey bridge is continuous and consists of two equal spans, each 130 ft. long. The girders are double-webbed; the thickness of each web plate is $\frac{1}{4}$ in. at the centre and $\frac{3}{8}$ in. at the abutments and centre pier; the total depth of the girders is 10 ft., and the depth from centre to centre of the flanges is 9 ft. $4\frac{3}{8}$ ins. Find (a) the reactions at the supports, and also (b) the points of inflection, when 200 tons of live load cover *one* span, the total dead load upon each span being 180 tons uniformly distributed. The top flange is cellular; its *gross* sectional area at the centre of each span is 51 sq. ins., and the corresponding *net* sectional area of the bottom flange is 55 sq. ins. Determine (c) the flange stresses in tons per square inch, and (d) the position of the neutral axis. ($I=372{,}500$.) Also (e) determine the reactions when, *first*, B and, *second*, C are lowered 1 in. ($E=16{,}900$ tons, and depth of top flange $=11.818$ ins.)

Ans. (a) 155, 350 and 55 tons; (b) $106\frac{1}{19}$ and $79\frac{4}{9}$ ft. from end supports; (c) 6.7 and 7.3 in loaded span, 1.13 and 1.22 in unloaded span; (d) 58.3 ins. from centre line of top flange; (e), *first*, 155.415, 349.17, and 55.415 tons; *second*, 154.793, 305.414, 54.793 tons.

284. A continuous girder $ABCDE$ of four spans, each of length l, rests upon supports at A, B, C, D, and E, and carries a uniformly distributed load of intensity w. By how much must the supports at B, C, and D be lowered, so that the reactions at the five supports may be equal?

Ans. $\frac{23}{40}\frac{wl^4}{EI}$; $\frac{4}{5}\frac{wl^4}{EI}$; $\frac{23}{40}\frac{wl^4}{EI}$.

285. A continuous girder of two spans of 60 ft. and 90 ft. respectively is loaded with a uniformly distributed load of 2 tons per foot-run. Determine the reactions at the piers and the B.M. over the centre support (a) when the ends rest upon the supports; (b) when the end of the short span is fixed horizontally.

Ans. (a) $33\frac{3}{4}$, $193\frac{3}{4}$, and $72\frac{1}{2}$ tons; 1575 ft.-tons.
(b) $36\frac{1}{4}$, $190\frac{35}{36}$, and $72\frac{7}{9}$ tons; -125 and -1550 ft.-tons.

286. A rolled steel joist, 40 ft. in length, depth 10 ins., breadth 5 ins., thickness throughout $\frac{1}{2}$ in., is continuous over three supports, forming two spans of 20 ft. each. What uniformly distributed load would produce a maximum stress of $5\frac{1}{2}$ tons/sq. in.? Sketch the diagrams of bending moments and shear force. *Ans.* $15\frac{5}{18}$ tons.

287. A continuous girder of three spans carries a load of 1 ton per lineal foot. The two side spans are 28 and 84 ft. in length, and the intermediate span is 56 ft. in length. Find the reactions and the B.M.'s at the supports (*a*) when the two ends rests upon supports; (*b*) when the end of the 28-ft. span is fixed to the support.

Ans. (*a*) *Reactions*, $12\frac{11}{28}$, $32\frac{9}{28}$, $89\frac{29}{84}$, and $33\frac{79}{84}$ tons; *B.M.'s*, -45 and -677 ft.-tons.

(*b*) *Reactions*, $1\frac{3}{11}$, $100\frac{44}{55}$, $-3\frac{2}{11}$, $69\frac{6}{55}$ tons; *B.M.'s*, $+53\frac{5}{11}$, $-302\frac{10}{11}$, and $-625\frac{23}{55}$ ft.-tons.

288. A continuous girder fixed at one end has three equal spans each of 104 ft. Determine the B.M.'s and reactions at the supports when a load of 104 tons is uniformly distributed, (*a*) on the span next the fixture; (*b*) on the middle span; (*c*) on the remaining span. Hence deduce (*d*) the corresponding results when the whole girder carries a uniformly distributed load of 1 ton per lineal foot.

Ans. *B.M.'s*, (*a*) -1144, -416, $+104$ ft.-tons; (*b*) $+312$, -624, -520 ft.-tons; (*c*) -104, $+208$, -728 ft.-tons; (*d*) -936, -832, -1144 ft.-tons.

Reactions, (*a*) 59, 1, -6, 50 tons; (*b*) -9, $+62$, -5, $+56$ tons; (*c*) 3, -12, $+68$, $+45$ tons; (*d*) 53, 100, 118, 41 tons.

289. A continuous girder 180 ft. long consists of two spans of 100 ft. and of 80 ft. The smaller span carries a uniformly distributed load of 80 tons. Find the force required to hold the outer end of the unloaded span upon its support, and also determine the remaining reactions and the B.M. at the intermediate pier.

Ans. Required force $=3\frac{5}{9}$ tons; $35\frac{5}{9}$ tons, 48 tons; $355\frac{5}{9}$ in.-tons.

290. A continuous girder of three spans, the outside spans being equal, is uniformly loaded. What must be the ratio of the lengths of the centre and a side span so that the neutral axis may be horizontal over the intermediate supports? What should the ratio be if the centre span is hinged (*a*) at the centre; (*b*) at the points of trisection? *Ans.* $\sqrt{3}:\sqrt{2}$; (*a*) $1:1$; (*b*) $3:2\sqrt{2}$.

291. A girder carrying a uniformly distributed load is continuous over four supports, and consists of a centre span (l_2) and two equal side spans (l_1). Find the ratio of l_1 to l_2, so that the neutral axis at the intermediate supports may be horizontal. Also find the value of the ratio when a hinge is introduced (*a*) at the middle point of the centre span; (*b*) at the points of trisection of the centre span; (*c*) at the middle points of the half lengths of the centre span.

Ans. $\frac{l_1^2}{l_2^2}=\frac{2}{3}$; $\frac{l_1^2}{l_2^2}=\frac{1}{1}$; $\frac{l_1^2}{l_2^2}=\frac{8}{9}$; $\frac{l_1^2}{l_2^2}=\frac{3}{4}$.

292. A continuous girder *ABCD* is fixed at *A*, rests upon supports at *B*, *C*, and *D*, and carries a uniformly distributed load. If the reactions at *A*, *B*, *C*, *D* are equal show that the ratio of the length of *CD* to *AB* must be

greater than unity and less than $\frac{4}{3}$. Also, if CD is equal to five fourths of AB, show that $AB:BC:CD::4:7:5$.

293. A girder consists of two spans AB, BC, each of length l, and is continuous over a centre pier B. A uniform load of length $2a(<l)$ and of intensity w travels over AB. Find the reactions at the supports for any given position of the load, and show that the bending moment at the centre pier is a maximum and equal to $\frac{awl}{3\sqrt{3}}\left(1-\frac{a^2}{l^2}\right)^{\frac{3}{2}}$ when the centre of the load is at at distance $\left(\frac{l^2-a^2}{3}\right)^{\frac{1}{2}}$ from A.

294. Show that a uniformly loaded and continuous girder of two equal spans with both ends fixed is 2.08 times as stiff as if the ends were free and merely rested on the supports.

295. A continuous girder of two spans AB, BC rests upon supports at A, B. A uniformly distributed load EF travels over the girder. G_1 is the centre of gravity of the portion BE upon AB, and G_2 that of the portion BF upon BC. If the bending moment at B is a *maximum*, show that

$$\frac{AE\cdot EB}{CF\cdot FB}=\frac{AG_1}{CG_2}.$$

296. A continuous girder rests upon three supports and consists of two unequal spans $AB\ (=l_1)$, $BC\ (=l_2)$. A uniform load of intensity w travels over AB, and at a given instant covers a length $AD\ (=r)$ of the span. If R_1, R_3 are the reactions at A and C respectively show that

$$R_1l_1{}^2+R_3l_2{}^2=wr\left(l_1{}^2-\frac{3}{4}rl_1+\frac{1}{8}\frac{r^3}{l_1}\right).$$

Draw a diagram showing the shearing force in front of the moving load as it crosses the girder.

If the live load may cover both spans, show that the shearing force at any point D is a maximum when AD and BC are loaded and BD unloaded. Illustrate this force *graphically*, taking into account the dead load upon the girder.

297. A girder of uniform section rests on two supports at its ends on the same level and is loaded in such a manner that the area of the bending-moment diagram is A, and the distance of the centre of gravity of that diagram from the middle of the span (measured horizontally) is c, the span of the beam being $2a$. If the beam, instead of merely being supported at each end, has its ends built in horizontally, show that, with the same loading as before, the bending moments at the two ends are given by

$$A\frac{a\pm 3c}{2a^2}.$$

298. A continuous girder of two spans AB, BC has its two ends A and

C *fixed* to the abutments. The load upon AB is a weight P distant p from A, and that upon BC a weight Q distant q from C. The length of $AB = l_1$, of $BC = l_2$. The bending moments at A, B, C are M_1, M_2, M_3, respectively. The areas of the bending-moment curves for the spans AB, BC *assumed* to be independent girders are A_1, A_2, respectively. Show that

$$M_1 l_1 + M_2(l_1 + l_2) + M_3 l_2 = -2(A_1 + A_2)$$

and

$$M_2(l_1 + l_2) = -2(A_1 p + A_2 q).$$

If $l_1 = l_2 = l$, show that M_2 is a *maximum* if

$$2l(Pp - Qq) = 3(Pp^2 - Qq^2).$$

299. A continuous girder AC consists of two equal spans AB, BC of 15 m. each. Determine the bending moments at the supports, the maximum intermediate bending moments, and the reactions (*a*) when the load upon each span is 3000 k./m.; (*b*) when the load per metre is 3000 k. upon AB and 1000 k. upon BC. Consider three cases, viz., (I) when both ends of the girder are free; (II) when both ends are fixed; and (III) when one end is free and the other fixed.

Ans. Case I. (*a*) B.M.'s, −84,375 k.m., 47,460.9375 k.m.
Reaction, 16,875 k., 56,250 k.
(*b*) *B.M.'s*, −56,250 k.m., 58,593.75 k.m., 7031.25 k.m.
Reaction, 18,750 k., 37,500 k., 3750 k.
Case II. (*a*) *B.M.'s*, −56,250 k.m., 28,125 k.m.
Reactions, 22,500 k.
(*b*) *B.M.'s*, −65,625 k.m., −37,500 k.m., −9375 k.m., 33,398.4375 k.m., 64,453,125 k.m.
Reactions, 24,375 k., 30,000 k., 5625 k.
Case III. (*a*) *B.M.'s*, $-48{,}214\frac{2}{7}$ k.m., $-72{,}321\frac{3}{7}$ k.m., $24{,}537\frac{123}{196}$ k.m., $52{,}088\frac{127}{196}$ k.m.
Reactions, $20{,}892\frac{6}{7}$ k., $51{,}428\frac{4}{7}$ k., $17{,}678\frac{4}{7}$ k.
(*b*) *B.M.'s*, $-64{,}285\frac{5}{7}$ k.m., $-40{,}178\frac{4}{7}$ k.m., $-32{,}573\frac{67}{196}$ k.m., $11{,}623\frac{17}{196}$ k.m.
Reactions, $24{,}107\frac{1}{7}$ k., $31{,}071\frac{3}{7}$ k., $4821\frac{3}{7}$ k.

300. A viaduct over the Garonne at Bordeaux consists of seven spans, viz., two end spans, each of 57.375 m., and five intermediate spans, each of 77.06 m.; the main girders are continuous from end to end, and are each subjected to a dead load of 3050 k. per lineal metre. Under a live load of 405 k. per metre determine the absolute maximum bending moment at the third support from one end. Also find the corresponding reactions, the points of inflection, and the maximum deflection in the first and second spans.

Ans. 4,125,659 k.m.; *reactions*, 46,479 k., 381,008 k.; *points of inflection*, 15.24 m. from end support for first span; 7.88 m. and 6374 m. from second support in second span.

301. A girder of two spans each of 100 ft. is continuous over the three supports A, B, and C. Draw the S.F. and B.M. diagrams for a uniformly distributed load of 100 tons on AB and 50 tons on BC (*a*) when B.M. is *nil* at A and at C, (*b*) when A is fixed horizontally.

302. A horizontal continuous girder of three equal spans, resting upon supports at A, B, C, D, carries a uniformly distributed load. Show that the bending moments at the intermediate supports will be unaffected when the supports B, C, D are depressed below the horizontal, provided that the amounts of the depressions are in the ratios of 1 to 2 to 3.

303. The bridge over the Grande Baise consists of two equal spans of 19.8 m.; each of the main girders is continuous and rests upon abutments at the ends. Find the position of the points of inflection, the bending moment at the centre support, the maximum intermediate bending moment, and the maximum flange stress (*a*) under the dead load of 1700 k. per lineal metre; (*b*) under the same dead load together with an additional proof load of 2000 k. per lineal metre on one span. The depth of the girder $=3.228$ m., and $I=.093929232444$.

Ans. (*a*) 14.85 m. from the abutments; 83,308.5 k.m.; $46{,}861\frac{1}{32}$ k.m.; 1.4315 k./sq. mm.

(*b*) 16.18 m. from abutment on loaded side; 11.138 m. from abutment on unloaded side; 132,313.5 k.m.; 121,196.32 k.m.; 2.27356 k./sq. mm.

304. A continuous girder of three equal spans is uniformly loaded. By how much must the two intermediate supports be depressed to produce the same reactions at all the supports? *Ans.* $\frac{7}{24}\frac{wl^4}{EI}$.

305. A horizontal girder ABC is fixed at A, rests upon supports at B and C, and carries a uniformly distributed load $2W$. If an error $\varDelta$ is made (1) in AB, (2) in BC, find the consequent errors in the B.M.'s at A and B, and if $AB=BC$, show that these errors are

$$(1)\ \frac{43}{196}W\varDelta,\ \frac{12}{196}W\varDelta; \qquad (2)\ \frac{18}{196}W\varDelta,\ \frac{36}{196}W\varDelta.$$

306. A uniform beam is supported by four equidistant props, of which two are terminal. Show that the two points of inflection in the middle segment are in the same horizontal plane as the props.

307. The horizontal girder ABC is fixed at A and rests upon supports at B and C, AB being equal to BC. If the depression of C is n times that of B, and if R_1, R_2, R_3 are the reactions at A, B, C, respectively, show that $R_1(48-17n)+R_2(8-n)=R_3(80-23n)$.

308. A continuous girder of two equal spans (l) is uniformly loaded. Show that the ends will just touch their supports if the centre support is raised $\frac{wl^4}{8EI}$.

309. The horizontal girder ABC is fixed at A, rests upon supports at B and C, and carries weights W_1 and W_2 concentrated at the middle points of AB and BC respectively. Find the reactions and the bending moments

at A and B. If $AB=BC$ and if $W_2=3W_1$, show that the moment of fixture is *nil*, that the bending moment at B is $-\frac{3}{8}W_1AB$, and that the reactions at A, B, and C are $\frac{1}{8}W_1$, $\frac{11}{4}W_1$, and $\frac{9}{8}W_1$, respectively.

How much must B be lowered so that the reaction at B may be *nil?* Find the corresponding reactions at A and C. How much must C now be lowered so that the reactions may be the same as before?

$$Ans.\ \frac{AB^3}{EI}\left(\frac{25W_1+43W_2}{768}\right);\ \frac{AB}{128}(117W_1+47W_2);\ \frac{AB}{128}(11W_1+81W_2);$$

4×depression of B.

310. A continuous girder consists of two spans, each 50 ft. in length; the effective depth of the girder is 8 ft. If one of the end bearings settles to the extent of 1 in., find the maximum increase in the flange and shearing stress caused thereby, and show by a diagram the change in the distribution of the stresses throughout the girder. (Assume the section of the girder to be uniform, and take $E=25{,}000{,}000$ lbs.)

Ans. Increase of maximum B.M. $=2\frac{625}{16}\left(\frac{I}{216w}-1\right)$,

" " shearing force $=\frac{25}{72}I$,

w being weight per foot of length and I the moment of inertia.

311. The horizontal girder ABC is fixed at A and rests upon supports at B and C. If AB is n times BC, show that the bending moments at A and B are in the ratio of n^2+2n^2-1 to n^3+2.

312. A continuous girder of two equal spans is *fixed* at one of the end supports. The girder carries a uniformly distributed load of intensity w. If the length of each span is l, find the reactions and moment of fixture. How much must the intermediate support be lowered so that it may bear none of the load? How much should the free end support then be lowered to bring upon the supports the same loads as before?

$$Ans.\ \frac{11}{28}wl,\ \frac{16}{14}wl,\ \frac{13}{28}wl;\ -\frac{wl^2}{14};\ \frac{4}{48}\frac{wl^4}{EI};\ \frac{4}{24}\frac{wl^4}{EI}.$$

313. A continuous-girder bridge has a centre span of 300 ft. and two side spans, each of 200 ft. The dead load upon each of the main girders is 1250 lbs. per lineal foot. In one of the side spans there is also an additional load of 2500 lbs. per lineal foot upon each girder. Find (a) the reactions and points of inflection. How much (b) must the third support from the loaded end be lowered so that the pressure upon it may be just zero?

Ans. (a) *Reactions,* 305,460 lbs.; 655,850 lbs.; 322,515 lbs.; 91,175 lbs.

B.M.'s, 13,908,000 ft.-lbs.; 6,765,100 ft.-lbs.

Points of inflection. For side spans, $162\frac{83}{91}$ and $145\frac{80}{91}$ ft. from end supports; for centre span, distance x from end support is given by $x^2-\frac{13900}{21}x+\frac{4025000}{39}=0$.

(b) $\frac{503125000000}{EI}$.

314. A continuous girder AC consists of two equal spans AB, BC, each of length l, and carries a uniformly distributed load of intensity w upon AB, and of intensity nw upon BC. Determine the bending moments at the supports, the maximum intermediate bending moments, and the reactions (*a*) when both ends of the girder are fixed; (*b*) when one end A is fixed and the other free. If $w_1 = w_2 = w$, find (*c*) the points of inflection and (*d*) maximum deflection for each span in each case.

Ans. (*a*) *B.M.'s*, $\frac{wl^2}{48}(n-5)$; $-\frac{wl^2}{24}(n+1)$; $\frac{wl^2}{48}(1+5n)$;

$\frac{wl^2}{1536}(83-22n+3n^2)$; $\frac{wl^2}{1536}(243n^2-214n+35)$.

Reactions, $\frac{wl}{16}(9-n)$; $\frac{wl}{2}(1+n)$; $\frac{wl}{16}(-1+9n)$.

(*b*) *B.M.'s*, $-\frac{wl^2}{28}(3-n)$; 0; $-\frac{wl^2}{28}(1+2n)$;

$\frac{wl^2}{1568}(88-30n+9n^2)$; $\frac{wl^2}{1568}(-1+12n)^2$.

Reactions, $\frac{wl}{28}(16-3n)$; $\frac{wl}{28}(13+19n)$; $\frac{wl}{28}(-1+12n)$.

(*c*) *Points of inflection* at $.212l$ and $.788l$ from fixed ends.

Maximum deflection at middle point of span and $= \frac{1}{384}\frac{wl^4}{EI}$.

(*d*) *Points of inflection* for AB at $.195l$ and $.734l$ from A; for BC at $\frac{11}{14}l$ from C.

Maximum deflection for AB in (*c*) at $.64l$ from A and $= \frac{1}{473}\frac{wl^4}{EI}$.

" " for BC in (*d*) at $\frac{7}{16}l$ (approximately from C and $= \frac{1}{155}\frac{wl^4}{EI}$ (approximately).

315. A vertical row of water-tight sheet piling 30 ft. high is supported by a series of uprights placed 8 ft. centre to centre and securely fixed at the base, while the upper ends are kept in the vertical by struts sloping at 45°. If the water rises to the top of the piling, find (*a*) the thrust on a strut; (*b*) the maximum intensity of stress in an upright; (*c*) the amount and position of the maximum deviation of an upright from the vertical. If the piling is strengthened by a second series of struts sloping at 45° from the points of maximum deviation, find (*d*) the normal reactions upon an upright and the bending moment at its foot. What (*e*) will be the reactions and bending moment if the second row of struts starts from the middle of the uprights?

Ans. (*a*) $45{,}000\sqrt{2}$ lbs.;

(*b*) $\frac{1}{A}(5625 \pm \frac{c}{I}72{,}000\sqrt{5})$ lbs./sq. in. at $6\sqrt{5}$ ft. from upper end;

(*c*) $\frac{62208000\sqrt{5}}{EI}$ at $6\sqrt{5}$ ft. from upper end;

(*d*) 3645, 110,587.5 lbs., 3,677,400 in.-lbs.

(*e*) 8839.3 lbs., 115,714.3 lbs., 2,989,285.7 in.-lbs.

316. A continuous girder *ABCD* is fixed at *A*, rests upon supports at *B*, *C*, and *D*, and carries a uniformly distributed load. If the reactions at *A*, *B*, *C*, and *D* are equal, show that the ratio of the length of *CD* to *AB* must be greater than unity and less than $\frac{4}{3}$. Also, if $CD=\frac{5}{4}AB$, show that $AB:BC:CD::4:7:5$.

317. The Osse iron viaduct consists of seven spans, viz., two end spans of 28.8 m. and five intermediate spans of 38 m.; each main girder is continuous and carries a dead load of 1450 k. per lineal metre. Find (*a*) the bending moments at the supports when a proof load of 2250 k. per lineal metre for each girder covers all the spans; and also find (*b*) the absolute maximum bending moment at the *fourth* support. Is (*c*) the following section of sufficient strength?—two equal flanges, each composed of a 600-mm.×8-mm. plate riveted by means of two 100-mm.×100-mm.×12-mm. angles to a 600-mm.×10-mm. vertical web plate and two 80-mm.×80-mm.×11-mm. angles riveted to each horizontal plate with the ends of the horizontal arms 15 mm. from the edges of the plates, the whole depth of the section being 4.016 m., and the distance between the web plates, which is open, being 2.8 m. If insufficient, how would you strengthen it?

Ans. (*a*) *B.M.'s*, 416,518 k.m., 452,790 k.m., 443,722 k.m.; (*b*) 542,199 k.m.

(*c*) $I=.14074440467$ and max. flange stress/sq. mm. $=7.73$ k.

This is much too large. The section may be strengthened by adding two 600-mm.×8-mm. plates to each flange. I is thus increased by .0783425536, and the flange unit stress becomes 5 k./sq. mm.

318. The Estressol viaduct consists of four spans of 25 m.; the main girders are continuous and their ends rest upon abutments; the dead load upon each girder is 1700 k. per lineal metre. Determine the position of points of inflection in each span, the reactions and bending moments at the supports when an additional load of 2000 k. per lineal metre crosses (*a*) the first span; (*b*) the first and second spans; (*c*) all the spans. Also, find (*d*) the *absolute* maximum bending moments at the intermediate supports, and (*e*) determine the maximum flange stress at the piers, I being .093929232404.

Ans. Let x_1, x_2, x_3, x_4 be the distances of the points of inflection in the first, second, third, and fourth spans from the first, second, fourth, and fifth spans, respectively.

(*a*) $x_1=19.4$ m., $x_2^2-28x_2+140=0$, $x_3^2-24.16x_3+120.8=0$, $x_4=20.168$ m.

Reactions, $38,348\frac{3}{14}$ k., $81,160\frac{5}{7}$ k., $34106\frac{2}{7}$ k., $49911\frac{4}{7}$ k., $16,473\frac{3}{14}$ k.

B.M.'s, $197,544\frac{9}{14}$ k.m., $53,571\frac{3}{7}$ k.m., $119,419\frac{9}{14}$ k.m.

(*b*) $x_1=20.72$ m.; $x_2^2-31.78x_2+232.4=0$, $x_3^2-28.1x_3+140.5=0$, $x_4=19.38$ m.

Reactions, 36,178$\frac{4}{7}$ k., 107,821$\frac{3}{7}$ k., 62,964$\frac{2}{7}$ k., 45,892$\frac{6}{7}$ k., 17,142$\frac{6}{7}$.

B.M.'s, 258,928$\frac{4}{7}$ k.m., 120,535$\frac{5}{7}$ k., 102,678$\frac{4}{7}$ k.m.

(*c*) $x_1 = 19.64$ m. $= x_4$; $x_2{}^2 - 26.8x_2 + 134 = 0$ and $x_2 = x_3$.

Reactions, 36,339$\frac{2}{7}$ k., 105,714$\frac{2}{7}$ k., 85,892$\frac{6}{7}$ k.

B.M.'s, 247,767$\frac{6}{7}$ k.m., 165,178$\frac{4}{7}$ k.m.

(*d*) 264,508$\frac{13}{14}$ k.m. at second support when first, second, and fourth spans are loaded. 209,821$\frac{3}{7}$ k.m. at third support when second and third spans are loaded.

(*e*) 4.5 k./mm.^{2}.

319. If the three supports of any two equal consecutive spans of a continuous girder of any number of spans are depressed below the horizontal show that the relation between the three bending moments at the supports will be unaffected if the depression of the centre support is a mean between the depressions of the other two supports.

320. Four weights, each of 6 tons, follow each other at fixed distances of 5 ft. over a continuous girder of two spans, each equal to 50 ft. If the second and third supports are 1 in. and 1$\frac{1}{2}$ ins., respectively, vertically below the first support, find the maximum B.M. at the intermediate support.

Ans. $\left(49.275 - \frac{EI}{40000}\right)$ ft.-tons.

321. A continuous girder of two equal 50-ft. spans is fixed at one of the end supports. The girder carries a uniformly distributed load of 1000 lbs. per lineal foot. Find the reactions and bending moments at the points of support. How much must the intermediate support be lowered so that it may bear none of the load? How much should the free end be *then* lowered to bring upon the supports the same loads as at the first?

Ans. Reactions = 23,21$\frac{2}{7}$4, 57,142$\frac{6}{7}$, 19,642$\frac{6}{7}$ lbs.;
Bending moments = 178,571$\frac{3}{7}$, 267,857$\frac{1}{7}$ ft.-lbs.

322. Each of the main girders of a railway bridge resting upon two end supports and five intermediate supports is fixed at the centre support, is 3 ft. deep throughout, and is designed to carry a uniformly distributed *dead* load of $\frac{1}{4}$ ton and a live load of $\frac{1}{2}$ ton per lineal foot. The end spans are each 51 ft. 8 ins. and the intermediate spans each 50 ft. in the clear. Find (*a*) the reactions at the supports. The girders are single-webbed and double-flanged; the flanges are 12 ins. wide and equal in sectional area, the areas for the intermediate spans being 13 sq. ins. and 17 sq. ins. at the centre and piers respectively. Find (*b*) the corresponding moments of resistance and flange stresses, the web being $\frac{3}{8}$ in. thick.

Ans. (*a*) 15.356, 43.465, 35.740, 38.378 tons.

(*b*) *At centre*, .1757, 714, 978 in.-tons; 3.2, 1.3, and 1.78 tons/sq. in.

" *piers*, 2488, 1698, 1692 in.-tons; 3.59, 2.45, and 2.83 tons/sq. in.

323. In a certain Howe-truss bridge of eight panels the timber cross-ties are directly supported by the lower chords, and are placed sufficiently close to distribute the load in an approximately uniform manner over the whole length of these chords, thus producing an additional stress due to flexure. Assuming that the chords may be regarded as girders supported at the ends and continuous over seven intermediate supports coincident with the panel-points, and that these panel-points are in a truly horizontal line, determine (*a*) the bending moments and reactions at the panel-points; (*b*) the maximum intermediate bending moments; and (*c*) the points of inflection corresponding to a load of w per unit of length, l being the length of a panel. $\left(\text{Take } \frac{l}{r} = 388.\right)$

Ans. (*a*) *B.M.'s*, 0, $-41rwl^2$, $-30rwl^2$, $-33rwl$, $-32rwl^2$.
Reactions, $153rwl$, $440rwl$, $374rwl$, $392rwl$, $386rwl$.
(*b*) $11{,}704.5\ w(rl)^2$; $5104.5w(rl)^2$; $6600.5w(rl)^2$; $6208.5w(rl)^2$
(*c*) The points are defined by the values of x in $x = 306rl$;

$$R_1(l+x) + R_2x - \frac{w}{2}(l+x)^2 = 0;\ R_1(2l+x) + R_2(l+x),\ \text{and}$$

$$R_3x - \frac{w}{2}(2l+x)^2 = 0;$$

$$R_1(3l+x) + R_2(2l+x) + R_3(l+x) + R_4x - \frac{w}{2}(3l+x)^2 = 0.$$

324. If d_1, d_2, d_3, d_4 are respectively the deflections of the first, second, third, and fourth panel-points in question 323, show that the bending moment at the middle panel-point (M_4) is given by

$$-93M_4 = -\frac{6EI}{l^2}(69d_4 - 88d_3 + 24d_2 - 6d_1) + \tfrac{15}{2}wl^2.$$

325. Four loads, each of 12 tons and spaced 5, 4, and 5 ft. apart, travel in order over a continuous girder of two spans, the one of 30 and the other of 20 ft. Place the wheels so as to throw a maximum B.M. upon the centre support, and find the corresponding reactions. Draw a diagram of B.M. and find the maximum deflection of each span.

326. The loads upon the wheels of a truck, locomotive, and tender, counting in order from the front, are 7, 7, 10, 10, 10, 10, 8, 8, 8, 8 tons, the intervals being 5, 5, 5, 5, 5, 9, 5, 4, 5 ft. The loads travel over a continuous girder of two 50-ft. spans *AB*, *BC*. Place the locomotive, etc., (*a*) on the span *AB*, so as to give a maximum B.M. at *B*; (*b*) so as to give an *absolute maximum* B.M. at *B*.

327. An eight-wheel locomotive travels over a continuous girder of two 100-ft. spans; the truck-wheels are 6 ft. centre to centre, the load upon each pair being 8000 lbs.; the driving-wheels are 8½ ft. centre to centre, the load upon each pair being 16,000 lbs.; the distance centre to centre between the front drivers and the nearest truck-wheels is also 8½ ft. Place the locomotive so as to throw a maximum B.M. upon the centre support, and find the corresponding reactions.

328. A single weight travels over the span *AB* of a girder of two equal spans, *AB*, *BC*, continuous over a centre pier *B*. Show that the reaction

at C is a maximum when the distance of the weight from A is $\frac{AB}{\sqrt{3}}$ if the ends A and C rest upon their supports, and when the distance is $\frac{2}{3}AB$ if the two ends are fixed. Find the corresponding bending moments at the central pier.

Ans. $\frac{Pl}{6\sqrt{3}}$; $\frac{2}{27}Pl$.

329. The weights 7, 7, 10, 10, 10, 10, 8, 8, 8, 8 tons, taken in order, pass over a continous girder of two spans, each of 50 ft. and fixed at both ends, the successive intervals being 5, 5, 5, 5, 5, 9, 5, 4, 5 ft. Place the wheels so as to give the maximum bending moment at the centre support, and find its value. *Ans.* First wheel 25.8399 ft. from nearest abutment;
Max. B.M. = 306.62 ft.-tons.

330. A swing-bridge of 200 ft. length, with arms each of 90 ft., revolves a turntable of 20 ft. diameter. When the bridge is closed find the reactions at the four supports due to a load of 3 tons at 30 ft. from an end support.

CHAPTER VIII.

PILLARS.

1. Classification.—The manner in which a material fails under pressure depends not merely upon its *nature* but also upon its *dimensions* and *form*. The actual compressive strength of a material must be determined by the use of very short specimens, in which there is no tendency to buckle or bend. An accurate determination of the relation between the stress and the strain cannot, however, well be made unless the length of the specimen exceeds four or five times its least transverse dimension. The breaking load does not change much until the length is about ten times this dimension, and, if bending is prevented by means of slight lateral restraints, will remain practically the same whatever the length may be. If lateral restraints are not employed, the tendency to buckle or bend increases with the ratio of the length to the least transverse dimension. Denoting this ratio by s, all pillars (or columns) *with ends truly flat and firmly secured* may in general be divided into three classes, viz.:

A. *Short pillars*, in which s does not exceed 4 or 5, and which fail under a direct pressure.

B. *Medium pillars*, in which s lies between 4 and about 30 for timber, cast iron, and hard steel, and between 4 and about 60 for wrought iron and mild steel. The failure of pillars of this class is due partly to a direct pressure and partly to bending.

C. *Long pillars*, in which s exceeds about 30 for timber, cast iron, and hard steel, and about 60 for wrought iron and mild steel. The failure of these pillars is practically wholly due to flexure.

2. Manner of Failure.—The manner in which a short pillar fails depends upon the character of the material.

A granular or crystalline mass may give way suddenly under compression and be reduced to powder.

More frequently hard and brittle substances like glass, bricks, stones, dry timber, cast iron, etc., which possess little ductility, split into fragments, and sometimes a hard vitreous material breaks up into prisms (Fig. 552).

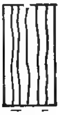


Fig. 552.

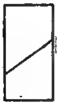


Fig. 553.

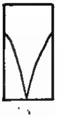


Fig. 554.

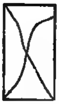


Fig. 555.

The failure of fibrous and granular materials like timber, bricks, cement and concrete cubes, artificial and certain natural stones, cast iron, etc., is usually by shearing along planes oblique to the line of thrust, and the more homogeneous the material is the more nearly do these planes coincide with the planes of least resistance to shear, which theoretically makes an angle of 45° with the axis of the pillar (Chapter V). Thus the portions of the fractured pillar assume the forms of wedges or pyramids (Figs. 553–558), and in the testing of cement and concrete cubes it not infrequently happens that a very nearly perfect double pyramid (Fig. 556) is produced.

None of the materials of construction, however, are truly homogeneous, and in the case of cast iron the irregularity of the texture and the hardness of the skin cause the angle between the plane of shear and the direction of the thrust to vary from 32° to 42°. Brick chimneys again sometimes fail by the shearing of the mortar, the upper portion sliding over an oblique plane.

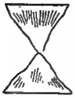


Fig. 556.

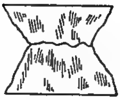


Fig. 557.

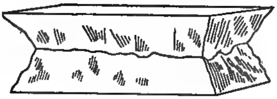


Fig. 558.

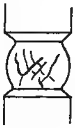


Fig. 559.

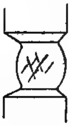


Fig. 560.

Bulging, i.e., a lateral spreading out (Figs. 559, 560), is characteristic of blocks of fibrous materials, e.g., wrought iron, copper, lead, and timber, and fracture occurs in the form of longitudinal cracks.

All substances, even the most crystalline, will bulge slightly before they fail, if they possess some degree of ductility.

Buckling is characteristic of fibrous materials, and the resistance of a pillar to buckling is always less than its resistance to direct crushing, and is independent of length.

Thin malleable plates usually fail by the bending, puckering, wrinkling, or crumpling up of the fibres, and the same phenomena may be observed in the case of timber and of long bars.

In the transverse testing of timber beams the initial failure is usually made evident by the crippling of the fibrous layers on the compression side.

Timber offers about twice the resistance to crushing when dry that it does when wet, as the presence of moisture diminishes the lateral adhesion of the fibres.

Long plate tubes, when compressed longitudinally, first bend and eventually fail by the buckling of a short length on the concave side.

The ultimate resistance to buckling of a well-made and well-shaped tube is about 27,000 lbs. per square inch section of metal, which may be increased to 33,000 or 36,000 lbs. per square inch by dividing the tube into two or more compartments.

A rectangular wrought-iron or steel tube offers the greatest resistance to buckling when the mass of the material is concentrated at the angles, while the sides consist of thin plates or lattice-work sufficiently strong to prevent the bending of the angles.

3. Hodgkinson's Formulæ for the Ultimate Strength of Long and Medium Pillars.—When a *long* pillar is subjected to a crushing force it first yields sideways, and eventually breaks in a manner apparently similar to the fracture of a beam under a transverse load. This similarity, however, is modified by the fact that an initial longitudinal compression is induced in the pillar by the super-imposed load.

Hodgkinson deduced experimentally that the strength of *long solid* round iron and square timber pillars *with flat ends* is given by an expression of the form

$$W = A\frac{d^n}{l^m},$$

W being the breaking weight in tons of 2240 lbs.;
d " " diameter or side of the pillar in *inches;*
l " " length of the pillar in *feet;*
n and m being numerical indices;

A being a constant varying with the material and with the sectional form of the pillar.

Such pillars, which are more than *thirty* times the least transverse dimension in length, fail by bending and not by simple crushing. The load is distributed more or less uniformly over the ends, which are finished in planes at right angles to the axis of the pillar.

For iron pillars. . . . . $n = 3.55$ and $m = 1.7$
" timber pillars . . . . $n = 4$ and $m = 2$
" cast iron. . . . . $A = 44.16$
" wrought iron. . . . . $A = 133.75$
" dry Dantzic oak. $A = 10.95$
" dry red deal. . . . . $A = 7.81$
" dry French oak. . . . . $A = 6.9$

The strength of *long hollow* round cast-iron pillars was found to be given by

$$W = 44.34 \frac{d^{3.6} - d_1{}^{3.6}}{l^{1.7}},$$

d being the external and d_1 the internal diameter, both in inches.

Thus the strength of a *hollow* cast-iron pillar is approximately equal to the difference between the strengths of two *solid* cast-iron pillars whose diameters are equal to the external and internal diameters of the hollow pillar.

The strength of *medium* pillars, i.e., of pillars which are *less* than thirty times the least transverse dimension in length, may be obobtained by the formula

$$W' = \frac{WfS}{W + \frac{3}{4}fS},$$

W' being the breaking weight in tons of 2240 lbs.;
W " " " " " " " " " as derived from the formula for *long* pillars;
f being the ultimate crushing strength in tons per square inch;
S " " sectional area of the pillar in square inches.

Again, if the ends of a cast-iron pillar are rounded, the above formulæ may be still employed to determine its strength, A being 14.9 for a *solid* and 13 for a *hollow* pillar.

According to Hodgkinson, the relative strengths of long cast-iron pillars of equal weight and length may be tabulated as follows:

(*a*) Pillars with *flat* ends:

The strength of a solid round pillar being 100,
" " " " square " is 93,
" " " " triangular pillar is 110.

(*b*) Pillars with round ends, i.e., ends for hinging or pin connections:

The strength of a hollow cylindrical pillar being 100,
" " " an H-shaped " is 74.6,
" " " a +-shaped " " 44.2.

The strengths of a long solid round pillar with flat ends and a long hollow cylindrical pillar with round ends are approximately in the ratio of 2.3 to 1.

The *stiffest* kind of wrought-iron strut is a built tube, the section consisting of a cell or of cells, which may be circular, rectangular triangular, or of any convenient form.

In experimenting upon hollow tubes, Hodgkinson found that, other conditions remaining the same, the *circular* was the strongest, and was followed in order of strength by the *square* in *four* compartments ⊞; the *rectangular* in *two* compartments, ◫; the *rectangular*, ▭; and the *square*.

The addition of a diaphragm across the middle of the rectangle *doubled* its resistance to crippling.

As the results of his experiments Hodgkinson also made the following inferences:

A pillar with both ends rough from the foundry so that a load can be applied only at a few isolated points, and a pillar with a rounded end so that the load can be applied only along the axis, are each *one third* of the strength of a pillar of class B, and from *one third* to *two thirds* of the strength of a pillar of class C, the pillars being of the same dimensions.

The strength of a pillar with one end flat and the other round is an arithmetical mean between the strengths of two pillars of the same dimensions, the one having both ends flat and the other both ends round.

Disks at the ends of pillars only slightly increase their strength, but facilitate the formation of connections.

An enlargement of the middle section of a pillar sometimes increases its strength in a small degree, as in the case of solid cast-iron pillars with rounded ends, which are made stronger by about *one seventh;* hollow cast-iron pillars are not affected. The strength of a disk-ended pillar is increased by about *one eighth* or *one ninth* when the middle diameter is lengthened by 50 per cent, but for slight enlargements the increase is imperceptible.

The strength of hollow cast-iron pillars is not affected by a slight variation in the thickness of the metal, as a thin shell is much harder than a thick one. The excess above or deficiency below the average thickness should not exceed 25 per cent.

In *metric measurement,* if d is the diameter and l the height of a *solid* iron column in *millimeters,* the safe load in *kilogrammes*

$$=1900\frac{d^4}{l^2} \text{ for cast iron,}$$

and

$$=3800\frac{d^4}{l^2} \text{ for wrought iron.}$$

Again, if a is a side of a square timber post in *centimeters* and l its height in *decimeters,* the safe load in *kilogrammes*

$$=256.5\times\frac{a^4}{l^2} \text{ for first-quality well-seasoned oak}$$

$$=180\frac{a^4}{l^2} \quad \text{`` oak of ordinary quality}$$

$$=214.2\times\frac{a^4}{l^2} \quad \text{`` first-quality well-seasoned pine}$$

$$=160\frac{a^4}{l^2} \quad \text{`` pine of ordinary quality.}$$

Ex. 1. *A solid cast-iron pillar* 9 *ft. in height and* 4 *ins. in diameter supports a load of* 55,000 *lbs. Find the normal and shearing intensity of stress per square inch in a plane section inclined at* 30° *to the axis.*

If the ends of a pillar are flat and firmly bedded, determine its breaking weight by Hodgkinson's formula.

$$\text{Axial intensity of stress/sq. in.} = \frac{55000}{\frac{22}{7}\,\frac{1}{4}(4)^2 \operatorname{cosec} 30^\circ} = 2187\tfrac{1}{2} \text{ lbs.}$$

Therefore

$$\text{normal intensity of stress} = 2187\tfrac{1}{2} \sin 30^\circ = 1093\tfrac{3}{4} \text{ lbs./sq. in.}$$

and $$\text{tangential intensity of stress} = 2187\tfrac{1}{2} \cos 30^\circ = 1894\tfrac{3}{8} \text{ lbs./sq. in.}$$

By Hodgkinson's formula for short pillars,

$$\text{the breaking weight in tons} = 44.16\frac{(4)^{3.6}}{(9)^{1.7}} = 154.97.$$

Hence, since pillar is of medium length,

$$\text{its actual breaking weight in tons} = \frac{154.97 \times \frac{80000}{2240}\,\frac{22}{7}\,4}{154.97 + \frac{3}{4}\frac{80000}{2240}\,\frac{22}{7}\,4} = 141\tfrac{1}{2},$$

taking 80,000 lbs. per sq. in. as the ultimate crushing strength of the cast iron.

Ex. 2. *Find the side of a square post of ordinary oak which is to be* 5 *metres high and to carry a load of* 10,000 *kilogrammes.*

$$10000 = 180\frac{a^4}{50^2}.$$

Therefore $$a = 19 \text{ cm.}$$

Ex. 3. *Determine the diameter of a solid cast-iron column which is to be* 5 *metres high and to carry a load of* 50,000 *kilogrammes.*

$$50000 = 1900\frac{d^4}{(5000)^2}.$$

Therefore $$d = 160 \text{ mm.}$$

4. Gordon's Formula for the Strength of a Pillar of Medium Length.—Consider a pillar with *flat ends,* of length l, and sectional area S, h being the least transverse dimension in the plane of flexure.

The effect of P is *twofold.*

In the first place, P produces along the pillar a direct thrust of intensity $f_1 = \dfrac{P}{S}$.

In the second place, it tends to bend the pillar and develops a skin compressive stress f_2 which is necessarily greatest at the section in which the neutral axis has deviated farthest from its initial vertical position.

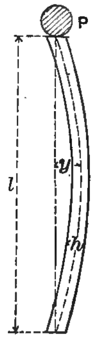


FIG. 561.

Denoting this deviation by y,

$$Py = \text{B.M.} = f_2\frac{I}{c},$$

c being the distance of the most compressed fibre from the neutral axis and I the moment of inertia of the section in the plane of flexure.

Therefore $$f_2=P\frac{cy}{I}=\frac{P}{S}\,\frac{cy}{r^2}=f_1\frac{cy}{r^2},$$

r being the least radius of gyration.

But $c\propto h$, $r\propto h$, and by Chapter VII, $y\propto\frac{l^2}{h}$.

Therefore $$f_2\propto f_1\frac{h}{h^2}\,\frac{l^2}{h}=f_1a\frac{l^2}{h^2},$$

a being a coefficient which is to be determined by experiment.

Hence, denoting by f the *total maximum stress* in the most strained fibre

$$f=f_1+f_2=f_1\left(1+a\frac{l^2}{h^2}\right)=\frac{W}{S}\left(1+a\frac{l^2}{h^2}\right),$$

and $$\frac{W}{S}=f_1=\frac{f}{1+a\frac{l^2}{h^2}},$$

which is known as *Gordon's formula.*

The coefficient a is by no means constant, and not only varies with the nature of the material, with the length of the pillar, with the condition of its ends, etc., but also with the sectional form of the pillar. The variation due to this latter cause may be eliminated, and the formula rendered somewhat more exact, by substituting the least *radius of gyration* for the least transverse dimension.

It has been shown that

$$f_2=f_1\frac{cy}{r^2},$$

and since $c\propto h$ and $y\propto\frac{l^2}{h}$,

therefore $$f_2\propto f_1\frac{l^2}{r^2}=f_1a_1\frac{l^2}{r^2}.$$

Hence Gordon's formula now becomes

$$\frac{W}{S}=f_1=\frac{f}{1+a_1\frac{l^2}{r^2}},$$

in which a_1 is independent of the sectional form, all variations of the latter being included in r^2. This modified form of Gordon's formula was first suggested by Rankine.

A pillar (or column) is *square-bearing* or flat-ended when it has square ends which butt against or are securely connected with an immovable surface; it is *pin- and square-bearing* when one end is square-bearing while the other presses against a close-fitting pin; it is *pin-bearing* when both ends are pin-jointed, the axes of the pins being parallel. In pin- and square-bearing pillars $\frac{9}{5}a$ and $\frac{9}{5}a_1$ should be substituted in the preceding formulæ for a and a_1, while the coefficients should be $4a$ and $4a_1$ if the pillar is pin-bearing.

5. Values of a, a_1 and f.—The following table, giving the values of the constants a, a_1, and f, has been prepared by taking an average of the best-known results, and is applicable to *square-bearing* round and square pillars:

Material and Form of Section.	f in Lbs. per Sq. In.	$\frac{1}{a}$.	$\frac{1}{a_1}$.
For *cast-iron* solid rectangular pillars.	80,000	450	6,400
" " " round pillars.	80,000	400	
" " hollow rectangular pillars.	80,000	500	
" " " round pillars.	80,000	600	
For *wrought-iron* solid rectangular pillars.	36,000	3,000	36,000
" " " round pillars.	36,000	2,250	
" " thick hollow round pillars.	36,000	5,500	
For *mild-steel* solid rectangular pillars.	67,200	2,000	36,000
" " " round pillars.	67,200	1,400	
" " hollow round pillars.	67,200	2,500	
For *strong-steel* solid rectangular pillars.	114,000	1,400	
" " solid round pillars.	114,000	900	
" " hollow round pillars.	114,000	1,500	
For *pine-timber* solid rectangular pillars.	5,000	250	3,000
" " " round pillars.	5, 00	250	
For *dry oak* timber.	7,200	250	

Graphical comparison of the crushing unit strength of solid round cast-iron, wrought-iron, and mild-steel pillars.

The crushing unit stress is given by $p = \dfrac{f}{1+a\dfrac{l^2}{h^2}}$.

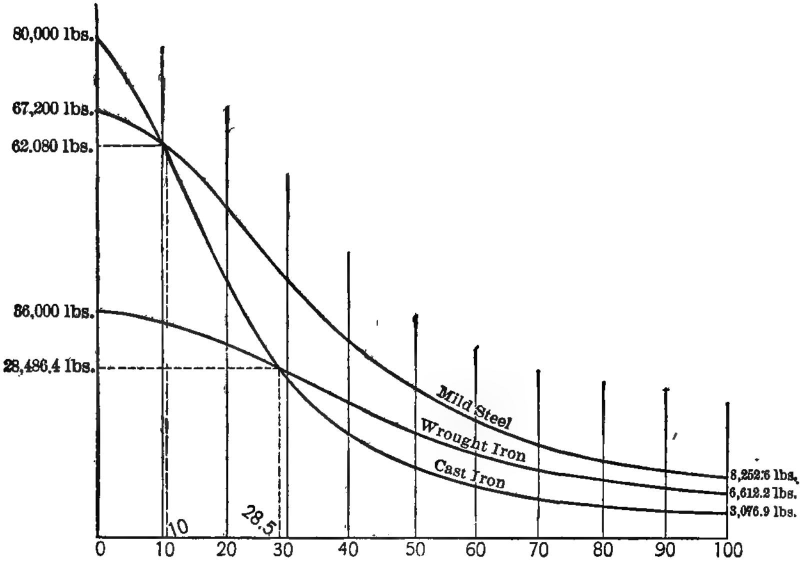


FIG. 562

Take the different values of $\frac{l}{h}$ as abscissæ, and the corresponding values of p as ordinates; the resulting curves for the values of f given in the table are shown in Fig. 562.

Hence the strength of a mild-steel pillar exceeds that of a wrought-iron pillar, but is less than that of a cast-iron pillar when $\frac{l}{h} < 10.7$; a wrought-iron pillar is stronger or weaker than a cast-iron pillar according as $\frac{l}{h} >$ or < 28.5.

Ex. 4. *Compare the breaking weights of round cast-iron, wrought-iron, and mild-steel pillars with flat ends, each being* 9 *ft. in length and* 6 *ins. in diameter. How will the results be modified if both ends are round?*

$$\frac{l}{h} = \frac{12\times 9}{6} = 18.$$

First, both ends flat:

For cast-iron, $\quad \text{B.W.} = \dfrac{80000\times\frac{22}{7}\times\frac{1}{4}(6)^2}{1+\frac{1}{400}(18)^2} = 1{,}250{,}197$ lbs.

For wrought-iron, $\text{B.W.} = \dfrac{36000 \times \frac{22}{7} \frac{1}{4}(6)^2}{1 + \frac{1}{2250}(18)^2} =$ 890,109 lbs.

" mild-steel, $\text{B.W.} = \dfrac{67200 \times \frac{22}{7} \frac{1}{4}(6)^2}{1 + \frac{1}{1400}(18)^2} =$ 1,543,572 "

Second, both ends round:

For cast-iron, $\text{B.W.} = \dfrac{80000 \times \frac{22}{7} \frac{1}{4}(6)^2}{1 + \frac{4}{400}(18)^2} =$ 533,693 "

" wrought-iron, $\text{B.W.} = \dfrac{36000 \times \frac{22}{7} \frac{1}{4}(6)^2}{1 + \frac{4}{2250}(18)^2} =$ 646,120 "

" mild-steel, $\text{B.W.} = \dfrac{67200 \times \frac{22}{7} \frac{1}{4}(6)^2}{1 + \frac{4}{1400}(18)^2} =$ 987,060 "

Ex. 5. *A hollow mild-steel pillar* 10 *ft. high and* 4 *ins. outside diameter has to carry a load of* 33,000 *lbs. Taking* 6 *as a factor of safety and assuming that both ends are to be pin-connected, find the proper thickness of the metal.*

$$6 \times 33000 = \frac{67200 \times \frac{22}{7} \frac{1}{4}(4^2 - \overline{4-2t}^2)}{1 + \dfrac{4}{2500}\left(\dfrac{10 \times 12}{4}\right)^2},$$

t being the required thickness. This reduces to

$$t^2 - 4t + 2.2875 = 0,$$

and therefore t, the thickness, $= .69$ in.

Ex. 6. *A pillar of diameter D supports a given load; if N pillars, each of diameter d, are substituted for this single pillar, show that d must lie between $\dfrac{D}{N^{\frac{1}{2}}}$ and $\dfrac{D}{N^{\frac{1}{4}}}$. Is it more economical to employ few or many pillars of given height to support a given load?*

$$\frac{f\pi\frac{1}{4}D^2}{1 + a\left(\dfrac{l}{D}\right)^2} = \text{the given load} \times \text{factor of safety} = \frac{fN\pi\,\frac{1}{4}d^2}{1 + a\left(\dfrac{l}{d}\right)^2},$$

which reduces to $al^2(D^4 - Nd^4) = D^2d^2(Nd^2 - D^2)$.

Therefore $D^4 > Nd^4$ and $Nd^2 > D^2$.

Hence $d < \dfrac{D}{N^{\frac{1}{4}}}$ and $> \dfrac{D}{N^{\frac{1}{2}}}$.

If w is the specific weight of the material,

the weight of the N pillars $=N\pi\frac{1}{4}d^2lw=P_1$,

" " " " single pillar $=\pi\frac{1}{4}D^2lw=P_2$,

and $P_1 > P_2$, since $Nd^2 > D^2$.

It is therefore more economical to use a single pillar.

Ex. 7. *Find the side of a square post of ordinary quality oak which is to be* $12\frac{1}{2}$ *ft. high and is to carry a load of* 20,000 *lbs.,* 10 *being a factor of safety. The ends are to be flat.*

If x ins. is a side of the square section,

$$10\times 20000=\frac{7200x^2}{1+\frac{1}{250}\left(\frac{12\times 12\frac{1}{2}}{x}\right)^2}=\frac{7200x^4}{x^2+90}.$$

Therefore
$$x^4-\frac{250}{9}x^2-2500=0$$

and
$$x=8.12 \text{ ins.}$$

6. Values of a and a₁ for Shapes and Built-up Members.—Further experiments are still needed for the proper determination of the coefficients a and a_1 in the case of built-up members and of such shapes as are shown by Figs. 563 to 568. At present every engineer uses the values for which his own experience gives him a preference, but it is usually found that there is no practical difference in the sectional areas of pillars (or struts) designed in accordance with any good standard specification.

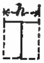


FIG. 563.

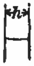


FIG. 564.

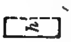


FIG. 565.

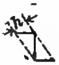


FIG. 566.

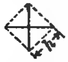


FIG. 567.

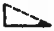


FIG. 568.

The least transverse dimension (i.e., h) is to be measured in the plane of greatest flexure.

Thus it may be taken as the *least* side of the rectangle circumscribing *tee* (Fig. 563), H *channel* (Fig. 565), and *cruciform* (Fig. 567) sections, and as the perpendicular from the angle to the opposite side of a triangle circumscribing *angle* (Fig. 568) sections.

The following table gives the values of f and a for the shapes in question, the members being square-bearing. These values have been deduced experimentally, but must be regarded as approximate only:

Material.	f in lbs./sq. in.	$\frac{1}{a}$
Cast iron.	80,000	134
Wrought iron.	42,500	900
Mild steel.	45,000	1300

The effect of a change of form is best provided for by the use of the formula which involves the least radius (r) of gyration. Thus the ultimate strength in pounds per square inch of medium-steel columns is

$$\frac{50{,}000}{1+a_1\left(\frac{l}{r}\right)^2},$$

$\frac{1}{a_1}$ being 36,000, 24,000, or 18,000 according as the column is square-bearing, pin- and square-bearing, or pin-bearing.

The factor of safety may be 4 for quiescent loads, as in buildings, and 5 for moving loads, as in bridges.

Ex. 8. *Find the breaking weight of a square-bearing cast-iron column* 10 *ft. high and of cruciform section, the metal being* 2 *ins. thick throughout and the outside length of each arm being* 8 *ins.* Then

$$A=28 \text{ sq. ins.} \quad \text{and} \quad h^2=3^2+3^2=18.$$

Therefore

$$\text{the breaking weight}=\frac{80000\times 28}{1+\frac{3}{400}\left(\frac{120}{h}\right)^2}=320{,}000 \text{ lbs.}$$

Ex. 9. *Determine the allowable stress in pounds per square inch of a* 10-*ft. pin-bearing strut composed of two* 5″×3″×$\frac{5}{16}$″ *angles placed back to back,* 5 *being a factor of safety.*

First, let the long legs be back to back and spread $\frac{1}{2}$ in. to admit of a single line of $\frac{1}{2}$-in. lattice bars along the central plane. Then

$$I=7.6205; \quad A=4.8 \text{ sq. ins}; \quad r^2=\frac{7.6205}{4.8}=1.5876,$$

and the allowable stress in pounds per square inch

$$=\frac{10000}{1+\frac{1}{18000}\left(\frac{120}{r}\right)^2}=6649.$$

Second, let the short legs be back to back. Then I with respect to neutral axis through centre of gravity $=3.468$. Therefore

$$r^2=\frac{3.468}{4.8}=.7225,$$

and the allowable stress in pounds per square inch

$$=\frac{10000}{1+\frac{1}{18000}\left(\frac{120}{r}\right)^2}=4746.$$

Again, the American Bridge Co.'s specifications limit the length of wind-bracing pin-bearing struts to 120 times the least radius of gyration, while the lengths of other pin-bearing members must not exceed 100 times this radius. They allow a working stress in pounds per square inch of

$$\frac{15000}{1+\frac{1}{13500}\left(\frac{l}{r}\right)^2} \text{ for soft steel,}$$

and of

$$\frac{17000}{1+\frac{1}{11000}\left(\frac{l}{r}\right)^2} \text{ for mild steel.}$$

Straight Line Formula.—The coefficients in pillar formulæ, generally speaking, have been determined by experiments in which the pillar has been ultimately crushed under a steadily applied and gradually increased load. Recent experience has shown that the working stress of framed and shaped compression members may be conveniently expressed by the "straight-line" formula

$$\text{working stress per square inch}=\mathbf{f}-\mu\frac{\mathbf{1}}{\mathbf{r}},$$

where $f=12{,}000$ lbs. and $\mu=45$, 53, or 60, according as the member is square-bearing, pin- and square-bearing, or pin-bearing.

No provision is made in this formula for the detrimental effect of the repeated stresses to which a member is often subjected by the action of a live load, as when a train passes over a bridge. This "impact" effect might be allowed for by reducing the live load to an equivalent dead load and making a corresponding change in the coefficients. Thus, in bridge-work, take $f=18{,}000$ lbs., and $\mu=60$, 70, and 80, instead of the values just given.

Again, for top chords (square ends with or without play) take $f=18{,}000$ lbs. and $\mu=70$.

For end posts or batter braces (with pin ends or ends tending to rotate) take $f=18{,}000$ lbs. and $\mu=80$.

For intermediate posts (allowance being made for pin ends, distortion, and greater liability to shock) take $f=16{,}000$ lbs. and $\mu=80$.

Ex. 10. *Design a medium-steel top-chord section, the unsupported length being* 26 *ft. and the total thrust along the chord being* 800,000 *lbs.* Try a section (Fig. 569) made up of

One $24'' \times \frac{1}{2}''$ cover plates	$=12$	sq. ins.
Two $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{3}{8}''$ top angles	$=4.96$	" "
" $18'' \times \frac{5}{8}''$ webs	$=22.50$	" "
" $6'' \times 4'' \times \frac{3}{4}''$ bottom angles	$=13.88$	" "
so that the total sectional area	$=53.34$	" "

For practical reasons a clearance of $\frac{1}{8}$ in. may be left between the edges of the web plates and the backs of the top and bottom angles, so that the depth over all $=\frac{1}{2}+2(\frac{1}{8})+18=18\frac{3}{4}$ ins.

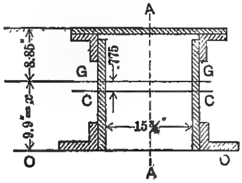


FIG. 569.

Let x be the distance of the centre of gravity of the section from OO. Then

$$x(53.34)=12\times18.5+4.96\times17.24+22.5\times9.125+13.88\times1.08=527.81,$$

and therefore $x=9.9$ ins.

The moment of inertia about the horizontal axis GG through the centre of gravity

$$\begin{aligned}&=\tfrac{1}{12}\times24(\tfrac{1}{2})^3+12(8.6)^2+2\times2.87+4.96(7.34)^2\\&+2\times\tfrac{5}{8}\times18^3\times\tfrac{1}{12}+22.5(.775)^2+2\times8.68+13.88(8.82)^2\\&=2878.9,\end{aligned}$$

and the radius of gyration with respect to GG is given by

$$r=\sqrt{\frac{2878.9}{53.34}}=7.346 \text{ ins.}$$

The moment of inertia with respect to the vertical axis AA will be found to be greater than that with respect to GG, so that 7.346 is the *least* radius of gyration, and therefore

$$\text{the allowable stress per square inch}=18000-70\frac{26\times12}{7.346}=15{,}026 \text{ lbs.}$$

Hence the sectional area required $=\frac{800000}{15026}=53.24$ sq. ins., which agrees very closely with the area of the assumed section.

In Cooper's bridge specifications *for medium steel and for live loads*

$f=10{,}000$ and $\mu=45$ for chord segments;
$f=8{,}500$ " $\mu=45$ " all posts of through-bridges;
$f=9{,}000$ " $\mu=40$ " all posts of deck bridges and trestles;
$f=13{,}000$ " $\mu=60$ " wind (i.e., live) loads on laterals and rigid bracing.

These results are to be doubled for *dead loads.*

The ratio $\frac{l}{r}$ should not exceed 100 for main members or 120 for laterals.

The allowable stresses for *soft steel* should be about 15 per cent less than those for medium steel.

7. Bending of Struts.—The strut of length l and constant sectional area S is assumed to be homogeneous and its weight is disregarded (Euler's Theory).

CASE I. *Strut Hinged at Both Ends.*—Let OBA, Fig. 570, represent the neutral axis of the bent strut and assume that the line of action of the resultant thrust P on the strut coincides with the axis before bending commences.

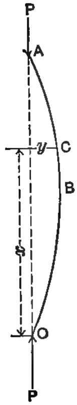


FIG. 570.

Let x, y be the vertical and horizontal distances respectively of any point C with respect to O. Then

$$Py=\text{B.M. at } C=-EI\frac{d^2y}{dx^2},$$

which may be written in the form

$$\frac{d^2y}{dx^2}+a^2y=0,$$

where

$$a^2=\frac{P}{EI}.$$

The complete solution of any such equation is

$$y=b\sin ax+c\cos ax,$$

b and c being arbitrary constants whose values are governed by the conditions of the particular problem under consideration.

In the present case $y=0$ when $x=0$, i.e., at O, so that c is also *nil*,

and $$y=b \sin ax.$$

Again, $y=0$ when $x=OA=OBA$, approximately $=l$. Therefore

$$0=b \sin al.$$

Now, b cannot be *nil*, as in such case y would be always *nil*, and lateral bending would be impossible. Hence

$$\sin al=0$$

and $$al=n\pi=l\sqrt{\frac{P}{EI}},$$

so that $$P=n^2EI\frac{\pi^2}{l^2},$$

n being a whole number.

The least value of P corresponds to $n=1$, and therefore

$$\mathbf{P}=\mathbf{EI}\frac{\pi^2}{\mathbf{l}^2}$$

is the *minimum* thrust which will produce bending. It is often called the *buckling load* of the strut.

The deviation of the axis from the vertical, i.e., the deflection, is greatest when

$$\frac{dy}{dx}=0=ab \cos ax,$$

or when $$ax=\frac{\pi}{2},$$

and then $$y=b \sin\frac{\pi}{2}=b,$$

so that the coefficient b is the maximum deflection, which, in the present case, is the deflection at the middle of the strut.

The meaning of the values 2, 3, 4, . . . , for n in the equation

$$n\pi = al$$

is that y is assumed to be 0 at one or more points between O and A, so that the strut has one or more points of inflection.

If the column is made to pass through N points dividing the vertical OA into $N+1$ equal divisions, then

$$y=0 \text{ when } x=\frac{l}{N+1},$$

and therefore, by eq. (4),
$$0=b\sin\frac{al}{N+1},$$

or
$$\frac{al}{N+1}=n\pi,$$

and hence
$$P=n^2EI\frac{\pi^2}{l^2}(N+1)^2.$$

As before, the least value of P corresponds to $n=1$, and

$$\mathbf{P}=\mathbf{EI}\frac{\pi^2}{\mathbf{l}^2}(\mathbf{N}+1)^2$$

is the least force which will bend the column laterally.

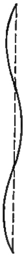


FIG. 571.

Hence the strength of the column is increased in the ratio of 4, 9, 16, etc., by causing it to pass through points which divide its length into 2, 3, 4, etc., equal parts, respectively.

Effect of a Lateral Load.—Let M_x be the B.M. at C due to a lateral load on the strut. Then

$$\frac{d^2y}{dx^2}+a^2y+\frac{M_x}{EI}=0,$$

which may be easily integrated as soon as M_x has been determined. For example, if the lateral load is of intensity w and uniformly distributed

$$M_x=\frac{w}{2}x(l-x),$$

a law which may be approximately and more conveniently expressed in the form

$$M_x=\frac{Ql}{8}\sin\frac{\pi x}{l},$$

Q being the total lateral load. Therefore

$$\frac{d^2y}{dx^2}+a^2y+\frac{1}{8}\frac{Ql}{EI}\sin\frac{\pi x}{l}=0.$$

Integrating,
$$(P_1-P)y=\frac{Ql}{8}\sin\frac{\pi x}{l},$$

where $$P_1=EI\frac{\pi^2}{l^2}.$$

The deflection (y) is greatest at the middle, and therefore

$$y_{\text{max.}}=\frac{Ql}{8(P_1-P)}.$$

The B.M. is also greatest at the middle, and therefore, since

$$M_x=\frac{Ql}{8}\text{ at the middle,}$$

$$\text{the max. B.M.}=EIa^2y_{\text{max.}}+\frac{Ql}{8}$$

$$=\frac{1}{8}\frac{PQl}{P_1-P}+\frac{1}{8}Ql$$

$$=\frac{1}{8}\frac{P_1Ql}{P_1-P}=\pm f_2Z,$$

Z being the least strength modulus of the section and f_2 the maximum stress due to bending. The upper sign denotes a compression and the lower a tension. Thus the total maximum and minimum stress p to which any part of the strut is subjected is given by

$$p=\frac{P}{S}\pm f_2=\frac{P}{A}\pm\frac{Ql}{8Z}\frac{P_1}{P_1-P.}$$

Let $\frac{P}{S}=f$ =the true breaking load per unit of area;

$\frac{P_1}{S}=f_1$ =Euler's " " " " " "

Then $$p=f\pm\frac{Ql}{8Z}\frac{f_1}{f_1-f},$$

which may be written in the form

$$\left(1-\frac{f}{p}\right)\left(1-\frac{f}{f_1}\right)=\pm\frac{Ql}{8Z}.$$

Greenhill has shown (Proc. Inst. Mech. Eng., 1883) that if the column is also subjected to a torsional couple T, the maximum thrust required to bend the column is

$$P=EI\frac{\pi^2}{l^2}-\frac{T^2}{4EI}.$$

In practice the last term is generally so small that it may be disregarded.

CASE II. *Strut Fixed at the Ends.*—Let μ be the moment of fixture at O. The line of action of the resultant thrust P no longer passes through the centre of gravity of the end section. The B.M. equation is now

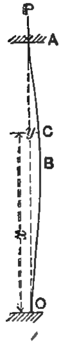


FIG. 572.

$$-EI\frac{d^2y}{dx^2}=\text{B.M.}=Py-\mu,$$

or $$\frac{d^2y}{dx^2}=a^2(b-y),$$

where $$a^2=\frac{P}{EI} \text{ and } b=\frac{\mu}{P}.$$

Put $b-y=z$.

Then $$\frac{d^2z}{dx^2}+a^2z=0,$$

of which, as before, the complete solution is

$$z=c \sin ax+d \cos ax=b-y,$$

c and d being arbitrary coefficients whose values are to be determined.

Differentiating, $$-\frac{dy}{dx}=ca \cos ax-da \sin ax.$$

But $$\frac{dy}{dx}=0 \quad \text{when} \quad x=0 \text{ or } =\frac{l}{2} \text{ or } =l.$$

Therefore $$0=ca, \quad \text{so that} \quad c=0,$$

and $$d \cos ax=b-y.$$

Also, $$0=+da \sin\frac{al}{2}=da \sin al,$$

and therefore $al = 2n\pi,$

n being a whole number. Hence

$$P = n^2 4EI\frac{\pi^2}{l^2}.$$

The least value of P corresponds to $n = 1$, and

$$\mathbf{P} = 4\mathbf{EI}\frac{\pi^2}{\mathbf{l}^2}$$

is the minimum thrust, or the *buckling load,* which will produce bending when both ends of the strut are *fixed.*

The deflection (y) is greatest when

$$\frac{dy}{dx} = 0 = -da \sin ax,$$

i.e., when $ax = 0$, and therefore

$$y_{\text{max.}} = b - d = \frac{\mu}{P} - d.$$

Case III. *Strut Fixed at One End and Hinged at the Other.*—As in Case II,

$$\frac{d^2y}{dx^2} = a^2(b - y),$$

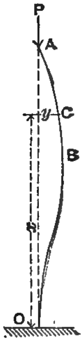


Fig. 573.

the complete solution being of the form

$$b - y = c \cos ax + d \sin ax.$$

At A, i.e., when $x = l$, B.M. $= 0 = b - y$. Therefore

$$0 = c \cos al + d \sin al.$$

At O, i.e., when $x = 0$, $\frac{dy}{dx} = 0$, and therefore

$$0 = -ca \sin al + da \cos al.$$

Hence $d = 0$ and also $c \cos al = 0 = -ca \sin al$, so that

$$b - y = c \cos ax$$

and
$$al=\frac{2n+1}{2}\pi,$$

or
$$P=\left(\frac{2n+1}{2}\right)^2 EI\frac{\pi^2}{l^2}.$$

The least value of P corresponds to $n=1$, and

$$\mathbf{P}=\frac{9}{4}\mathbf{EI}\frac{\pi^2}{\mathbf{l}^2}$$

is the minimum thrust, or *buckling load*, which will produce bending when one end of the strut is fixed and the other hinged.

For the maximum deflection (y),

$$\frac{dy}{dx}=0=ca\sin ax,$$

so that
$$ax=0,$$

and
$$y_{\text{max.}}=b-c=\frac{\mu}{P}-c.$$

The results of Cases I, II, and III show that the buckling load of a long strut hinged at both ends is the same as that of a strut of 1½ *times the length hinged at one end and fixed at the other, and is also the same as that of a strut of twice the length fixed at both ends.*

If f is the compressive strength per unit of area of a strut material, then a short strut of sectional area S theoretically fails by direct crushing under the load fS. It is impracticable, however, to insure, even with the greatest care, that the material is homogeneous, that the load is properly distributed over the end, or that the strut is perfectly straight. Thus the strut will actually fail under a less load than fS, and for the same reasons the actual buckling load is less than $\mu EI\frac{\pi^2}{l^2}$, the coefficient μ being 1, $\frac{9}{4}$, or 4, according as the strut is hinged at both ends, hinged at one end and fixed at the other, or fixed at both ends. Experiment shows that the strength of struts intermediate in length between these two extremes depends in some way upon the length, and the empirical formula

$$P=\frac{fS}{1+\dfrac{fS}{\mu EI\dfrac{\pi^2}{l^2}}}=\frac{fS}{1+\dfrac{f}{\mu E\pi^2}\left(\dfrac{l}{r}\right)^2}$$

has been found to give, with a fair degree of accuracy, the strength of all struts.

It may be noted that in the case of very long struts, the first term of the denominator may be disregarded as compared with the second, and the *buckling load* is obtained, while in the case of very short struts the second term of the denominator becomes small and may be disregarded as compared with *unity*.

This empirical formula is identical with Rankine's modification of Gordon's formula if $a_1=\dfrac{f}{\mu E\pi^2}$.

Ex. 11. *Find by Euler's method the buckling load of a* 160″×3″×2″ *steel strut when* (*a*) *the two ends are hinged* (*b*) *one end is hinged and the other fixed,* (*c*) *both ends are fixed, taking* $E=29{,}400{,}000$ *lbs. per sq. in.*

$$\text{The buckling load}=\mu EI\frac{\pi^2}{l^2}=\mu\times 29{,}400{,}000\times\frac{3.2^3}{12}\,\frac{484}{49}\,\frac{1}{(160)^2}$$

$$=\mu\times 22{,}687\tfrac{1}{2}\text{ lbs.}$$

For case a, $\mu=1$ and buckling load $=22{,}687\frac{1}{2}$ lbs.
" " b, $\mu=\frac{9}{4}$ " " " $=51{,}046\frac{7}{8}$ "
" " c, $\mu=4$ " " " $=90{,}750$ "

Ex. 12. *Find the breaking weight of a strut of the same section and of the same quality of steel as that in the preceding example but only* 80 *ins. in length, the compressive strength of the steel being* 60,000 *lbs. per sq. in.*

$$fS=60{,}000\times 6=360{,}000\text{ lbs.,}$$

and therefore

$$\text{the breaking weight}=\frac{360000}{1+\dfrac{360000}{\mu\times 22687\frac{1}{2}}}.$$

Thus, in case a, $\mu=1$ and breaking weight $=21{,}342$ lbs.
" " " b, $\mu=\frac{9}{4}$ " " " $=44{,}707$ "
" " " c, $\mu=4$ " " " $=72{,}479$ "

Ex. 13. *Find the limiting ratio of the length* (l) *to the thickness* (d) *of a flat pin-bearing steel bar, E being* 30,000,000 *lbs. per sq. in. and the allowable working stress* 10,000 *lbs. per sq. in.*

The bar will not bend laterally so long as

$$10000 < \frac{30000000}{S}\frac{Sd^2}{12}\frac{\pi^2}{l^2} < \frac{12100000000}{49}\left(\frac{d}{l}\right)^2,$$

$$\text{i.e., so long as } \frac{l}{d} < \frac{110}{7}\sqrt{10} < 49.69.$$

Hence the length of a flat bar in compression seems to be comparatively limited. If, however, both ends are securely *fixed*, the strength is *quadrupled* and the admissible length of the bar is *doubled*, while it may be still further increased by fixing the bar at intermediate points as indicated in Case I. This shows the marked advantage to be gained by riveting together the diagonals of lattice girders at the points where they cross each other.

Long Struts.—Let OBA be the bent axis of a long strut of length l under the buckling load P, the line of action of P coinciding with OA the position of the axis before bending commences. If y is the maximum deflection, the maximum B.M.

$$= Py = \mu EI\frac{\pi^2}{l^2}y.$$

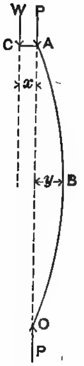


FIG. 574.

Allowance may be made for any slight initial curvature in the strut, for an uneven distribution of the load over the end, and for a want of homogeneousness in the material by assuming that the same maximum deflection is produced by a load $W(<P)$ at C, where AC is necessarily a small quantity.

Let $AC = x$.

Then
$$W(x+y) = Py$$

and
$$y = \frac{Wx}{P-W}.$$

If the line of action of the load W coincides with the axis of the strut, then x is *nil.* So long as the load is less than P, $d=0$, and the failure of the pillar would be due to direct crushing. If the load is equal to P, d becomes *indeterminate* $\left(=\frac{0}{0}\right)$ and the strut remains in a state of neutral equilibrium at any inclination to the vertical. It is impossible that W should exceed P, as d would then be negative; and therefore a load greater than P would cause the strut to bend over laterally until it broke.

Thus $P=\mu EI\frac{\pi^2}{l^2}$ must be the theoretical maximum buckling strength of the strut.

Take $\frac{W}{S}=f$ and $\frac{P}{S}=f_1$. Also, let f_2 be the maximum stress developed in the strut by bending. Then

$$W(x+y)=Py=f_2Z,$$

Z being the strength modulus of the section.

Hence, if P is the total maximum or minimum stress in the most deflected section,

$$p=f\pm f_2=f\pm\frac{W}{Z}(x+y)=f\pm\frac{W}{Z}\frac{Px}{P-W}$$

$$=f\pm\frac{W}{Z}\frac{f_1x}{f_1-f},$$

which may be written in the form

$$\left(1-\frac{f}{p}\right)\left(1-\frac{f}{f_1}\right)=\pm\frac{W}{Z}\frac{x}{p}=\pm\frac{S}{Z}\frac{f}{p}x.$$

Ex. 14. *Find the crushing load of a solid mild-steel pillar* 3 *ins. in diameter and* 10 *ft. long, with two pin ends. Also find the deviation* (x) *of the line of action of a load of* 20,000 *lbs. from the axis of the pillar, so that the maximum intensity of stress may not exceed* 10,000 *lbs. per square inch.*

By Gordon's formula and the table, Art. 4,

$$\text{the crushing load}=\frac{67200\times\pi\times\frac{9}{4}}{1+\frac{4}{1400}\left(\frac{120}{3}\right)^2}=85{,}292.3 \text{ lbs.}$$

Again, the theoretical maximum buckling strength P

$$=EI\frac{\pi^2}{l^2}=28000000\times\frac{22}{7}\frac{3^4}{64}\frac{484}{49}\frac{1}{(120)^2}$$

$$=80{,}216 \text{ lbs.}$$

Therefore $$\frac{P}{P-W}=\frac{80216}{80216-20000}=\frac{80216}{60216}=1.332.$$

Hence $$10000=\frac{20000}{\frac{22}{7}\frac{1}{4}(3)^2}\left(1+1.332\times\frac{8}{3}x\right)$$

and $$x=.714 \text{ in.}$$

8. Uniformly Varying Stress.—The load upon a pillar is rarely, if ever, uniformly distributed, and in practice it is often considered sufficient to assume that the pressure in any transverse section varies *uniformly.*

Any variable external force applied normally to a plane surface *AA* of area *S* may be graphically represented by a cylinder *AABB*, the end *BB* being the locus of the extremities of ordinates erected upon *AA*, each ordinate being proportional to the intensity of pressure at the point on which it is erected.

Let *P* be the total force upon *AA*, and let the line of its resultant intersect *AA* in *C*; *C* is the *centre of pressure* of *AA*, and the ordinate *CC* necessarily passes through the centre of gravity of the cylinder.

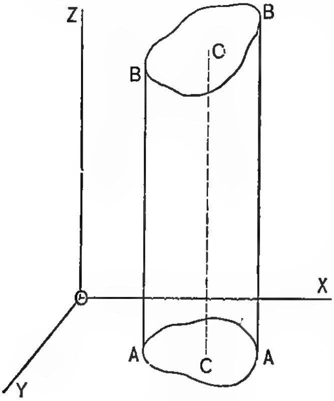


FIG. 575.

Assume that the pressure upon *AA* varies uniformly; the surface *BB* is then a plane inclined at a certain angle to *AA*.

Take *O*, the centre of figure of *AA*, as the origin, and *AA* as the plane of x, y.

Let *OY*, the axis of y, be parallel to that line *EE* of the plane *BB* which is parallel to the plane *AA*.

Through *EE* draw a plane *DD* parallel to *AA* and form the cylinder *AADD*.

The two cylinders *AABB* and *AADD* are evidently equal in volume, and *OF*, the average ordinate, represents the mean pressure over *AA*; let it be denoted by p_0.

At any point *R* of the plane *AA* erect the ordinate *RQP*, intersecting the planes *DD*, *BB*, in *Q* and *P* respectively.

Let x, y be the co-ordinates of R.

The pressure at R

$$=p=PR=\pm PQ+QR=\pm PQ+OF=\pm ax+p_0,$$

a being a constant depending upon the variation.

Let x_0, y_0 be the co-ordinates of the centre of pressure C.

Let ΔS be an elementary area at any point R.

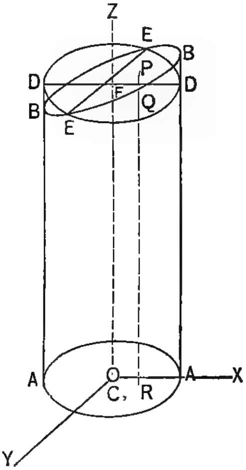


FIG. 576.

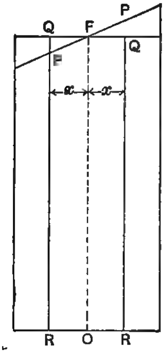


FIG. 577.

Then $p\Delta S$ is the pressure upon ΔS, and $\Sigma(p\Delta S)$ is the total pressure upon the surface AA, Σ being the symbol of summation.

Hence

$$x_0\Sigma(p\Delta S)=\Sigma(px\Delta S) \quad \text{and} \quad y_0\Sigma(p\Delta S)=\Sigma(py\Delta S).$$

But $p=p_0\pm ax$.

Therefore $$x_0\Sigma\{(p_0\pm ax)\Delta S\}=\Sigma\{(p_0x\pm ax^2)\Delta S\}$$

and $$y_0\Sigma\{(p_0\pm ax)\Delta S\}=\Sigma\{(p_0y\pm axy)\Delta S\}.$$

Now O is the centre of figure of AA, and therefore $\Sigma(x\Delta S)$ and $\Sigma(y\Delta S)$ are each zero.

Also, $\Sigma(\Delta S)=S$, $\Sigma(x^2\Delta S)$ is the *moment of inertia* (I) of AA with respect to OY, and $\Sigma(xy\Delta S)$ is the *product of inertia* (K) about the axis OZ.

Therefore $$x_0 p_0 S = aI = x_0 P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $$y_0 p_0 S = aK = y_0 P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In any symmetrical section y_0 is zero, and x_0 is the deviation of the centre of pressure C from the centre of figure O.

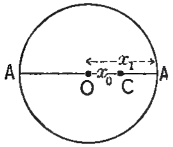


FIG. 578.

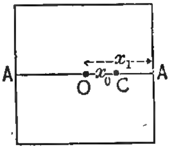


FIG. 578.

Let x_1 be the distance from O of the extreme points A of the section.

The greatest stress in AA is $p_0 + ax_1 = p_1$, suppose.

But $a = \frac{x_0 S p_0}{I}$, by eq. (1).

Therefore $$p_0 + \frac{x_0 x_1 S p_0}{I} = p_1,$$

or $$\frac{p_0}{p_1} = \frac{1}{1 + \frac{x_0 x_1}{I} S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is generally advisable, especially in masonry structures, to limit x_0 by the condition that the stress shall be nowhere *negative*, i.e., a tension. Now the minimum stress is $p_0 - ax_1$, so that to fulfil this condition

$$p_0 > \text{ or } = ax_1.$$

But $$p_1 = ax_1 + p_0; \quad \text{therefore } p_1 < \text{or } = 2p_0.$$

Hence, by eq. (3), $$\frac{p_0}{2p_0} < \text{ or } = \frac{1}{1 + \frac{x_0 x_1}{I} S},$$

and $$\frac{x_0 x_1 S}{I} < \text{ or } = 1; \quad \text{i.e., } x_0 < \text{ or } = \frac{I}{x_1 S}.$$

The uniformly varying stress is equivalent to a single force P along the axis, and a couple of moment

$$P \times CO = P\sqrt{x_0^2 + y_0^2} = a\sqrt{I^2 + K^2}.$$

The line CO is said to be *conjugate* to OY.

If the angle $COX = \theta$, then $\cot \theta = \frac{x_0}{y_0} = \frac{I}{K}$.

9. Weyrauch's Theory of the Resistance to Buckling.—In order to make allowance for buckling, Weyrauch proposes the two following methods:

METHOD I. Let F_1 be the necessary sectional area, and b_1 the admissible unit stress for a strut subjected to loads varying from a maximum compression B_1 to a minimum compression B_2.

Let F' be the necessary sectional area, and b' the admissible unit stress for a strut subjected to loads which vary between a given maximum tension and a given maximum compression, B' being the numerically absolute maximum load and B'' the maximum load of the opposite kind.

According to Chapter IV, if there is no tendency to buckling, and putting $m=\frac{t-u}{u}$, $m'=\frac{u-s}{u}$, and $v'=u\div$factor of safety,

$$F_1=\frac{B_1}{b_1}=\frac{B_1}{v'\left(1+m_1\frac{B_2}{B_1}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$F'=\frac{B'}{b'}=\frac{B'}{v'\left(1-m'\frac{B''}{B'}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If there is a tendency to buckling, let l be the length of the strut, F its required sectional area, and T the mean unit stress at the moment of buckling.

Then, according to the theory of long struts,

$$TF\propto\frac{EI}{l^2}=\delta\frac{EI}{l^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

δ being a coefficient depending upon the method adopted for securing the ends, E the coefficient of elasticity, and I the least moment of inertia of the section.

Also, let t be the statical compressive strength of the material of the strut, and take $t=\mu T$. Then

$$\mu=\frac{t}{T}=\frac{tFl^2}{\delta EI}=\frac{Fl^2}{\lambda I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$\lambda=\frac{\delta E}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If the strut under a pressure B were not liable to buckling, it would be subjected to a direct thrust only. The required sectional area of the strut would then be $\frac{B}{t}$, and the unit stress for an area F would be $\frac{B}{F}$.

If the strut under the pressure B is liable to buckling, its required sectional area will be $\frac{B}{T}$, since T is the mean unit stress at the moment of buckling. Let x be the unit stress, at the moment of buckling, for the area F.

Assuming that the unit stresses in the two cases are in the same ratio as the required sectional areas, then

$$x : \frac{B}{F} : : \frac{B}{T} : \frac{B}{t}.$$

Therefore
$$x = \frac{B}{F}\frac{t}{T} = \mu\frac{B}{F}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The force which, when uniformly distributed over the area F will produce this stress, is $Fx = \mu B$.

Hence allowance may be made for buckling by substituting for the *compressive* forces in equations (1) and (2) their values multiplied by μ. Thus equation (1) becomes

$$F = \frac{\mu B_1}{b_1} = \frac{\mu B_1}{v'\left(1 + m_1\frac{\mu B_2}{\mu B_1}\right)} = \frac{\mu B_1}{v'\left(1 + m_1\frac{B_2}{B_1}\right)} = \mu F_1, \quad . \quad . \quad (7)$$

and equation (2) becomes

$$F' = \frac{\mu B'}{b_1} = \frac{\mu B'}{v'\left(1 - m'\frac{B''}{\mu B'}\right)}; \text{ if } B' \text{ is a compression}, \quad . \quad . \quad (8)$$

and
$$F = \frac{B'}{b'} = \frac{B'}{v'\left(1 - m'\frac{\mu B''}{B'}\right)}, \text{ if } B'' \text{ is a compression}. \quad . \quad . \quad (9)$$

If $\mu<1$, equations (1) and (2) give larger sectional areas than equations (7), (8), and (9), so that the latter are to be applied only when $\mu>1$.

METHOD II. General formulæ applicable to all values of μ may be obtained by following the same line of reasoning as that adopted in the proof of Gordon's formula. It is there assumed that the total unit stress in the most strained fibre is $p_1\left(1+a\frac{l^2}{h^2}\right)$, p_1 being the stress due to direct compression, and $p_1a\frac{l^2}{h^2}$ that due to the bending action.

So, instead of employing equations (1) and (2) when $\mu<1$, and equations (7), (8), and (9) when $\mu>1$, formulæ including *all* cases may be obtained by substituting for the compressive forces in equations (1) and (2) their values multiplied by $1+\mu$.

Thus equation (1) becomes

$$F=\frac{(1+\mu)B_1}{v'\left(1+m_1\frac{B_2}{B_1}\right)}=(1+\mu)F_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and equation (2) becomes

$$F=\frac{(1+\mu)B''}{v'\left(1-m'\frac{B''}{(1+\mu)B'}\right)} \text{ if } B' \text{ is a compression,} \quad . \quad . \quad (11)$$

or

$$F=\frac{B'}{v'\left(1-m'\frac{(1+\mu)B''}{B'}\right)} \text{ if } B'' \text{ is a compression.} \quad . \quad . \quad (12)$$

Equations (7), (8), (9), respectively, give larger values of F than the corresponding equations (10), (11), and (12).

For wrought-iron bars it may be assumed, as in Chapter IV, that $v_1=v'=700$ k. per sq. cm., and $m_1=m'=\frac{1}{2}$.

The value of λ is given by formula (5), but is unreliable, and varies in practice from 10,000 to 36,000 for struts with *fixed* ends.

When the ends are fixed, $\delta=4\pi^2$, according to theory. Hence

$$\lambda=4\pi^2\frac{E}{t}.$$

Therefore if $E=2{,}000{,}000$ k. per sq. cm., and $t=3300$ k. per sq. cm., $\delta=23{,}926$, or in round numbers 23,900; 24,000 is the value usually adopted by Weyrauch.

Ex. 15. *The load upon a wrought-iron column* 360 *cm. long varies between a compression of* 50,000 *k. and a compression of* 25,000 *k. Calculate the sectional area of the column, assuming it to be* 1st *solid and* 2d *hollow, allowance being made for buckling.*

First. By eq. (1),

$$F_1=\frac{50000}{700(1+\frac{1}{2}\times\frac{25000}{50000})}=\frac{400}{7}=\pi r^2,$$

r being the radius of the section.

Also, $$I=\frac{\pi r^4}{4}.$$

Therefore $$\frac{F_1}{I}=\frac{4}{r^2}=\frac{11}{50}.$$

Hence, by eq. (4),

$$\mu=\frac{360\times360}{24000}\times\frac{11}{50}=1.188.$$

Thus $\mu>1$, and by eq. (7) the required sectional area is

$$F_1\times1.188=\tfrac{400}{7}\times1.188=67.9 \text{ sq. cm.}$$

Second. $$F_1=\tfrac{400}{7}=\pi(r_1^2-r_2^2),$$

r_1 being the external and r_2 the internal radius of the section.

Let $r_1=9$ cm. and $r_2=7.92$ cm. Then

$$\pi(r_1^2-r_2^2)=57.43 \text{ sq. cm.}$$

Also, $$I=\pi\frac{(r_1^4-r_2^4)}{4}.$$

Therefore $$\frac{F}{I}=\frac{4}{r_1^2+r_2^2}=\frac{4}{143.7264}.$$

Hence, by eq. (4),

$$\mu=\frac{360\times360}{24000}\times\frac{4}{143.7264}=.15$$

Thus, in the latter case, since $\mu<1$, there is no tendency to buckling.

If the area is determined by equation (10), its value becomes $1.15\times\frac{400}{7}=$ 65 sq. cm.

10. Flexure of Columns.—In Art. 7 the moment equation has been expressed in the form

$$-EI\frac{d^2y}{dx^2}=M,$$

and this is sufficiently accurate if the deviation of the axis of the strut from the vertical is so small that $\left(\frac{dy}{dx}\right)^2$ may be neglected without sensible error.

The more correct equation is

$$-\frac{EI}{\rho}=M,$$

ρ being the radius of curvature.

Consider, e.g., the strut in Art. 7, Case I. Then

$$-a^2y=-\frac{P}{EI}y=\frac{1}{\rho}=\frac{d\theta}{ds}=\frac{d\theta}{dy}\sin\theta,$$

θ being the inclination of the tangent at M to the axis of x, and ds an element of the bent strut at M. Then

$$-a^2y\,dy=\sin\theta\,d\theta.$$

Integrating, $$\frac{a^2y^2}{2}=\cos\theta-\cos\theta_0, \quad \ldots \ldots \ldots \quad (1)$$

θ_0 being the value of θ at a strut end.

Let $\sin\frac{\theta_0}{2}=\mu$ and $\sin\frac{\theta}{2}=\mu\sin\phi$. Then

$$\frac{a^2y^2}{2}=2\mu^2(1-\sin^2\phi),$$

or $$y=\frac{2\mu}{a}\cos\phi. \quad \ldots \ldots \ldots \quad (2)$$

Let Y be the maximum deviation of the axis of the strut from the vertical, i.e., the value of y when $\theta=0$ or $\phi=0$. Then

$$Y=\frac{2\mu}{a}=\frac{2\sin\frac{\theta_0}{2}}{a}. \qquad \ldots\ldots\ldots \quad (3)$$

Again, $$ds=\rho d\theta=-\frac{1}{a}\frac{d\phi}{\sqrt{1-\mu^2\sin^2\phi}}.$$

Hence, if l is the length of the strut,

$$l=\frac{2}{a}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\mu^2\sin^2\phi}}=\frac{2}{a}F_\mu(\phi), \qquad \ldots\ldots \quad (4)$$

$F_\mu(\phi)$ being an elliptic integral of the *first* kind.

Let P' be the *least* thrust which will make the strut bend. As shown in Art. 7,

$$a^2=\frac{P'}{EI}=\frac{\pi^2}{l^2},$$

and, by eq. (4), the corresponding value of the modulus μ is given by

$$F_\mu(\phi)=\frac{\pi}{2}. \qquad \ldots\ldots\ldots \quad (5)$$

Let the actual thrust on the strut be

$$P=n^2P', \qquad \ldots\ldots\ldots \quad (6)$$

n^2 being a coefficient $>$ unity.

The corresponding value of the modulus is given by

$$F_\mu(\phi)=\frac{l}{2}\sqrt{\frac{P}{EI}}=\frac{l}{2}na=n\frac{\pi}{2}. \qquad \ldots\ldots\ldots \quad (7)$$

By reference to Legendre's Tables it is found that a large increase in the value of μ, i.e., of $\sin\frac{\theta_0}{2}$ or θ_0, is necessary in order to produce even a small increase in the value of $F_\mu(\phi)$ and therefore of $n\left(=\sqrt{\frac{P}{P'}}\right)$. Hence as soon as the thrust P exceeds the least thrust which will bend the column, viz., P', θ_0 rapidly increases.

The total maximum intensity of stress in the skin of the strut at the most deflected point

$$=\frac{P}{A}+\frac{Mz}{I}=\frac{P}{A}+\frac{PYz}{I}=f+\frac{2z}{k}\sin\frac{\theta_0}{2}\sqrt{fE}, \quad . \quad . \quad . \quad (8)$$

z being the distance of the skin from the neutral axis, and f being equal to $\frac{P}{A}$.

The last term of this equation includes the product fE, which is very large, and also the factor $\sin\frac{\theta_0}{2}$, which increases with θ_0 so that the ultimate strength of the material is rapidly approached, and, in fact, rupture usually takes place *before* the column has assumed the position of equilibrium defined by the slope θ_0 at the ends.

If there were no limit to the flexure, the column would take its position of equilibrium only after a number of oscillations about this position, and the maximum stress in the material would be necessarily greater than that given by eq. (8).

Again, $$dx=ds\cos\theta=-\frac{1}{a}\frac{(1-2\mu^2\sin^2\phi)d\phi}{\sqrt{1-\mu^2\sin^2\phi}}.$$

Let X be the vertical distance between the strut ends. Then

$$X=\frac{2}{a}\int_0^{\frac{\pi}{2}}\frac{1-2\mu^2\sin^2\phi}{\sqrt{1-\mu^2\sin^2\phi}}d\phi$$

$$=\frac{2}{a}\left\{\int_0^{\frac{\pi}{2}}2\sqrt{(1-\mu^2\sin^2\phi)}d\phi-\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\mu^2\sin^2\phi}}\right\}$$

$$=\frac{2}{a}\{2E_\mu(\phi)-F_\mu(\phi)\},$$

$E_\mu(\phi)$ being an elliptic integral of the *second* kind.

Hence the diminution in the length of the strut

$$=L-X=\frac{4}{a}\{F_\mu(\phi)-E_\mu(\phi)\}.$$

If the column has an initial curvature of $\frac{1}{\rho_0}$, the moment equation may be expressed in the form

$$EI\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)=M=-EI\left(\frac{d^2y}{dx^2}-\frac{d^2y_0}{dx^2}\right), \quad . \quad . \quad . \quad . \quad (1)$$

ρ_0 and $\frac{d^2y_0}{dx}$ being the values of ρ and $\frac{d^2y}{dx}$ when $M=0$.

Hinged Ends.—It is *assumed* that the line of action of the thrust P is at a distance d from the axis of the strut. Then

$$-EI\left(\frac{d^2y}{dx^2}-\frac{d^2y_0}{dx^2}\right)=P(y+d)=\frac{p-f}{z}I, \quad . \quad . \quad . \quad . \quad (2)$$

or

$$\frac{d^2y}{dx^2}-\frac{d^2y_0}{dx^2}=-a^2(y+d)=-\frac{p-f}{zE}, \quad . \quad . \quad . \quad . \quad (3)$$

where $a^2=\frac{P}{EI}$, p=total stress at the distance z from the neutral axis, and f=stress due to direct thrust $\left(=\frac{P}{S}\right)$, so that the stress due to *bending*$=p-f$.

It is also *assumed* that the form of the axis of the column before it is acted upon by the thrust P is a *curve of sines* defined by the equation

$$y_0=\Delta\cos\frac{\pi x}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the origin being half-way between the ends of the strut, and Δ being the maximum *initial* deviation of the axis from the vertical, i.e., the value of y_0 when $x=0$.

Therefore

$$\frac{d^2y_0}{dx^2}=-\frac{\Delta\pi^2}{l^2}\cos\frac{\pi x}{l},$$

and hence, by eq. (3),

$$\frac{d^2y}{dx^2}=-a^2(y+d)-\Delta\frac{\pi^2}{l^2}\cos\frac{\pi x}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

A solution of this equation is

$$y+d=d\frac{\cos ax}{\cos\frac{al}{2}}+\Delta\frac{\cos\frac{\pi x}{l}}{1-\frac{a^2l^2}{\pi^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Now $\frac{al}{2}$ is always small for such values of f as would constitute a safe working load, and therefore

$$\cos\frac{al}{2}=1-\frac{a^2l^2}{8}, \text{ approximately,}$$

so that eq. (6) becomes

$$y+d=d\cos ax\left(1-\frac{a^2l^2}{8}\right)^{-1}+\Delta\cos\frac{\pi x}{l}\left(1-\frac{a^2l^2}{\pi^2}\right)^{-1},$$

or

$$y+d=d\cos ax\left(1+\frac{a^2l^2}{8}\right)+\Delta\cos\frac{\pi x}{l}\left(1+\frac{a^2l^2}{\pi^2}\right), \text{ approximately.} \qquad (7)$$

Let Y be the maximum value of y, i.e., the value of y when $x=0$. Then

$$Y=a^2l^2\left(\frac{d}{8}+\frac{\Delta}{\pi^2}\right)+\Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Hence, by eq. (3), the total maximum intensity of stress

$$=p=f+a^2zE(Y+d)=f+f\left(\frac{z}{r}\right)^2\left\{c+b\frac{f}{E}\left(\frac{l}{r}\right)^2\right\}, \quad . \quad . \qquad (9)$$

where $\qquad b=\frac{1}{z}\left(\frac{d}{8}+\frac{\Delta}{\pi^2}\right)$ and $c=\frac{d+\Delta}{z}$.

Eq. (9) is a quadratic from which f may be found in terms of p. As a first approximation, p may be substituted for f in the last term of the portion within brackets, the error being in the direction of safety.

Fixed Ends.—Let M_1 be the moment of fixture.

Eq. (3) now becomes

$$\frac{d^2y}{dx^2}-\frac{d^2y_0}{dx^2}=-a^2(y+d)-\frac{M_1}{EI}=-a^2\left(y+d+\frac{M_1}{P}\right). \quad . \quad . \quad (10)$$

Assuming again that the initial form of the axis is a curve of sines, the solution of the last equation is

$$y+d+\frac{M_1}{P}=\left(d+\frac{M_1}{P}\right)\frac{\cos ax}{\cos\frac{al}{2}}+\Delta\frac{\cos\frac{\pi x}{l}}{1-\frac{a^2l^2}{\pi^2}}. \quad . \quad . \quad . \quad (11)$$

Initially, $$y_0=\Delta\cos\frac{\pi x}{l},$$

and $\frac{dy}{dx}$ is equal to $\frac{dy_0}{dx}$ when $x=\frac{l}{2}$ or $=-\frac{l}{2}$.

Hence $$-\Delta\frac{\pi}{l}=-\left(d+\frac{M_1}{P}\right)\frac{a\sin\frac{al}{2}}{\cos\frac{al}{2}}-\frac{\Delta\frac{\pi}{l}}{1-\frac{a^2l^2}{\pi^2}},$$

or $$d+\frac{M_1}{P}=-\frac{2\Delta}{\pi}-\Delta a^2l^2\frac{12-\pi^2}{6\pi^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Again, the value of y at the point $x=0$ is

$$Y=\left(d+\frac{M_1}{P}\right)\frac{a^2l^2}{8}+\Delta\left(1+\frac{a^2l^2}{\pi^2}\right). \quad . \quad . \quad . \quad . \quad (13)$$

Also, if p_1, p_2 are the total maximum intensities of stress at the end and at the most deflected point, then

$$\frac{p_1-f}{zE}=-a^2\left(d+\frac{M_1}{P}\right)=\text{etc.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and $$\frac{p_2-f}{zE}=-{}^2\left(Y+d+\frac{M_1}{P}\right)=\text{etc.}; \quad . \quad . \quad . \quad . \quad . \quad (15)$$

two equations from which f may be found as before.

The following conclusions are drawn from the above investigation:

First. The *actual* strength of a column depends *partly* upon known facts as to dimensions, material, etc., and *partly* upon *accidental* circumstances.

Second. Experiments upon the crippling or destruction of columns cannot be expected to give coherent results when applied to the determination of the constants in such an equation as (9).

Third. It is a question whether p should be made the *elastic limit* of the material and the working stress a definite fraction of the corresponding value of f derived from eq. (9), or whether p should be the allowable skin working stress, and the value of f be found by means of the same equation. The former seems to be the more logical assumption.

Fourth. It would appear that the strength of hinged columns is likely to be much more variable than the strength of columns with fixed ends, as it depends upon two variable elements, d and Δ, while the end fixture eliminates d.

FIRST ELLIPTIC INTEGRAL, $F_\mu(\phi)$.

ϕ	$\mu=0$	$\mu=.1$	$\mu=.2$	$\mu=.3$	$\mu=.4$	$\mu=.5$	$\mu=.6$	$\mu=.7$	$\mu=.8$	$\mu=.9$	$\mu=1$
0°	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5°	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087
10°	0.175	0.175	0.175	0.175	0.175	0.175	0.175	0.175	0.175	0.175	0.175
15°	0.262	0.262	0.262	0.262	0.262	0.263	0.263	0.263	0.264	0.264	0.265
20°	0.349	0.349	0.349	0.350	0.350	0.351	0.352	0.353	0.354	0.355	0.356
25°	0.436	0.436	0.437	0.438	0.439	0.440	0.441	0.443	0.445	0.448	0.451
30°	0.524	0.524	0.525	0.526	0.527	0.529	0.532	0.536	0.539	0.544	0.549
35°	0.611	0.611	0.612	0.614	0.617	0.620	0.624	0.630	0.636	0.644	0.653
40°	0.698	0.699	0.700	0.703	0.707	0.712	0.718	0.727	0.736	0.748	0.763
45°	0.785	0.786	0.789	0.792	0.798	0.804	0.814	0.826	0.839	0.858	0.881
50°	0.873	0.874	0.877	0.882	0.889	0.898	0.911	0.928	0.947	0.974	1.011
55°	0.960	0.961	0.965	0.972	0.981	0.993	1.010	1.034	1.060	1.099	1.154
60°	1.047	1.049	1.054	1.062	1.074	1.090	1.112	1.142	1.178	1.233	1.317
65°	1.134	1.137	1.143	1.153	1.168	1.187	1.215	1.254	1.302	1.377	1.506
70°	1.222	1.224	1.232	1.244	1.262	1.285	1.320	1.370	1.431	1.534	1.735
75°	1.309	1.312	1.321	1.336	1.357	1.385	1.426	1.488	1.566	1.703	2.028
80°	1.396	1.400	1.410	1.427	1.452	1.485	1.534	1.608	1.705	1.885	2.436
85°	1.484	1.487	1.499	1.519	1.547	1.585	1.643	1.731	1.848	2.077	3.131
90°	1.571	1 575	1.588	1.610	1.643	1.686	1.752	1.854	1.993	2.275	∞

SECOND ELLIPTIC INTEGRAL, $E_\mu(\phi)$.

ϕ	$\mu=0$	$\mu=.1$	$\mu=.2$	$\mu=.3$	$\mu=.4$	$\mu=.5$	$\mu=.6$	$\mu=.7$	$\mu=.8$	$\mu=.9$	$\mu=1$
0°	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5°	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087
10°	0.1 5	0.175	0.174	0.174	0.174	0.174	0.174	0.174	0.174	0.174	0.174
15°	0.262	0.262	0.262	0.262	0.261	0.261	0.261	0.260	0.260	0.259	0.259
20°	0.349	0.349	0.349	0.348	0.348	0.347	0.347	0.346	0.345	0.343	0.342
25°	0.436	0.436	0.436	0.435	0.434	0.433	0.431	0.430	0.428	0.425	0.423
30°	0.524	0.523	0.523	0.521	0.520	0.518	0.515	0.512	0.509	0.505	0.500
35°	0.611	0.610	0.609	0.607	0.605	0.602	0.598	0.593	0.588	0.581	0.574
40°	0.698	0.698	0.696	0.693	0 690	0.685	0.679	0.672	0.664	0.654	0.643
45°	0.785	0.785	0.782	0.779	0.773	0.767	0.759	0.748	0.737	0.723	0.707
50°	0.873	0.872	0.869	0.864	0.857	0.848	0.837	0.823	0.808	0.789	0.766
55°	0.960	0.959	0.955	0.948	0.939	0.928	0.914	0.895	0.875	0.850	0.819
60°	1.047	1.046	1.041	1.032	1.021	1.008	0.989	0.965	0.940	0.907	0.866
65°	1.134	1.132	1.126	1.116	1.103	1.086	1.063	1.033	1.001	0.960	0.906
70°	1.222	1.219	1.212	1.200	1.184	1.163	1.135	1.099	1.060	1.008	0.940
75°	1.309	1.306	1.297	1.283	1.264	1.240	1.207	1.163	1.117	1.053	0.966
80°	1.396	1.393	1.383	1.367	1.344	1.316	1.277	1.227	1.172	1.095	0.985
85°	1.484	1.480	1.468	1.450	1.424	1.392	1.347	1.289	1.225	1.135	0.996
90°	1.571	1.566	1.554	1.533	1.504	1.467	1.417	1.351	1.278	1.173	1.000

EXAMPLES.

1. The sectional area of a pillar is 144 sq. ins., and the pillar carries a load of 4000 lbs. Find the normal and tangential intensities of stress on a plane inclined at 20° to the axis. *Ans.* 3.25 lbs.; 8.93 lbs.

2. A short cast-iron column of 6 ins. external and 4 ins. internal diameter carries a load of 20 tons, the line of action of the resultant being 12 ins. from axis of column. Find maximum and minimum stresses.
Ans. 15.37 and 12.52 tons/sq. in.

3. A cylindrical pillar 6 ins. in diameter supports a load of 400 lbs., of which the centre of gravity is $\frac{5}{8}$ in. from the axis. Determine the greatest and least intensities of stress upon any transverse section of the pillar.
Ans. $25\frac{25}{27}$ lbs.; $2\frac{106}{297}$lbs.

4. Calculate the breaking weight by Hodgkinson's formula of a solid round cast-iron pillar 20 ft. in length and 10 ins. in diameter, (1) both ends being securely fixed; (2) both ends being hinged.
Ans. (1) 951.4 tons; (2) 414.04 tons.

5. Determine by Hodgkinson's formula the diameter of a solid wrought-iron pillar equal in length and strength to that in the preceding question.
Ans. 7.35 ins.

6. Calculate the breaking weight by Hodgkinson's formula of a square-bearing hollow cast-iron column of 4 ins. external and 3 ins. internal diameter.

7. A square-bearing timber post .18 m.×.18 m. in section is 4 m. high. What load will it safely bear? (Use Hodgkinson's formula.) *Ans.* 10,594 k.

8. Find by Hodgkinson's formula the load which can be carried by a cast-iron column, with square ends, 5 m. high and 120 mm. in diameter.
Ans. 16,071 k.

9. Find by Hodgkinson's method the breaking weight of a square-bearing solid cast-iron pillar 6 ins. in diar. and 20 diars. in length. *Ans.* 177 tons.

10. A hollow cast-iron square-bearing column of 9 ins. external and 7 ins. internal diar. is 24 ft. long. Determine by Hodgkinson's method the load it will safely carry, 10 being a factor of safety. *Ans.* 32 tons.

11. What is the length of a solid cast-iron pin-bearing column of 3 ins. diameter which fails under a load of 12,320 lbs.? *Ans.* 19.5 ft.

12. With 8 as a factor of safety, the safe working load on a solid square-bearing pillar 10 ft. long is 5 tons. Find its diameter. *Ans.* 2.8 ins.

13. Find the breaking weight of a solid square-bearing cast-iron column $7\frac{1}{2}$ ins. in diameter and 16 diameters in length, the crushing strength of the iron being 40 tons. *Ans.* 838 tons.

14. A hollow cast-iron column of 8 ins. external and 6 ins. internal diameter is 18 diameters long. If 10 is a factor of safety, find the load it will safely carry when both ends are (*a*) square-bearing, (*b*) pin-bearing.
Ans. (*a*) 45.6 tons; (*b*)

15. Determine the breaking weight of a solid cast-iron pillar 9 ft. in height and 4 ins. in diameter when the ends are (*a*) square-bearing, (*b*) hinged.
Ans. (*a*) 159 tons; (*b*)

16. Determine the breaking weight of a solid round pillar with both ends firmly secured, 10 ft. in length and 2 ins. in diameter, (1) if of cast iron; (2) if of wrought iron; (3) if of steel (mild).
Ans. 25,142.8 lbs.; 43,516.48 lbs.; 59,136 lbs.

17. A solid pin-bearing cast-iron strut 15 ft. long is to carry a load of 20 tons. If 6 is a factor of safety, find the diameter of the strut.
Ans. $6\frac{1}{4}$ ins.

18. A solid or hollow pillar of cast iron, wrought iron, or mild steel is to be designed to carry a *steady* load of 30,000 lbs. Determine the necessary diameter in each case, 6 being a factor of safety. (The pillar is to be 12 ft. high, and the metal of the hollow pillar is to be $\frac{3}{8}$ in. thick.) Determine the load that will produce a maximum stress of 9000 lbs. per square inch in the solid steel pillar.
Ans. *Solid:* 3.68 ins.; 3.39 ins.; 3 ins.; 24,055 lbs.
Hollow: 4.96 ins.; 8.5 ins.; 3.9 ins.

19. Design a square-bearing pine pillar 20 ft. long to carry a load of 6000 lbs., 4 being a factor of safety. *Ans.* Required area $=35.6$ sq. ins.

20. Determine the strength of a mild-steel pillar 7 ins. in diameter and 20 diameters in length. ($a=\frac{1}{1400}$, $f=60{,}750$ lbs.) Find the diameter of each of two pillars equivalent to the single pillar.

21. Determine the breaking weight of an oak pillar 9 ft. high, 11 ins. wide. and 5 ins. thick. *Ans.* 138.160 lbs.

22. What weight will be safely borne by a pillar of dry oak subject to vibration, 10 ft. high and 6 ins. square, 10 being a factor of safety?
Ans. 9969 lbs.

23. Calculate the breaking weight of a solid cast-iron pillar 20 ft. in length

and 10 ins. in diameter when (*a*) both ends are square-bearing, (*b*) one end is square-bearing and the other pin- and square-bearing.

Ans. (*a*) 1150.05 tons; (*b*)

24. Find the diameter of a wooden column 20 ft. long to support a load of 10,000 lbs., 10 being a factor of safety and both ends of the column being absolutely fixed. *Ans.* 8.55 ins.

25. Find the breaking stress per square inch of a 4″×4″ solid wrought-iron pillar for lengths of 5, 10, 15, and 20 ft., the two ends being absolutely fixed. *Ans.* 33,488 lbs.; 27,692 lbs.; 21,492 lbs.; 16,363 lbs.

26. Compare the breaking weight of a solid square pillar of wrought iron 20 ft. long and 6 ins. square with that of a solid rectangular pillar of the same material, the section being 9 ins. by 4 ins. *Ans.* 845,217 lbs.; 589,090 lbs.

27. A solid round pillar of mild steel 16 ft. high supports a *steady* load of 20,000 lbs. If the factor of safety is 6, what is its diameter? *Ans.* 3 ins.

28. Find the diameter of each of four pillars of the same material which may be substituted for the single pillar in the preceding example.

Ans. 2.04 ins.

29. Find the breaking weight of a cylindrical strut of 3 ins. diameter and 40 diameters in length, both ends being hinged. *Ans.* 67,430 lbs.

30. A hollow cast-iron pillar with an external diameter of 9 ins. is to be substituted for the solid pillar in the preceding example. Determine the thickness of the metal. *Ans.* $\frac{3}{4}$ in.

31. A hollow cast-iron pillar 12 ft. in height has to support a steady load of 33,000 lbs.; its internal diameter is $5\frac{1}{2}$ ins. Find the thickness of the metal, the factor of safety being 6. *Ans.* .28 in.

32. A solid wrought-iron pillar is to be substituted for the pillar in the preceding example. Find its diameter. *Ans.* $3\frac{1}{2}$ ins.

33. The external and internal diameters of a hollow cast-iron column 12 ft. in length are D and $\frac{5}{6}D$ respectively; the load upon the column is 25,000 lbs. If the factor of safety is 4, find D, (*a*) when both ends of the column are absolutely fixed; (*b*) when both ends are hinged.

Ans. (*a*) 4.03 ins.; (*b*) 5.4 ins.

34. A hollow cast-iron pillar of 6 ins. external diameter and 20 diameters in length has to carry a load of 67,200 lbs. If 8 is a factor of safety, find the thickness of the metal? *Ans.* .42 in.

35. A hollow circular mild-steel square-bearing column 28 ft. long and of 6 ins. external diameter has to carry a load of 50 tons. Find the thickness of the metal, 4 being a factor of safety. *Ans.* .9 in.

36. A square-bearing hollow wrought-iron cylindrical strut 10 ft. long and of $1\frac{1}{2}$ ins. external diameter fails under a load of 2.7 tons. Find the thickness of the metal. *Ans.* 2.7 tons.

37. A solid rectangular wrought-iron strut 10 ft. long and 2 ins. thick fails under a load of 30 long tons. If the ends are pin-bearing, what should be the width of the strut? *Ans.* 5.4 ins.

38. The least breadth of an angle-iron is 3 ins., and its length is 90 ins. It is fixed at both ends. What is the safe load, 6 being a factor of safety?

Ans. $3\frac{1}{2}$ tons.

39. What is the breaking weight of a cast-iron stanchion of a regular cruciform section and 15 ft. in height, the arms being 24 ins. by 1 in.? *Ans.* 2,811,215 lbs.

40. Each of the pillars supporting the lowest floor of a refinery is 16½ ft. high, is of a regular cruciform section, and carries a load of 240,000 lbs.; the total length of an arm is 14 ins. Determine its thickness, the factor of safety being 10. *Ans.* 2.598 ins.

41. Find the load which can be safely carried by a column 30 ft. high of the section shown by the figure, which consists of *two* 14″×¾″ side plates, *four* Z bars, each 6″×3½″×¾″, and *one* web plate 8″×¾″. Use straight-line formula $p=17,100-57\frac{l}{r}$. *Ans.* 726,000 lbs.

42. Find the safe load on a square-bearing column 20 ft. long composed of *two* 12-in.×20-lb. per lineal foot medium steel channels placed back to back and 10 ins. apart, 4 being a factor of safety. Sectional area of each channel =11.8 sq. ins. *Ans.* 109,740 lbs.

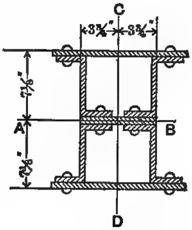


FIG. 580.

43. Design a medium steel strut 15 ft. long and consisting of four angles to carry a load of 170,000 lbs.; use the formula $p=16,000-60\frac{l}{r}$.

Ans. Try 6″×3½″×⅜″ angles with the short legs back to back and spaced ¾ in. apart for lacing. Section required =13.58 sq. ins.

44. What should be the distance between the faces of the long legs of the angles in the preceding example so that the moments of inertia with respect to the axes through the centre of gravity of the section, parallel to the short and long legs, may be equal? *Ans.* 7.62 ins.

45. The section of a chord segment 22 ft. long consists of *one* 18″×½″ cover and two 12-in.×44-lb. per lineal foot channels. If the allowable stress p is that given by $p\left\{1+\frac{1}{24000}\left(\frac{l}{r}\right)^2\right\}=10,000$, find the load which may be applied at the centre of gravity of the section. *Ans.* 304,325 lbs.

46. What would be the additional stress if the load were applied at the centre of the channels? *Ans.* 5410 lbs.

47. Determine the buckling strength of a 2-in. round steel bar 60 diameters in length, with pin ends. If the stress is nowhere to exceed 12,000 lbs. per square inch, find the load for which the maximum deviation of line of load from axis will not exceed .125 in.

48. Calculate the buckling load of a piece of cast-iron pipe, length 24 ins.; external diameter, 4.4 ins.; internal diameter, 3.9 ins.; rounded ends.

Ans. 158 tons for hard to 98 tons for soft cast iron.

49. A steel strut 10 ft. long consists of two tees back to back, each 4″×4″×½″. Taking $f=60,000$ lbs., $a_1=\frac{1}{40000}$, and 6 as a factor of safety find the working load (*a*) when the strut has two pin ends; (*b*) when it has two fixed ends. ($E=29,000,000$ lbs.)

Also find the deviation of the axis of the load from the axis of the strut so that the maximum stress in the metal may not exceed 10,000 lbs. per square inch. *Ans.* (*a*) 25,019 lbs.; (*b*) 50,019 lbs.
Deviation =.513 in. in (*a*) and .158 in. in (*b*).

50. A solid wrought-iron strut 20 ft. high and 4 ins. in diameter has one end fixed and the other perfectly free. Find the deviation of the line of action of a load of 10,000 lbs. from the axis, so that the stress may not exceed 10,000 lbs. per square inch, E being 27,000,000 lbs. *Ans.* 1.8 ins.

51. Find the buckling load of a pin-bearing steel strut of 3 ins. solid square section and 108 ins. long, E being 30,000,000 lbs. per square inch.
Ans. By Euler, 173,600 lbs.; by Gordon, 176,000 lbs.

52. Find the safe load on a rolled tee-iron strut $6'' \times 4'' \times \frac{1}{2}''$, 10 ft. long, fixed at one end, free at the other.

53. Find the strength of a square-bearing double-tee section column 22 ft. in length. The equal $25'' \times .96''$ flanges are connected by a $12'' \times 1''$ web. If the allowable stress is not to exceed 10,000 lbs. per square inch, find the greatest deviation of the axis of the load from the axis of the column. (E =31,500,000 lbs./sq. in.; $44{,}000a_1=1$ and f =53,770 lbs./sq. in.)

54. A round steel strut 1 in. in diameter and 100 diameters in length is pivoted at both ends. Find Euler's value for its ultimate strength, taking E =29,400,000 lbs. per square inch.

If an end load acting by itself produces a maximum fibre stress of 1000 lbs. per square inch, and a lateral load acting by itself produces a maximum fibre stress of 500 lbs. per square inch, what will be the maximum fibre stress when the two loads act together?

55. A hollow cast-iron column with two pin ends is 24 ft. high and has a mean diameter of 12 ins.; it carries a load of 80,000 lbs. Find the proper thickness of the metal, 10 being a factor of safety. If the deviation of the line of action of the load from the axis is 1 in., find the maximum stress per square inch in the metal, E being 17,000,000 lbs.
Ans. 1.28 ins.; 2236 lbs. per sq. in.

56. Find the crushing load of a solid wrought-iron pillar 3 ins. in diameter, 10 ft. high, and fixed at both ends. Calculate the deviation which will produce a maximum stress in the metal of 9000 lbs. per square inch under loads of (*a*) 15,000 lbs., (*b*) 30,000 lbs., E being 29,000,000 lbs.
Ans. 148,775 lbs.; (*a*) 1.158 ins.; (*b*) .38 in.

57. Solve the preceding example on the assumption that the column has two pin ends. *Ans.* 66,218 lbs.; (*a*) .985 in.; (*b*) .261 in.

58. A hollow cast-iron column of 19 ins. external and 16 ins. internal diameter has rounded ends, is 80 ft. in length, and is stayed laterally at intervals of 20 ft. Find the least force which will cause the column to bend.
Ans. 2,182,300 lbs., E being 16,000,000 lbs.

59. Find the breaking weight of a wrought-iron column 3 ins. in diameter, 160 ins. in length, and hinged at both ends.

Also find the deviation for which the stress in the metal will not exceed 10,000 lbs. per square inch when the load upon the column is 40,000 lbs.

Ans. 117,720 lbs.; .99 in.

60. An 8-in. channel section strut with $4.02'' \times .56''$ flanges and a web .42 in. thick has both ends fixed. Taking $E = 30{,}000{,}000$ lbs. per square inch, and the ultimate stress at 45,000 lbs. per square inch, find the buckling strength for lengths of 8, 10, 12, and 15 ft. What should the corresponding lengths be if both ends are hinged?

61. Find the buckling strengths of the following tees:

(*a*) Area $=1.67$ sq. ins.; $\frac{l}{r} = 8.2$;

(*b*) " $=1.95$ " " $\frac{l}{r} = 18$;

(*c*) " $=2.04$ " " $\frac{l}{r} = 19.8$.

Ans. (*a*) 26.1 tons; (*b*) 21.5 tons; (*c*) 20.3 tons.

62. The 22-ft. panel length of the top chord for a deck-span has to carry a load of 350,000 lbs. in direct compression and a uniformly distributed load of 26,400 lbs. Design the section, the allowable stress p in compression being given by $p\left\{1 + \frac{1}{18000}\left(\frac{l}{r}\right)^2\right\} = 10{,}000$.

Ans. Try a section consisting of *one* $24'' \times \frac{3}{8}''$ cover, *two* upper $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{3}{8}''$ angles, *two* lower $6'' \times 4'' \times \frac{7}{8}''$ angles, *two* $24'' \times \frac{1}{2}''$ webs, allowing play between web ends and faces of angles. Section required $=50$ sq. ins.

63. *Reinforced concrete columns less than 25 diameters in length, which includes all usually found in practice, fail by direct crushing, and the effect of bending need not therefore be considered. Find the safe load upon a concrete column of sectional area S reinforced by four steel rods each of a square inch section, f_c being the safe unit stress in the concrete and x the ratio of elastic moduli of the steel and concrete.* Ex. $S = 8'' \times 8''$; $a = 1$ sq. in.; E for steel $=30{,}000{,}000$, and for concrete $=3{,}000{,}000$ lbs./sq. in.; $f_c = 600$ lbs.

Ans. $f_c(S + 4ra)$; 60,000 lbs.

64. A bridge diagonal 10 ft. long consists of *two* $6'' \times 3'' \times \frac{1}{2}''$ tee bars placed back to back. Find the maximum allowable stress, 4 being a factor of safety, (*a*) when the ends are square-bearing, (*b*) when one end is square- and the other pin-bearing. *Ans.* (*a*) 28 tons; (*b*) $14\frac{1}{2}$ tons.

65. The web members of a Warren girder are bars of rectangular section and 10 ft. in length. One of the bars has to carry loads varying between a steady minimum tension of 20.2 tons and a maximum tension of 40.4 tons, and another to carry loads varying between a maximum compression of 8.7 tons and a maximum tension of 14.4 tons. Find the sectional area in each case, allowance being made for buckling in the latter.

66. Determine the sectional area of a double-tee strut which is to carry a load varying between a maximum tension of 80,000 lbs. and a maximum compression of 60,000 lbs. Each flange consists of two $6'' \times 6'' \times \frac{3}{8}''$ angle-

irons riveted to a 12″×⅜″ web plate. The length of the strut is to be (*a*) 6 ft.; (*b*) 12 ft.

67. What safe load may be borne by a 3″×3″×⅜″ angle wrought-iron strut 18 ins. long when both ends are (*a*) square-bearing, (*b*) pin-bearing, 4 being a factor of safety? *Ans.* (*a*) 9.3 tons; (*b*) 7.6 tons.

68. The 12½-ft. column for an elevated railway consists of a 10″×⅜″ web riveted on each side to two equal 6″×3½″ Z bars. The total vertical load due to train and superstructure = 113,280 lbs.; the weight of the column and its attachments = 1600 lbs.; the wind load is the equivalent of two horizontal forces of 6000 and 2400 lbs. acting 22¼ ft. and 16¼ ft. respectively above the column's base. The maximum deviation of the upper end from the mean position due to a change of temperature = ¼ in. The column is fixed at the base, and the top is so constrained that the axis is vertical at that point. $E = 27{,}000{,}000$ lbs. Find the maximum fibre stresses due to the several forces, and also the total maximum fibre stress.

Ans. 5041 lbs.; 890 lbs.; 71 lbs.; 2686 lbs.; 10,100 lbs.

69. Calculate the strength of a compression-chord panel member 352 ins. long and composed of two 18″×⅜″ web plates 13 ins. apart, riveted to an upper 21″×$\frac{7}{16}$″ cover by means of two 3½″×3½″×⅜″ angles, and also riveted at the bottom to two 5″ (horizontal)×3½″×$\frac{9}{16}$″ angles.

70. A brace hinged at both ends is subject to stresses varying from a maximum compression of 58,100 lbs. to a maximum tension of 144,900 lbs. The length is 537 ins., and the brace is composed of two 12″×3.173″×.513″ channels, back to back. Is there a sufficient sectional area of metal?

71. An intermediate bridge vertical 336 ins. long consists of two 12″×3.05″×.39″ channels kept 10⅝″ apart by 2″×⅜″ lacing. What force will such a strut safely bear? (Neglect the effect of the lacing.)

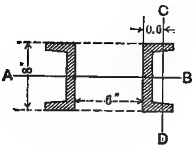


FIG. 581.

72. A column 20 ft. long consists of two 8-in. channels spaced 6 ins. back to back, the flanges being turned outwards. The area of one channel is 4 sq. ins.; moment of inertia of one channel about axis AB is 36.0; moment of inertia about axis CD through centre of gravity of channel is 1.55; distance of CD from back of channel 0.6 in. Find the load which the column will safely carry if the allowable stress per square inch is given by the formula $p\left(1+\frac{l^2}{18000r^2}\right)=10000$.

Ans. 59,040 lbs.

73. A hollow cylindrical tower of steel plate having an external diameter of 3 ft., a thickness of ½ in., and a height of 60 ft. carries a central load of 50 tons and is subjected to a horizontal wind pressure of 56 lbs. per foot of its height. Calculate the vertical stresses at the fixed base of the tower on the windward and on the leeward side. (Allow for the weight of the tower itself.)

74. The cylinder of an inverted engine is supported on one vertical cast-iron column of variable rectangular section, the centre line of the cylinder being 8 ins. from the vertical face of the column. Find the maximum stress induced at a section of the column which is 10 ins. deep and 6 ins. wide, the thickness of metal being 1 in. all round and the driving force on the piston being 5 tons.

75. Find the limiting ratio of length to diameter for a steel pin-bearing strut strained to the elastic limit of 25,000 lbs. per square inch, E being 30,000,000 lbs. per square inch. *Ans.* 27.2.

76. Find the limiting ratio of length to thickness for a steel square-bearing strut strained to the elastic limit of 25,000 lbs. per square inch, E being 30,000,000 lbs. per square inch. *Ans.* 63.

77. The allowable stress in a square-bearing timber strut is 1000 lbs. per square inch. The strut is square in section and $E=1{,}500{,}000$. Find the limiting ratio of length to side of square. *Ans.* 70.28.

78. Find the limiting length of an I strut, fixed at both ends, 24 ins. deep, of 25 sq. ins. sectional area, and weighing 85 lbs. per lineal foot, the radius of gyration in the plane of flexure being 1.33 ins. The metal may be strained to 12,000 lbs. per square inch, and $E=30{,}000{,}000$ lbs. per square inch. *Ans.* 418 ins.

79. A 72-in. pin-bearing steel T strut has a 3-in. flange×$2\frac{1}{2}$-in. stem and weighs 7.2 lbs. per lineal foot. Taking $E=29{,}400{,}000$ lbs. per square inch, find the greatest stress which can be developed in the strut without lateral bending. *Ans.* 29,040 lbs.

80. A 7-in. square-bearing steel channel strut weighing $9\frac{3}{4}$ lbs. per lineal foot and 58.6 ins. in length has 2.09-in. flanges and a web .21 in. thick. Find the greatest stress which could be developed in the strut without lateral bending. *Ans.* 116,160 lbs., E being 29,400,000 lbs./sq. in.

81. The ends of a long strut are in the same vertical line, one end is fixed and the other is kept in position by a horizontal force. Show that the buckling load is $2.045\pi^2\dfrac{EI}{l^2}$.

82. The strut OA is fixed at O and carries a load P at the end B of a horizontal arm AB $(=q)$ rigidly attached to the strut at A. If the strut is bent into the form OA, the horizontal distance AC of A from the vertical through O being p, and if it is assumed that the difference in length between OA $(=l)$ and OC is sufficiently small to be disregarded, show that $q=(p+q)\cos al$. Also find the buckling load when q is small enough to be neglected.

Ans. $\frac{1}{4}EI\dfrac{\pi^2}{l^2}$.

83. In the preceding example, if in addition to P there is a horizontal force H at B, show that the deflection y of any point in OA at a vertical distance x from O is given by

$$y=p+q+\frac{H}{P}(l-x)+b\sin(ax+c),$$

the coefficients b and c being given by

$$0=p+q+\frac{H}{P}l+b\sin c \quad \text{and} \quad 0=-\frac{H}{P}+ab\cos c, \quad \text{where} \quad a^2EI=P.$$

84. In Art. 10, show how equations (3) and (6) will be modified if the line of action of P is distant $\alpha+\beta$ from one end and $\alpha-\beta$ from the other end of the column's axis. Also, if the coefficient of elasticity, E, is variable and equal to $m\pm n\frac{z}{z_0}$ at a point distant z from the axis, z_0 being the maximum value of z and m and n coefficients, show that $y+\frac{n}{m}\frac{r^2}{z_0}$ must be substituted for y in eq. (3).

85. In one of Christie's experiments an angle-bar 2″×2″×$\frac{5}{16}$″, with hinged ends, for which $\frac{l}{r}$ had the value 154, deflected .01 in. for an increase in the load of 3000 lbs. Show that $\frac{d}{8}+\frac{\Delta}{\pi^2}=.0048$ in.

86. A long column with pin ends is bent laterally until the angular deviation (θ_0) at the ends is 4°. Find the total maximum intensity of stress, the section of the column being (a) a circle, (b) a square. ($E=29{,}000{,}000$ lbs., and the stress due to direct thrust $=1500$ lbs. per squar inch.)

Ans. (a) 30,615 lbs.; (b) 26,715 lbs.

87. With the same *maximum* stress as in the last example, find the angular deviation at the ends so that the stress due to direct thrust may be 10,000 lbs. per square inch. *Ans.* (a) 1° 5′; (b) 1° 33′.

88. Show that the load required to produce an angular deviation of 14° at the two pin ends of a long column is only *one per cent* greater than that which *just produces flexure.*

CHAPTER IX.

TORSION.

1. Torsion is the force with which a thread, wire, or prismatic bar tends to recover its original state after having been twisted, and is produced when the external forces which act upon the bar are reducible to two equal and opposite couples (the ends of the bar being free), or to a single couple (one end of the bar being fixed), in planes perpendicular to the axis of the bar. The effect upon the bar is to make any transverse section turn through an angle in its own plane, and to cause originally straight fibres, as *DE*, to assume helicoidal forms, as *FG* or *DC*. This induces longitudinal stresses in the fibres and transverse sections become warped. It is found sufficiently accurate, however, in the case of cylindrical and regular polygonal prisms, to assume that a transverse section which is plane before twisting remains plane while being twisted. In order that the bar may not be bent, its axis must coincide with the axis of the twisting couple.

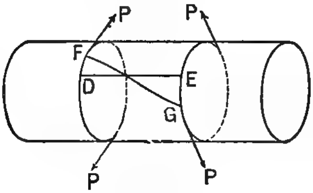


FIG. 582.

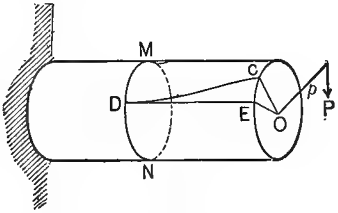


FIG. 583.

The angle turned through by one transverse section relatively to another is called the *angle of torsion* (or *twist*), and Coulomb, from experiments upon wires, made the following deductions:

(*a*) *That the angle of twist is directly proportional to the length of wire twisted;*

(*b*) *That the angle of twist is directly proportional to the twisting moment;*

(*c*) *That the angle of twist is inversely proportional to the fourth power of the diameter;*

(*d*) *That the angle of twist for wires of the same lengths and diameters, but of different materials, is inversely proportional to a certain coefficient G called the coefficient of rigidity of the material.*

These results may be expressed analytically by the formula

$$A = C\frac{LPp}{GD^4},$$

A being the angle (*in radians*) through which a wire of length L and diameter D is twisted by the application of a force P at the end of a lever-arm of length p. The value of the coefficient C depends upon the form of the section only.

Ex. 1. *A brass wire* 30 *ins. long and* 0.1 *in. in diameter is twisted through an angle of* 195° *by a twisting moment of* 4 *in.-lbs. If the coefficient of rigidity is* 3,600,000 *lbs. per square inch, find C. Hence find the moment which will twist a bar of the same material,* 300 *ins. in length and* 1 *in. in diameter, through an angle of* 5°.

$$\frac{195}{180}\pi = C\frac{30 \times 4}{3600000(.1)^4}.$$

Therefore $$C = \frac{143}{14} = 10.2 = \frac{32}{\pi}, \text{ very nearly.}$$

Again, $$\frac{5}{180}\pi = \frac{32}{\pi}\,\frac{300 \cdot Pp}{3600000 \times 1^4}.$$

Therefore $$Pp = 102.9 \text{ in.-lbs.}$$

2. **Torsional Strength of Shafts.**—Consider an elementary prism QR of the shaft between two cross-sections at a unit distance (*one inch*) apart. Assuming the section on the left to be fixed, the twisting moment moves the prism from QR to QR_1, the angle ROR_1 being the angle of twist per unit of length (*in one inch*).

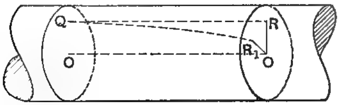


FIG. 584.

Let θ be this angle measured in radians.

The *shear strain* in the prism $=\frac{RR_1}{QR}=\frac{x\theta}{OO}=x\theta$, x being the length OR.

The *shear stress* at R is therefore $Gx\theta$, and if a is the sectional area of the prism, the *shear force* at R *at right angles* to $OR=Gx\theta a$.

This force is equivalent to a parallel force on the axis and a couple of moment $Gx^2\theta a$.

There is a similar force and a similar moment for every elementary area into which the cross-section may be divided.

The forces must necessarily neutralize each other and the total twisting moment (or torque) on the shaft $=\Sigma Gx^2\theta a$, the symbol Σ denoting algebraic sum.

Now G and θ are the same for each elementary area, and if T is the total torque,

$$T=G\theta\Sigma x^2a=G\theta J=Pp,$$

J being the moment of inertia with respect to the axis, or what is called the *polar* moment of inertia.

For a solid cylindrical shaft of diameter D,

$$J=\frac{\pi D^4}{32};$$

for a hollow cylindrical shaft of external diameter D and internal diameter D_1,

$$J=\frac{\pi(D^4-D_1^4)}{32}.$$

Again, the shear stress $G\theta x$ increases with x and is greatest at the surface, i.e., when $x=\frac{D}{2}$.

Denoting this maximum shear stress by f,

$$f=G\theta\frac{D}{2}.$$

Hence, for the *solid shaft*,

$$Pp=T=\frac{G\theta\pi D^4}{32}=\frac{f\pi D^3}{16}=\frac{fD^3}{5.1},$$

and for the *hollow shaft*

$$Pp=T=\frac{G\theta\pi(D^4-D_1^4)}{32}=\frac{f\pi}{16}\frac{D^4-D_1^4}{D}=\frac{f}{5.1}\frac{D^4-D_1^4}{D}.$$

If the thickness $t\left(=\frac{D-D_1}{2}\right)$ of the hollow shaft is small as compared with the diameter D, then, approximately,

$$Pp=T=\frac{G\theta\pi D^3t}{4}=\frac{f\pi D^2t}{2}.$$

Thus the *strength of the solid shaft* is defined by the value of f and depends on the cube of the diameter, while its *torsional rigidity*, which is measured by the ratio $\frac{T}{\theta}$, depends upon the fourth power of the diameter.

In order to diminish the amount of twist, or spring, a shaft is often made much stronger than is actually necessary, and a common rule is to specify that the twist is not to exceed 1° in 20 diameters. Then

$$f=G\theta\frac{D}{2}=G\frac{\pi\times1^\circ}{180}\frac{1}{20D}\frac{D}{2}=\frac{G\pi}{7200}.$$

Hence

for cast iron, taking $G=$ 6,300,000 lbs./sq. in., $f=2750$ lbs./in.sq.
" wrought iron, " $G=10,500,000$ " " $f=4583$ " "
" steel, " $G=12,000,000$ " " $f=5238$ " "

In shafts in which the spring is not of so much importance, as, e.g., when the shafting is in short lengths, higher values of f may be used. Thus

for cast iron, if $f=$ 4,500 lbs./sq. in., the twist is 1° in 12.2 diams.
" wrought iron, if $f=$ 7,200 " " " " " " 12.7 "
" steel, if $f=$11,200 " " " " " " 9.4 "

Another rule in accordance with good practice is that the twist must not exceed $\frac{1}{13}$ degree per lineal foot of length.

Again, let H.P. be the horse-power transmitted by a shaft making N revolutions per minute. Then, *for a solid shaft,*

$$T2\pi N = \frac{f\pi D^3}{16}\ 2\pi N = \text{work done in inch-pounds per minute}$$

$$= 33000\text{H.P.} \times 12,$$

and therefore

$$D = \sqrt[3]{\frac{3528000}{11f}} \times \sqrt[3]{\frac{\text{H.P.}}{N}}.$$

Taking $f = 4500$, 7200, and 11,200 lbs. per sq. in. for cast iron, wrought iron, and steel respectively, then, *approximately,*

$$D = 4\sqrt[3]{\frac{\text{H.P.}}{N}} \text{ for cast iron,}$$

$$D = 3.6\sqrt[3]{\frac{\text{H.P.}}{N}} \text{ `` wrought iron,}$$

$$D = 3\sqrt[3]{\frac{\text{H.P.}}{N}} \text{ `` steel,}$$

formulæ agreeing with the best practice in the case of shafts *subjected to torsion only.*

Ex. 2. *A steel shaft* 20 *ft. in length and* 3 *ins. in diameter makes* 200 *revolutions per minute and transmits* 50 *H.P. Through what angle is the shaft twisted?*

A wrought-iron shaft of the same length is to do the same work at the same speed. Find its diameter so that the stress at the circumference may not exceed three fifths of that at the circumference of the steel shaft.

If T is the twisting moment,

$$T \times 2\pi \times 200 = 50 \times 33000 \times 12 \text{ in.-lbs.}$$

Therefore
$$15750 = T = 12000000 \times \theta \times \frac{22}{7}\frac{3^4}{32},$$

and
$$\theta = \frac{49}{297000}.$$

Therefore the total angle of twist in degrees $= 20 \times 12 \times \frac{180}{\pi}\theta = 2°.26.$

Again,
$$f\frac{\pi 3^3}{16}=\frac{3}{5}f\frac{\pi D^3}{16},$$

if D is the diameter of the wrought-iron shaft.

Therefore $D^3=45$ and $D=3.556$ ins.

Ex. 3. *Show that a hollow shaft is both stiffer and stronger than a solid shaft of the same material, weight, and length.*

Let d be the diameter of the solid shaft, and d_1, d_2 the external and internal diameters respectively of the hollow shaft.

Let θ_1, θ_2 be the angles of torsion in radians of the solid and hollow shafts respectively.

Then, for the *same twisting couple* T,

$$\frac{\text{the rigidity of the solid shaft}}{\text{the rigidity of the hollow shaft}}=\frac{\dfrac{T}{\theta_1}}{\dfrac{T}{\theta_2}}=\frac{\theta_2}{\theta_1}$$

$$=\frac{G\pi d^4}{G\pi(d_1^4-d_2^4)}=\frac{d^2}{d_1^2+d_2^2},$$

since $\pi(d_1^2-d_2^2)=\pi d^2$.

But $d_1^2+d_2^2>d^2$, and therefore $\theta_1>\theta_2$, so that the solid shaft twists through a greater angle than the hollow shaft.

Again, if T_1 and T_2 are the twisting moments of the solid and hollow shafts respectively,

$$\frac{T_1}{T_2}=f\pi\frac{d^3}{16}\div f\frac{\pi}{16}\left(\frac{d_1^4-d_2^4}{d_1}\right)=\frac{d_1 d}{d_1^2+d_2^2}=\frac{d_1(d_1^2-d_2^2)^{\frac{1}{2}}}{d_1^2+d_2^2}.$$

Hence $T_2>T_1$, since it is evident that $d_1(d_1^2-d_2^2)^{\frac{1}{2}}<(d_1^2+d_2^2)$.

Ex. 4. *In a spinning-mill a cast-iron shaft* 8¼ *ins. in diameter makes* 27 *revolutions per minute. Find the work transmitted if the angle of torsion is not to exceed* $\frac{1}{13}$° *per lineal foot. What will the work be if the maximum shear stress in a section is not to exceed* 4500 *lbs. per square inch?*

First. If T is the twisting couple,

$$T\times 2\pi\times 27=33000\times\text{H.P.}$$

Therefore
$$\text{H.P.}=\frac{9}{1750}T=\frac{9}{1750}\,\frac{1}{12}\left(G\theta\,\frac{\pi d^4}{32}\right)$$

$$=\frac{9}{1750}\,\frac{1}{12}\times 6300000\times\frac{\pi}{180}\times\frac{1}{12}\times\frac{1^\circ}{13}\times\frac{\pi}{32}(8\tfrac{1}{4})^4$$

$$=137.49.$$

Second. $\text{H.P.}=\frac{9}{1750}T=\frac{9}{1750}\frac{1}{12}\left(f\frac{\pi d^3}{16}\right)=\frac{9}{1750}\frac{1}{12}\,4500\;\frac{22}{7}\;\frac{(8\frac{1}{4})^3}{16}=212.72.$

Ex. 5. *The external diameter of a hollow shaft is p times the internal diameter. Compare its torsional strength with that of a solid shaft of the same material and weight.*

Let pd and d be the external and internal diameters of the hollow shaft

Let D = the diameter of the solid shaft.

Then

$$\frac{\text{Twisting moment of hollow shaft}}{\text{Twisting moment of solid shaft}}=\frac{f\frac{\pi}{16}\left(\frac{p^4d^4-d^4}{pd}\right)}{f\frac{\pi}{16}D^3}=\frac{d^3}{D^3}\left(\frac{p^4-1}{p}\right).$$

But $$D^2=p^2d^2-d^2=(p^2-1)d^2.$$

Therefore the moments are in the ratio of $\dfrac{p^2+1}{p\sqrt{p^2-1}}$.

3. **Non-circular Sections.**—The polar moment of inertia, J, of any area with respect to an axis through the centre of inertia perpendicular to its plane is given by

$$J=I_1+I_2,$$

I_1 and I_2 being the moments of inertia with respect to any two axes through the same point at right angles to each other. *If it is assumed* that the angle of torsion (θ) is the same at all points of a non circular section, then

the torsional rigidity $=\dfrac{T}{\theta}=GJ=G(I_1+I_2)$

= the sum of the flexural rigidities in any two planes at right angles to one another.

St. Venant has shown that this is in excess of the torsional rigidity derived from the true theory, and it can be easily shown that the twisting couple produces a greater twist than that based upon Coulomb's laws and also warps the naturally plane sections of the shaft. Figs. 585, 586, 587, given by St. Venant, show the appearances presented by elliptic, square, and rectangular bars under exaggerated torsion as may be obtained with such substances as india-rubber. St. Venant also enunciated the important practical result that the ribs introduced to increase the *flexural* rigidity of a bar or beam has a really detrimental effect upon its *torsional* rigidity.

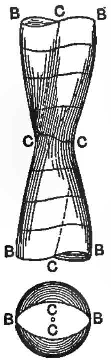


FIG. 585.

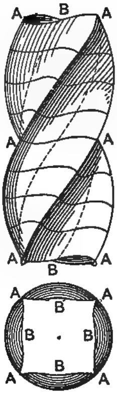


FIG. 586.

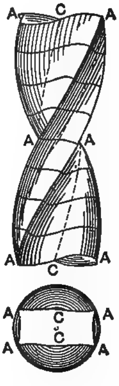


FIG. 587.

The following table gives a few of St. Venant's principal results, the third column giving the fraction which the torsional rigidity is of that of a circular cylinder of the same sectional area:

Section.	Torsional Rigidity.	Fraction.
An ellipse	$G\pi\frac{a^3b^3}{a^2+b^2}$	$\frac{2ab}{a^2+b^2}$
An equilateral triangle	$Ga^4\times\frac{\sqrt{3}}{80}$	.72581
A square	$Ga^4\times.140576$	.88362
A rectangle	$Gab^3\left(\frac{1}{3}-.21\frac{b}{a}\right)$	

Again, the true torsional rigidities of a shaft with a square section having curved corners and hollow sides (Fig. 588), of a shaft with

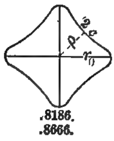


FIG. 588.

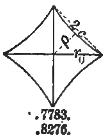


FIG. 589.

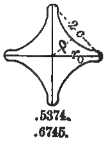


FIG. 590.

a square section having acute angles and hollow sides (Fig. 589), and of a shaft with a star section having rounded points (Fig. 590) are

.8666, .8276, and .6745 times, respectively, the corresponding torsional rigidities based upon Coulomb's laws.

St. Venant was the first to call attention to the fact that in non-circular sections the stress is more generally greatest at points in the bounding surface which are nearest to the axis and least at those points which are farthest from the axis. Thus the surface shear stress at any point x, y of an elliptic section is $\frac{2T}{\pi a^3 b^3}\sqrt{b^4x^2+a^4y^2}$, which is greatest and equal to $\frac{2T}{\pi ab^2}$, when $x=0$, i.e., at the end of the minor axis.

In the rectangle the shear is greatest at the middle point of the longest side, while for squares and equilateral triangles there are lines of maximum strain through the middle points of the sides. It may be remarked in general that any elastic solid bounded by surfaces with projecting edges or angles, or with re-entrant edges or angles, cannot experience any finite stress or strain near a projecting point unless acted upon by external forces at the point; the strain near an edge can only be in the direction of the edge, while the stress and strain are increased indefinitely in the neighborhood of a re-entrant edge or angle. This result is in accordance with the important and well-known practical rule that every re-entering edge or angle ought to be rounded to prevent risk of rupture in solid pieces designed to bear stress.

Ex. 6. *A square wooden shaft* 8 *ft. in length is acted upon by a force of* 200 *lbs., applied at the circumference of an* 8-*ft. wheel on the shaft. Find the length of the side of the shaft, so that the total torsion may not exceed* 2° ($G=400{,}000$). *What should be the diameter of a round shaft of equal strength and of the same material?*

First. $200\times4\times12=T$ in.-lbs. $=400000\,\frac{\pi}{180}\,\frac{2^\circ}{96}\,a^4\times.140576$, and $a=4.655$ ins.

Second. If d is the required diameter,

$$400000\,\frac{\pi}{180}\,\frac{2}{96}\,\frac{\pi d^4}{32}=400000\frac{\pi}{180}\,\frac{2}{96}\,a^4\times.140576.$$

Therefore $\qquad d^4=\frac{7\times32}{22}\,a^4\times.140576 \quad$ and $\quad d=5.1$ ins.

4. Variable Resistance.—The formulæ deduced for the twisting moment of a shaft is based on the assumption that the power is transmitted against a constant resistance. In practice the resistance varies between a maximum and a minimum limit, which are sometimes of widely different values, and the shaft must be designed for the maximum moment to which it may be subjected.

Ex. 7. *The wrought-iron screw shaft of a steamship is driven by a pair of cranks set at right angles and* 21.7 *ins. in length; the horizontal pull upon each crank-pin is* 176,400 *lbs., and the effective length of the shaft is* 866 *ins. Find the diameter of the shaft so that* (*a*) *the circumferential stress may not exceed* 9000 *lbs. per square inch and* (*b*) *the angle of torsion may not exceed* $\frac{1^\circ}{13}$ *per lineal foot, G being* 10,000,000 *lbs. The actual diameter of the shaft is* 14.9 *ins. What* (*c*) *is the actual torsion?*

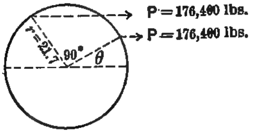


FIG. 591.

The ultimate tensile strength of the iron being 60,000 lbs. per square inch, find (*d*) the actual ultimate strength under unlimited repetitions of stress. What (*e*) is the torsion when one of the cranks passes a dead point?

(*a*) Twisting moment $= Pr(\sin\theta + \sin\overline{90^\circ+\theta}) = 2Pr\sin\overline{\theta+45^\circ}\cos 45^\circ$, which is a maximum and $= Pr\sqrt{2}$, when $\theta = 45^\circ$.

Then $$176400\times 21.7\times\sqrt{2} = \frac{9000\times\pi\times d^3}{16} \quad\text{and}\quad d = 14.53 \text{ ins.}$$

(*b*) $$176400\times 21.7\times\sqrt{2} = 10000000\,\frac{\pi}{180}\,\frac{1}{12}\,\frac{1}{13}\,\pi\frac{d^4}{32}$$

and $$d = 14.9 \text{ ins.}$$

(*c*) If A° is the total torsion,

$$176400\times 21.7\times\sqrt{2} = 10000000\,\frac{\pi}{180}\,\frac{A^\circ}{866}\,\pi\frac{(14.9)^4}{32}$$

and $$A^\circ = 5^\circ.5465.$$

(*d*) The twisting moment $= 2Pr\sin(\theta+45^\circ)\cos 45^\circ$, which is a maximum when $\theta = 45^\circ$ and a minimum when $\theta = 0$. Therefore

$$max.\text{ moment} = Pr\sqrt{2} = f_{\text{max.}}\,\frac{22}{7}\,\frac{(14.9)^3}{16} = 176400\times 21.7\times\sqrt{2}$$

and $$min.\text{ moment} = Pr \quad = f_{\text{min.}}\,\frac{22}{7}\,\frac{(14.9)^3}{16} = 176400\times 21.7,$$

so that $f_{\text{max.}} = 8331.24$ lbs./sq. in. and $f_{\text{min.}} = 5891.08$ lbs./sq. in.

Using Unwin's formula,

$$\text{the fluctuation } d = f_{\text{max.}} - f_{\text{min.}} = 2440.16,$$

and therefore

$$\text{the max. stress/sq. in.} = \frac{2440.16}{2} + \sqrt{60000(60000 - \tfrac{3}{2} \times 2440.16)} = 59361.2 \text{ lbs.}$$

(*c*) On passing a dead point the total torsion of $A°$ is given by

$$176400 \times 21.7 = 10000000 \frac{\pi}{180} \frac{A°}{866} \frac{22}{7} \frac{(14.9)^4}{32}$$

and

$$A° = 3°.92.$$

5. Distance Between Bearings.—The distance between the bearings of a line of shafting is limited by the consideration that the stiffness of the shaft must be such as will enable it to resist excessive bending under its own weight and under any other loads (e.g., pulleys, wheels, etc.) applied to it. For this reason the ratio of the maximum deviation of the axis of the shaft from the straight to the corresponding distance between bearings should not exceed a certain fraction whose value has been variously estimated by different authorities.

Let l be the distance in feet between bearings, d the diameter of the shaft in inches, w the weight of the material of the shaft per cubic foot, and let the applied load be equivalent to a load per lineal unit of length m times that of the shaft. Assume a *stiffness* $\frac{1}{1200}$, and that the axis of the shaft is truly in line at the bearings. The maximum deflection of the shaft is given by the formula

$$D = \frac{1}{384} \frac{(m+1)(\text{weight of shaft}) l^3 \times 1728}{EI}$$

$$= \frac{1}{384}(m+1)\frac{\pi d^2}{4}\frac{1}{144} wl \frac{64}{\pi d^4} \frac{l^3 \times 1728}{E}.$$

Therefore

$$\frac{D}{l} = \frac{1}{100} = \frac{(m+1)w}{2E} \frac{l^3}{d^2},$$

or

$$l = \sqrt[3]{\frac{Ed^2}{50w(m+1)}}.$$

For wrought iron, $E = 30,000,000$ lbs. and $w = 480$ lbs. Therefore

$$l = 12.7 \sqrt[3]{\frac{d^2}{m+1}}.$$

If the applied load, instead of being uniformly distributed is concentrated at the centre, the maximum deflection

$$=D\text{ ins.}=\frac{1}{192}\,\frac{(m+\frac{1}{2})(\text{weight of shaft})l^3\times 1728}{EI},$$

and hence
$$l=\sqrt[3]{\frac{Ed^2}{100w(m+\frac{1}{2})}}.$$

For wrought iron
$$l=8.5\sqrt[3]{\frac{d^2}{m+\frac{1}{2}}}.$$

6. Efficiency of Shafting.—Let it require the whole of the driving moment to overcome the friction in the case of a shaft of diameter d and length L. The *efficiency* of a shaft of the same diameter and length $l=1-\dfrac{l}{L}$.

But
$$\frac{f\pi d^3}{16}=(Pp)=\text{moment of friction}=\mu\frac{w\pi d^2}{4}L\frac{d}{2}=\frac{\mu w\pi d^3}{8}L,$$

w being the specific weight of the material of the shaft and μ the coefficient of friction. Hence

$$\frac{1}{L}=2\mu\frac{w}{f},$$

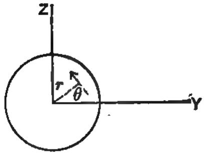


FIG. 592.

and the efficiency $=1-2\mu\dfrac{wl}{f}$.

The efficiency may also be found as follows: Let f_r, v_r be the stress and velocity at any point distant r from the axis of the shaft which is taken as the axis of x, the other axes being as shown in Fig. 592.

Then, by Art. 20, Chapter V, the work transmitted across a small element $dydz$ of the section

$$=udydz=(f_{xx}v_x+f_{xy}v_y+f_{xz}v_z)dydz.$$

In the present case

$$f_{xx}=0,\qquad f_{xy}=-f_r\sin\theta,\qquad f_{xz}=f_r\cos\theta,$$

$$v_x=0,\qquad v_y=-v_r\sin\theta,\qquad v_z=v_r\cos\theta.$$

Therefore

$$\text{the total work transmitted} = \int\int u dydz$$

$$= \int\int (f_r v_r \sin^2\theta + f_r v_r \cos^2\theta) dydz = \int\int f_r v_r dydz$$

$$= \frac{4fv}{d^2}\int\int r^2 dydz = \frac{4fv}{d^2}J = \frac{fvd^2\pi}{8},$$

f and v being the stress and velocity at the surface. Again,

$$\text{the work lost in friction} = \mu\frac{w\pi d^2}{4}lv.$$

Hence

$$\text{the efficiency} = \frac{f\frac{vd^2}{8}\pi - \mu\frac{w\pi d^2}{4}lv}{f\frac{vd^2}{8}\pi} = 1 - 2\frac{\mu lv}{f}.$$

Ex. 8. *The efficiency of an axle is* ½; *the working stress in the shaft is* 9000 *lbs. per square inch; the coefficient of friction is* .10. *How far may work be transmitted? The shaft is of wrought iron.*

$$\frac{1}{2} = 1 - 2\frac{1}{10}\,\frac{480}{1728}\,\frac{l}{9000},$$

and $$l = 81000 \text{ ins.} = 6750 \text{ ft.}$$

Ex. 9. *Determine* (*a*) *the profile of a shaft of length* l *which at every point is so proportioned as to be just able to bear the power it has to transmit plus the power required to overcome the friction beyond the point under consideration. Find* (*b*) *the efficiency of such a shaft, and* (*c*) *the efficiency of a shaft made up of a series of* n *divisions each of uniform diameter.*

(*a*) Let s be the maximum allowable skin stress. Then

$\frac{s\pi r^3}{2}$ is the max. allowable driving moment at the driving end of radius r, and

$\frac{s\pi r^3}{2}$ " " " " " " at x from the driving end, the corresponding radius being y.

Hence

$f w\pi y^2 \cdot dx \cdot y$ = moment used up in overcoming the friction for a length dx.,

and

$f w\pi \int^x y^3 dx =$ " " " " " " " the length x.

Therefore $$fw\pi\int_0^x y^3dx = s\frac{\pi r^3}{2} - s\frac{\pi y^3}{2}.$$

Differentiating, $$fw\pi y^3dx = -\frac{3\pi s}{2}y^2dy,$$

or $$\frac{dy}{y} = -\frac{2}{3}\frac{fw}{s}dx = -\frac{dx}{3L},$$

where $L\left(=\frac{s}{2fw}\right)$ is the length of a shaft of uniform diameter for which the whole driving moment is required to overcome the friction. Integrating,

$$\log_e y = -\frac{x}{3L} + c,$$

c being a constant of integration.

When $x=0$, $y=r$; therefore

$$\log_e r = c,$$

and $$\frac{y}{r} = e^{-\frac{x}{3L}}$$

is the profile required.

(*b*) Again,

$$\text{the useful moment} = \frac{s\pi y^3}{2},$$

$$\text{the total moment} = \frac{s\pi r^3}{2},$$

and $$\text{the efficiency} = \frac{y^3}{r^3} = e^{-\frac{l}{L}}.$$

$$\text{(c) The efficiency of each length} = \frac{L-\frac{l}{n}}{L} = \left(1-\frac{l}{nL}\right),$$

and therefore

$$\text{the efficiency of the shaft} = \left(1-\frac{l}{nL}\right)^n.$$

7. **Combined Bending and Torsion.**—It was shown in Chapter V that in a shaft subjected to a bending moment M_b and to a twisting moment M_t, acting simultaneously,

$$\text{the maximum tensile or compressive stress} = \frac{p}{2} + \sqrt{\frac{p^2}{4}+q^2},$$

and the maximum shear stress $=\sqrt{\frac{p^2}{4}+q^2}$,

where $p=\frac{M_b}{Z}$ and $q=\frac{M_t}{2Z}$,

Z being the strength modulus of the section.

For a solid section of diameter d, $Z=\frac{\pi d^3}{32}$,

" hollow " " external diameter d_1, and internal diameter d_2, $Z=\frac{\pi(d_1^4-d_2^4)}{32d_1}$.

Hence

the maximum tensile or compressive stress $=\frac{1}{2Z}(M_b+\sqrt{M_b^2+M_t^2})$,

and

$$\text{the maximum shear stress}=\frac{1}{2Z}\sqrt{M_b^2+M_t^2}.$$

Generally speaking, it is found that the first of these two equations gives the largest diameter, and thus the maximum tensile or compressive stress is the same as the shearing stress when the shaft is subjected to a twisting moment $M_b+\sqrt{M_b^2+M_t^2}$.

Ex. 10. *Power is taken from a shaft by means of a pulley 24 inches in diameter which is keyed on to the shaft at a point dividing the distance between two consecutive supports into segments of 20 and 80 ins.; the tangential force at the circumference of the pulley is 5500 lbs. If the shaft is of cast iron, determine its diameter, taking into account the bending action to which it is subjected.*

$$M_b=5500\,\frac{80\times 20}{100}=88{,}000 \text{ in.-lbs.}$$

$$M_t=5500\times 12=66{,}000 \text{ in.-lbs.}$$

Therefore M

Hence if d is th

and

Again, another expression for the maximum stress in the section of a shaft may be obtained as follows: The stresses at a point near the circumference are:

On a cross-section, (*a*) a normal stress $p\left(=\frac{M_b}{Z}\right)$;

(*b*) a shear " $q\left(=\frac{M_t}{2Z}\right)$;

and on a section parallel to the axis, (*a*) a normal stress 0;

(*b*) a shear stress $q\left(=\frac{M_t}{2Z}\right)$,

Z being the *strength modulus* of the section.

Hence, as in Chapter V, if p_1, p_2 are the principal stresses at the point,

$$p_1=\frac{p}{2}+\sqrt{\frac{p^2}{4}+q^2}$$

and
$$p_2=\frac{p}{2}-\sqrt{\frac{p^2}{4}+q^2}.$$

Let e_1, e_2 be the strains in the directions of the principal stresses,

then
$$Ee_1=p_1-\frac{p_2}{\sigma}=\frac{p}{2}\frac{\sigma-1}{\sigma}+\frac{\sigma+1}{\sigma}\sqrt{\frac{p^2}{4}+q^2}$$

and
$$Ee_2=p_2-\frac{p_1}{\sigma}=\frac{p}{2}\frac{\sigma-1}{\sigma}-\frac{\sigma+1}{\sigma}\sqrt{\frac{p^2}{4}+q^2},$$

σ be

must not exceed

th

therefore

A common value of σ is 4, and then

$$f=\tfrac{3}{8}p+\tfrac{5}{8}\sqrt{p^2+4q^2}=\frac{1}{Z}\left(\tfrac{3}{8}M_b+\tfrac{5}{8}\sqrt{M_b{}^2+M_t{}^2}\right)$$

or $$fZ=\tfrac{3}{8}M_b+\tfrac{5}{8}\sqrt{M_b{}^2+M_t{}^2}.$$

This formula is given by Grashof, Cauchy, and others for combined bending and twisting.

Ex. 11. *Solving Ex.* 10 *by the preceding method,*

$$5600\frac{22}{7}\frac{d^3}{16}=\frac{3}{8}(88000)+\frac{5}{8}\sqrt{(88000)^2+(66000)^2}$$

$$=134{,}750 \text{ in.-lbs.,}$$

and $$d^3=122.5,$$

or $$d=4.967 \text{ ins.}$$

8. Centrifugal Whirling of Shafts.—It is known that a shaft, however nearly balanced, when driven at a sufficiently high speed bends, or "whirls," as it is termed.

The particular or "critical" speed depends on the manner in which the shaft is supported, on its dimensions, its modulus of elasticity, and on the manner of loading.

In an unloaded shaft the period of whirl coincides with the natural period of vibration, as might be expected; but generally in a loaded shaft the period of whirl is less than the natural period of vibration. As in the lateral vibration, so in whirling, there is a series of periods at which the shaft whirls.

This torsional vibration often occurs in very small shafting and sometimes in long shafting of much larger diameter. If the impulses producing the vibration are repeated at the proper intervals, the vibration may continually increase until the torsion becomes of sufficient magnitude to cause rupture.

Consider the case of a uniform shaft weighing w lbs. per unit of length, subjected to an endlong thrust F and revolving with an angular velocity of ω radians per second.

Take the middle point of the shaft as the origin and let y be the deviation from straightness of a point distant x from the origin. Then if m is the B.M. at this point due to the centrifugal effect,

$$\frac{d^2m}{dx^2}=\frac{w}{g}\omega^2y.$$

Also, $$-EI\frac{d^2y}{dx^2}=\text{total B.M. at the point}$$

$$=Fy-m+\frac{w}{2}\left(\frac{l^2}{4}-x^2\right).$$

Differentiating twice,

$$-EI\frac{d^4y}{dx^4}=F\frac{d^2y}{dx^2}-\frac{d^2m}{dx^2}-w,$$

or $$\frac{d^4y}{dx^4}+\frac{F}{EI}\frac{d^2y}{dx^2}-\frac{w}{g}\frac{\omega^2}{EI}y-\frac{w}{EI}=0.$$

The general solution of this equation is

$$y=A\cos\alpha x+B\sin\alpha x+Ce^{\beta x}+De^{-\beta x}-\frac{g}{\omega^2},$$

where $$\alpha^2=\frac{1}{2EI}\left(F+\sqrt{F^2+4EI\frac{w}{g}\omega^2}\right)$$

and $$\beta^2=\frac{1}{2EI}\left(-F+\sqrt{F^2+4EI\frac{w}{g}\omega^2}\right).$$

Since y is the same for equal positive and negative values of x, $B=0$ and $C=D$. Therefore

$$y=A\cos\alpha x+C(e^{\beta x}+e^{-\beta x})-\frac{g}{\omega^2}.$$

Again, assuming that the bearings do not constrain the direction of the axis of the shaft,

$$\frac{d^2y}{dx^2}=0\text{ when }x=\frac{l}{2}.$$

Also, $$y=0 \text{ when } x=\frac{l}{2}.$$

Therefore $$0=-A\alpha^2\cos\frac{\alpha l}{2}+C\beta^2\left(e^{\frac{\beta l}{2}}+e^{-\frac{\beta l}{2}}\right)$$

and $$0=\ A\cos\frac{\alpha l}{2}+C\left(e^{\frac{\beta l}{2}}+e^{-\frac{\beta l}{2}}\right)-\frac{g}{\omega^2}.$$

Hence

$$A=\frac{g\beta^2}{\omega^2(\alpha^2+\beta^2)\cos\frac{\alpha l}{2}}$$

and $$C=\frac{g\alpha^2}{\omega^2(\alpha^2+\beta^2)\left(e^{\frac{\beta l}{2}}+e^{-\frac{\beta l}{2}}\right)}=\frac{g\alpha^2}{2\omega^2(\alpha^2+\beta^2)\cosh\frac{\beta l}{2}}.$$

The B.M. is greatest at the middle point and is the value of

$$-EI\frac{d^2y}{dx^2} \text{ when } x=0. \quad \text{Therefore}$$

$$M_{\max.}=A\alpha^2-2C\beta^2=\frac{g\alpha^2\beta^2}{\omega^2(\alpha^2+\beta^2)}\left(\frac{1}{\cos\frac{\alpha l}{2}}-\frac{1}{\cosh\frac{\beta l}{2}}\right).$$

The greatest stress at the middle is

$$\frac{M_{\max.}}{Z}+\frac{F}{S},$$

Z being the strength modulus of the section and S its area.

Ex. 12. *A propeller shaft* 13.4 *ins. in diameter,* 98 *ft. long, and making* 60 *revolutions per minute is subjected to an end thrust of* 50,000 *lbs. Show that the centrifugal force effect is* 280 *times greater than the effect due to the end thrust.*

$$\frac{\text{Centrifugal effect}}{\text{End-thrust effect}}=\frac{w}{g}\frac{\omega^2}{EI}\div\left(\frac{F}{2EI}\right)^2$$

$$=\frac{4w\omega^2EI}{F^2g}=\frac{4\times.28(2\pi)^2\times30000000\times\pi(13.4)^4}{(50000)^2\times32.2\times12}=279.85.$$

Suppose that there is no end thrust ($F=0$) and that the effect of the shaft's weight may be disregarded as compared with the centrifugal effect. The general equation and its solution now become

$$\frac{d^4y}{dx^4}-\frac{w}{g}\frac{\omega^2}{EI}y=0$$

and
$$y=A\cos\alpha x,$$

A being the deflection at the middle and $\alpha^2=\sqrt{\frac{w}{g}\frac{\omega^2}{EI}}$.

Also,
$$y=0 \text{ when } x=\frac{l}{2}.$$

Therefore
$$0=A\cos\frac{\alpha l}{2} \quad\text{and}\quad \alpha l=\pi.$$

Hence
$$l=\frac{\pi}{\alpha}=\pi\left(\frac{gEI}{w\omega^2}\right)^{\frac{1}{4}}.$$

Ex. 13. *A steel shaft of diameter d ins. and weighing .28 lb. per cubic inch makes n revolutions per second. Take $E=28{,}000{,}000$ lbs. per square inch.* Then

$$w=.28\frac{\pi d^2}{4}=.22\times d^2,$$

and
$$l=\pi\left(\frac{32\times12\times28000000\times\pi d^4}{\frac{\pi d^2}{4}\times.28\times(2\pi n)^2\times64}\right)^{\frac{1}{4}}=277\tfrac{1}{2}\left(\frac{d}{n}\right)^{\frac{1}{2}}.$$

9. Helical Springs.—Let the figure represent a cylindrical spiral spring of length L supporting a weight W. Consider a section of the spring at any point B.

At this point there is a shear W and a torque Wy, y being the distance of B from the axis of the spring, i.e., the radius of the coil.

The effect of W may generally be neglected as compared with the effect of the moment Wy, and it may therefore be assumed that the spring is under torsion at every point. Let there be n coils. Then

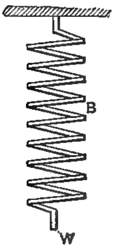


FIG. 593.

$$Wy=\frac{f\pi r^3}{2}=\frac{G\theta\pi r^4}{2},$$

r being the radius of the spring and θ the twist in radians per unit of length.

Also, $L = 2\pi yn$, approximately.

The change of length in the spring (i.e., the deflection)

$$= Ly\theta = \frac{2wy^2L}{G\pi r^4} = \frac{fLy}{Gr} = \frac{2\pi fny^2}{Gr}.$$

The work stored up in the spring, i.e., the work done in stretching or compressing the spring,

$$= \frac{Wy}{2}L\theta = \frac{W^2y^2L}{G\pi r^4} = \frac{f^2\pi r^2L}{4G}.$$

A weight at the end of the spring tends to turn as well as to change the length of the spring, and this is due to a slight bending action.

According to Hartnell, $f = 70{,}000$ lbs. per square inch for $\frac{1}{4}$-in. steel, $f = 60{,}000$ lbs. per square inch for $\frac{3}{8}$-in. steel, $f = 50{,}000$ lbs. per square inch for $\frac{1}{2}$-in. steel, and G varies from 13,000,000 lbs. for $\frac{1}{4}$-in. steel to 11,000,000 lbs. for $\frac{3}{8}$-in. steel.

Also, for wire less than $\frac{3}{8}$ in. in diameter he takes

$$\text{the safe load} = \frac{96000r^3}{y}, \text{ and the deflection} = \frac{Wny^3}{2880000r^4}.$$

Assuming that the laws upon which the torsion of rectangular and square sections is based are the same as for circular sections, then for a rectangular section of breadth b and depth h,

$$Wy = \frac{1}{12}G\theta bh(b^2 + h^2) = \tfrac{1}{6}fbh\sqrt{b^2 + h^2}$$

and the deflection

$$\delta = \frac{2Lyf}{G\sqrt{b^2 + h^2}} = \frac{4\pi y^2nf}{G\sqrt{b^2 + h^2}} = \frac{12Wy^2L}{Gbh(b^2 + h^2)}.$$

For a square section $b = h$.

These results must be modified in accordance with the deductions of Art. 3.

Helical Spring in Torsion.

Let a moment of M_t in.-lbs. twist the spring through an angle θ (measured in radians).

" n_1, n_2 be the number of free coils before and after twisting.
" y_1, y_2 " " radii of the spring " " " "

Then
$$2\pi y_1 n_1 = L = 2\pi y_2 n_2$$

and
$$\theta = 2\pi(n_1 - n_2).$$

Therefore
$$M_t = EI\left(\frac{1}{y_1} - \frac{1}{y_2}\right) = \frac{2\pi EI}{L}(n_1 - n_2) = \frac{EI\theta}{L}.$$

The angle of twist in degrees $= \dfrac{180\theta}{\pi}$.

For a *round* wire of diameter d, $I = \dfrac{\pi d^4}{64}$.

" *square* " " side d, $I = \dfrac{d^4}{12}$.

Open Coiled Helical Spring.—In the preceding discussion the coils are assumed to be so flat that the strain is taken to be one of torsion only. If the obliquity of the helix is large, the bending effect can no longer be disregarded. In addition to the load W, applied axially, and tending to elongate the spring, let a couple T, about the axis, tend to increase the number of coils.

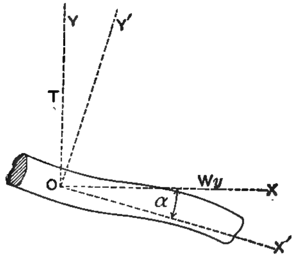


Fig. 594.

Consider the portion of the spring below a section at O. For equilibrium the molecular forces developed in this section must balance W and T.

Drawing OY parallel to the axis and OX at right angles, the load W produces a moment Wy about OX.

Draw OY' at right angles to OX', the axis of the wire. Then Wy and T may each be resolved into two components, viz.,

$Wy \cos\alpha$ and $T \sin\alpha$, producing bending about OY'.
$Wy \sin\alpha$ and $T \cos\alpha$, " twisting about OX'.

Thus, taking as positive the directions of closer winding and axial elongation,

$$\text{the total torque about } OY' = Wy \sin\alpha - T\cos\alpha$$
$$= G\theta J = B\theta,$$

and

$$\text{the total torque about } OX' = Wy\cos\alpha + T\sin\alpha$$
$$= EiI = Ai,$$

J being the polar moment of inertia and $A(=EI)$ and $B(=GJ)$ the flexural and torsional rigidities of the wire. Then

$$\theta = \text{angle of twist per unit of length}$$
$$= \frac{Wy\sin\alpha - T\cos\alpha}{B},$$

and

$$i = \text{angle of bending per unit of length}$$
$$= \frac{Wy\cos\alpha + T\sin\alpha}{A}.$$

Hence if ϕ and x are the total angular rotation and axial elongation,

$$\frac{\phi}{S} = i\sin\alpha - \theta\cos\alpha$$
$$= Wy\sin\alpha\cos\alpha\left(\frac{1}{A} - \frac{1}{B}\right) + T\left(\frac{\sin^2\alpha}{A} + \frac{\sin^2\alpha}{B}\right),$$

and

$$\frac{x}{Sy} = i\cos\alpha + \theta\sin\alpha$$
$$= Wy\left(\frac{\cos^2\alpha}{A} + \frac{\sin^2\alpha}{B}\right) + T\sin\alpha\cos\alpha\left(\frac{1}{A} - \frac{1}{B}\right).$$

A positive value of ϕ indicates that as the spring elongates the winding increases. If the coils are very flat and α therefore so small that it may be taken $=0$, then

$$\frac{\phi}{S} = \frac{T}{B} \quad \text{and} \quad \frac{x}{Sy} = \frac{Wy}{A},$$

indicating that an axial force does not tend to produce rotation, and that no axial elongation will be produced when the spring is subjected to a couple only.

Ex. 14. *A 2-in. helical spring with* 30 *coils is made of* ¼*-in. steel wire. Find the deflection under a load of* 1 *lb., the coefficient of distortion being* 12,000,000 *lbs.*

$$\text{Deflection} = \frac{2Wy^2L}{G\pi r^4} = \frac{4Wy^3n}{Gr^4}.$$

Therefore $$\text{deflection} = \frac{4\times1\times1\times30}{12000000(\frac{1}{8})^4} = .04096 \text{ in.}$$

EXAMPLES.

1. A steel shaft 4 ins. in diameter is subjected to a twisting couple which produces a circumferential stress of 15,000 lbs. What is the stress (shear) at a point 1 in. from the centre of the shaft? Determine the twisting couple. *Ans.* 7500 lbs.; 188,571$\frac{3}{7}$ lbs.

2. A weight of 2$\frac{2}{5}$ tons at the end of a 1-ft. lever twists asunder a steel shaft 1$\frac{2}{5}$ ins. in diameter. Find the breaking weight at the end of a 2-ft. lever, and also the modulus of rupture. *Ans.* 1$\frac{1}{5}$ tons; 23,510 lbs.

3. A couple of N ft.-tons twists asunder a shaft of diameter d. Find the couple which will twist asunder a shaft of the same material and diameter $2d$. *Ans.* $8N$.

4. Compare the couples required to twist two shafts of the same material through the same angle, the one shaft being l ft. long and d ins. in diameter, the other $2l$ ft. long and $2d$ in. in diameter. Compare the couples, the diameter of the latter shaft being $\frac{d}{2}$. *Ans.* 1 to 8; 32 to 1.

5. A shaft 15 ft. long and 4$\frac{1}{2}$ ins. in diameter is twisted through an angle of 2° under a couple of 2000 ft.-lbs. Find the couple which will twist a shaft of the same material 20 ft. long and 7$\frac{1}{5}$ ins. in diameter through an angle of 2$\frac{1}{2}$°. *Ans.* 12,288 ft.-lbs.

6. A wrought-iron shaft 20 ft. long and 5 ins. in diameter is twisted through an angle of 2°. Find the maximum stress in the material, G being 10,500,000 ft.-lbs. *Ans.* 3819.2 lbs. per sq. ins.

7. A shaft 1 in. in diameter can safely transmit a torque of 2400 lb.-ins. What diameter of shaft would be required for transmitting 15 H.P. at 200 revolutions per minute? *Ans.* 1$\frac{1}{4}$ ins.

8. The amount of twist in a solid shaft is to be limited to 1° for each 10 ft. of length. Find the diameter for a twisting moment of 50 in.-tons, the modulus of torsional rigidity being 10,000,000 lbs. per sq. in. *Ans.* 5.15 ins.

9. A crane chain exerts a pull of 6000 lbs. tangentially to the drum upon which it is wrapped. Find the diameter of a wrought-iron axle which will transmit the resulting couple, the effective radius of the drum being 7$\frac{1}{2}$ ins., and the safe working stress per square inch 7200 lbs. *Ans.* 3.17 ins.

10. Find the diameter and the total angle of torsion of a 12-ft. wrought-iron shaft driven by a water-wheel of 20 H.P., making 25 revolutions per minute, G being 10,000,000 lbs. and the working stress 7200 lbs. per square inch. *Ans.* 3.29 ins.; 3°.6.

11. A brass wire 20 ins. long, 0.1 in. diameter, twists through a total angle of 130° when a twisting moment of 4 in.-lbs. is applied. Find G for the material. What would be the twist of a shaft of the same material with a twisting moment of 600 in.-lbs., 20 ft. long, 1.2 ins. diameter?
Ans. 3,600,000 lbs./sq. in.; 7°.8.

12. A round iron shaft 15 ft. long is acted upon by a weight of 2000 lbs. applied at the circumference of a 24-in. wheel on the shaft. Taking $G=6{,}000{,}000$ lbs., find the diameter of the shaft so that the total angle of torsion may not exceed 2°. *Ans.* 3.7 ins.

13. A force of 200 lbs. at the circumference of an 8-ft. wheel twists a round wooden shaft 8 ft. long. The total angle of torsion is not to exceed 2°. Find the diameter of the shaft. *Ans.* 4.35 ins., G being 750,000 lbs./sq. in.

14. Calculate the diameter of a steel shaft to transmit 4000 H.P. at 200 revolutions per minute, when the allowable stress on the metal is 12,000 lbs. per square inch.

15. Deduce the diameter of a shaft to transmit 300 H.P. at 200 revolutions per minute when the allowable stress is 10,000 lbs. per square inch.

16. Find the diameter of a solid steel shaft which is to transmit a moment of 392,700 ft.-lbs., the maximum allowable shear stress being 11,200 lbs./sq. inch. *Ans.* 12.9 ins.

17. Calculate the diameter of a hollow shaft required to transmit 1000 H.P. at 50 revolutions per minute, the skin stress being 6,000 lbs. per square inch, and the internal diameter $\frac{3}{5}$ of the external diameter.

18. In a 4-in. shaft 10 ft. long the maximum shear stress is 10,000 lbs./sq. in. and $G=10{,}000{,}000$ lbs./sq. in. Find the twisting couple and the torsion.
Ans. 125,714$\frac{2}{7}$-in.-lbs.; 1°.374.

19. Determine the diameter of a wrought-iron shaft for a screw steamer, and the torsion per lineal foot; the indicated H.P. $=1000$, the number of revolutions per minute $=150$, the length of the shaft from thrust-bearing to screw $=75$ ft., and the safe working stress $=7200$ lbs. per square inch.
Ans. 6.67 ins.; 10°.5.

20. In a spinning-mill a cast-iron shaft 84 ft. long makes 50 revolutions per minute and transmits 270 H.P. Find its diameter (1) if the stress in the metal is not to exceed 5000 lbs. per square inch; (2) if the angle of torsion per lineal foot is not to exceed $\frac{1^\circ}{13}$. Also (3) in the first case find the total torsion. *Ans.* (1) 7.02 ins.; (2) 10.23 ins.; (3) 13°.048.

21. A shaft 2 ins. in diameter and 140 ft. long is used to transmit 30 H.P. at 300 revolutions per minute. Find the angle through which the shaft springs and the skin stress in the material. Modulus of rigidity $=5000$ tons per square inch.

22. A wrought-iron shaft in a rolling-mill is 220 feet in length, makes 95 revolutions per minute, and transmits 120 H.P. to the rolls; the main body of the shaft is 4 ins. in diameter, and it revolves in gudgeons 3¾ ins. in diameter. Find the greatest shear stresses in the actual shaft and also the necessary diameter of the shaft.

Ans. 6330 and 7682 lbs.; 3.56 ins. if $f=9000$ lbs./sq. in.; 5.12 ins. if $\frac{1}{\theta^\circ}=13\times12$.

23. A force of 5000 lbs. at the end of a 6-in. lever-arm twists a 60-in. shaft of 2 ins. diameter through an angle of 7°. Find the modulus of rigidity.
Ans. 9,697,000 lbs.

24. Determine the twisting moment and the torsion for a 4-in. shaft 10 ft. long and subjected to a maximum shear stress of 10,000 lbs./sq. in., G being 12,500,000 lbs./sq. in.

25. A 3-in. shaft 40 ft. long springs 6° when transmitting power at 150 revolutions per minute. Find the H.P. transmitted, taking $G=12,000,000$ lbs./sq. in.

26. A shaft transmits a given H.P. at N revolutions per minute without bending. Find the weight of the shaft in pounds per lineal foot.

Ans. $32.9\left(\frac{\text{H.P.}}{N}\right)^{\frac{2}{3}}$.

27. The shafting of the turbines at Niagara Falls consists of a steel tube 38 ins. in diameter and ¾ in. thick. Find what horse-power can be transmitted at 250 revolutions per minute when the working stress is limited to 9000 lbs. per square inch. Also find the diameter of a solid shaft which will be equivalent to the above.

28. A vertical cast-iron axle in the Saltaire works makes 92 revolutions per minute and transmits 300 H.P.; its diameter is 10 ins. Find the angle of torsion.
Ans. .02282° per lineal foot.

29. A solid shaft is subjected to a twisting moment of 50 in.-tons, the modulus of torsional rigidity being 5000 tons/square inch. Find its diameter, assuming (*a*) that the amount of twist is limited to 1° for each 10 ft. of length, (*b*) that the working stress is nowhere to exceed 3½ tons/square inch.

30. Find the diameter of a shaft which is to transmit 25 H.P. at 50 revolutions per minute, and in which the working stress is to be 7000 lbs. per square inch.

31. A line of steel shafting is 80 ft. long; if a twisting moment of 4000 lb.-ins. is applied at one end, what will be the total angle of twist, the diameter of the shaft being 2½ ins.? What horse-power will the shaft transmit at 220 revolutions per minute?
Ans. 5°.2; 14 H.P.

32. A turbine makes 114 revolutions per minute, and transmits 92 H.P. through the medium of a shaft 8 ft. 6 ins. in length. What must be the diameter of the shaft so that the total angle of torsion may not exceed $\frac{2^\circ}{3}$, G being 10,500,000 lbs.?
Ans. 4.7 ins.

Determine the side of a square pine shaft that might be substituted for the iron shaft. ($G=525{,}000$ lbs.) *Ans.* 8.45 ins.

33. Find the diameter of a shaft for a winding-drum which works under the following conditions: The load lifted is $1\frac{1}{2}$ tons; diameter of drum, 5 ft.; width of face of drum, 26 ins.; distance from inner face of drum to the middle of the bearing of shaft, 13 ins.; maximum stress, 7000 lbs. per square inch. *Ans.* 5.44 ins.

34. The diameter of one shaft is double that of another of the same material; the smaller gave way when subjected to a twisting moment of 2 ft.-tons. What twisting moment will be required to wrench the other? *Ans.* 16 ft.-tons.

35. An iron shaft of which the working stress must not exceed 548 k./cm. is acted upon by a couple equivalent to a force of 50 k. at the end of a lever 0.4 m. in length. Find the diameter of the shaft. *Ans.* 2.65 cm.

36. A malleable iron shaft 20 ft. long and 6 ins. diameter is subjected to a moment which twists the ends through an angle of 2°; taking G, the coefficient of transverse elasticity, as 9,000,000, find f, the stress at the skin.

37. Find the diameter of an iron shaft which is to transmit 120 H.P. at 60 revolutions per minute, the safe working strength being 548 k./cm.^{2}. *Ans.* 11 cm.

38. A water-wheel of 20 H.P. makes 5 revolutions per minute; find the diameter suitable for the maleable iron shaft which transmits this force. If the shaft is 12 ft. long, what is the angle of torsion? ($f=9000$ lbs., $G=9{,}000{,}000$ lbs./sq. in.)

39. How many H.P. may be transmitted by a shaft of 150 mm. diameter at 30 revolutions per minute, the safe working strength being reduced by shocks to 400 k./cm.^{2}? *Ans.* 118.

40. A shaft 12 ft. long and 6 ins. diameter is subjected to a twisting moment of 16 ft.-tons, and the two ends are thus twisted through a certain angle; a second shaft of the same material, 16 ft. long and 9 ins. diameter, is twisted so that its angle of torsion is exactly the same as that of the first; find the twisting moment required to do this. *Ans.* $60\frac{3}{4}$ in.-tons.

41. Find the maximum stress developed in a shaft of 120 mm. diameter which transmits 200 H.P. at 50 revolutions per minute. *Ans.* 422 k./cm.^{2}.

42. If f_1 is the safe torsional working stress of a shaft, and f_2 is the safe working stress when the shaft acts as a beam, show that the torsional resistance of the shaft is to its bending resistance in the ratio of $2f_1$ to f_2.

43. A circular shaft is twisted beyond the limit of elasticity. If the equalization of stress is perfect, show that for a given maximum stress the twisting couple is greater in the ratio of 4 to 3 than it would be if the elasticity were perfect.

44. Show that the resilience of a twisted shaft is proportional to its weight.

Ans. Resilience $=\dfrac{f^2}{G}\dfrac{\text{volume}}{4}$.

45. If a round bar of any material is subjected to a twisting couple, show that its maximum resilience is two thirds the maximum resilience of the material.

46. A shaft moving with a surface velocity of 10 ft. per second transmits 1000 H.P. Find the diameter if the shear stress is not to exceed 10,000 lbs. per square inch. *Ans.* 3.74 ins.

47. A winding-drum 20 ft. in diameter is used to raise a load of 5 tons. If the driving-shaft were in pure torsion, find the diameter for a stress of 3 tons per square inch. *Ans.* 10.1 ins.

48. Find the thickness of a hollow shaft when (*a*) its rigidity, (*b*) its strength is 25% greater than that of a solid shaft of diameter *d* of the same length and weight.

49. Find the percentage of weight saved by using a hollow instead of a solid shaft.

Ans. If of equal *stiffness* $=\frac{200}{m^2+1}$.

If of equal *strength* $=100\left\{1-\sqrt[3]{\frac{m^2(m^2-1)}{(m^2+1)^2}}\right\}$,

m being the ratio of the external to the internal diameter of hollow shaft.

50. A solid and a hollow cylindrical shaft of equal length contain the same amount of the same kind of metal, the solid one fitting the hollow of the other. Compare their torsional strengths when used separately.

Ans. Strength of solid shaft = .471 × that of hollow shaft.

51. Find the diameter of a hollow shaft required to transmit 5000 H.P. at 70 revolutions per minute; stress, 7500 lbs. per square inch; the external diameter being twice the inner; maximum twisting moment = $1\frac{1}{2}$ times the mean.

52. A steel bar having a diameter of .410 in. and a length under test of 4 ins. gave the following results in the testing-machine:

Torque in In.-lbs.	Angle of Twist.	Torque in In.-lbs.	Angle of Twist.
150	$1° 1'$	450	$3° 5'$
300	$2° 3'$	600	$4° 6'$

Calculate the modulus of rigidity.

53. The halves of a flange coupling for a shaft transmitting 60 H.P. at 100 revolutions per minute are bolted together with six bolts at 6 ins. from the centre. Find the diameter of the bolts, the safe shear stress being 8000 lbs. per square inch. *Ans.* .41 in.

54. A steel shaft 4 ins. in diameter and weighing 490 lbs. per cubic foot makes 100 revolutions per minute. If the working stress in the metal is 11,200 lbs. per square inch, find the twisting couple and the distance to which the work can be transmitted, the coefficient of friction being .05 and the efficiency of the shaft $\frac{3}{4}$. *Ans.* 140,800 in.-lbs.; $8228\frac{4}{7}$ ft.

55. If the shaft is of steel, and if the loss due to friction is 20 per cent, find the distance to which work may be transmitted, μ being .05.

Ans. $6582\frac{6}{7}$ ft.

56. A wrought-iron shaft 220 ft. between bearings and 4 ins. in diameter can safely transmit 120 H.P. at the rate of 95 revolutions per minute. What is the efficiency of the shaft? ($\mu=\frac{1}{10}$.) *Ans.* .976.

57. The efficiency of a wrought-iron shaft is $\frac{1}{2}$; the working stress in the metal is 7200 lbs. per square inch; the coefficient of friction is .125. How far can the work be transmitted? *Ans.* 4320 ft.

58. The working shear stress of a shaft is 15,000 lbs.; how far can work be transmitted with an efficiency of .5, the coefficient of friction being .5? *Ans.* 24,545.45 ft.

59. Take a round shaft 3 ins. in diameter and find the sizes of equivalent shafts of square, elliptic, and rectangular sections if the breadth and thickness of each of these latter are as 1 to 2. If these shafts are 20 ft. long, and they are transmitting 20 H.P. at 100 revolutions per minute, what is the total twist of each of them? ($G=10{,}500{,}000$.)

Ans. 2.73 ins., 1.78; 2.38 ins., 1°.05; 2.15 ins.×4.3 ins., .93°.

60. A wrought-iron shaft 200 ft. in length and weighing 440 lbs. per cubic foot is supported on bearings, the coefficient of friction being .05. The shaft is subjected to a uniform twisting couple which developes a stress of 10,000 lbs. per square inch. Find the efficiency of the shaft. *Ans.* .994.

61. An iron shaft of 1 in. diameter is subjected to a turning effort of 500 ft.-lbs. The shaft is 1000 ft. long; find its efficiency in so far as it is affected by its weight.

62. A hollow cast-iron shaft of 12 ins. external diameter is twisted by a couple of 27,000 ft.-lbs. Find the proper thickness of the metal, so that the stress may not exceed 5000 lbs. per square inch. *Ans.* .308 in.

63. What twisting moment can be transmitted by a hollow steel shaft of 8 ins. internal and 10 ins. external diameter, the working stress being 5 tons per square inch? *Ans.* $579\frac{6}{7}$ in.-tons.

64. The inner and outer diameters of a hollow steel shaft are 10 and 12 ins., and f_s =6 tons per square inch, is the working value of the resistance to shearing. What is the twisting moment this shaft is capable of transmitting?

65. What thickness of metal is required for a cast-iron hollow shaft of 10 ins. outer diameter so as to resist a twisting moment of 10 ft.-tons?

66. A hollow shaft, the external and internal diameters of which are 20 ins. and 8 ins. respectively, runs at 70 revolutions per minute with a surface stress of 6000 lbs. per square inch. Find the twisting moment, and horsepower transmitted.

67. A solid shaft is 10 ins. in diameter, and the internal diameter of a hollow shaft is 5 ins., find the external diameter and compare the torsional strengths, the shafts being of the same weight and material.

Ans. $5\sqrt{5}$ ins.; $\sqrt{5}$ to 3.

68. A hollow steel shaft has an external diameter d and an internal diameter $\frac{d}{2}$. Compare its torsional strength with that of (*a*) a solid steel shaft

of diameter d; (*b*) a solid wrought-iron shaft of diameter d; the safe working stresses of steel and iron being 5 tons and $3\frac{1}{4}$ tons respectively.
Ans. (*a*) $\frac{15}{16}$; (*b*) $\frac{75}{82}$.

69. A solid wrought-iron shaft is to be replaced by a hollow steel shaft of the same diameter. If the material of the latter is 30 per cent stronger than that of the former, what must be the ratio of internal to external diameter? What is the percentage saving in weight? *Ans.* 1.44; 46%.

70. A square steel shaft is required for transmitting power to a 30-ton overhead travelling-crane. The load is lifted at the rate of 4 ft. per minute. Taking the mechanical efficiency of the crane gearing as 35 per cent, calculate the necessary size of shaft to run at 100 revolutions per minute. The twist must not exceed 1° in a length equal to 30 times the side of the square. ($G=13,000,000$.) *Ans.* 2 ins. square.

71. A round cast-iron shaft 15 ft. in length is acted upon by a weight of 2000 lbs. applied at the circumference of a wheel on the shaft; the diameter of the wheel is 2 ft. Find the diameter of the shaft so that the total angle of torsion may not exceed 2°. *Ans.* 3.76 ins.

72. A wrought-iron shaft is subjected to a twisting couple of 12,000 ft.-lbs.; the length of the shaft between the sections at which the power is received and given off is 30 ft.; the total admissible twist is 4°. Find the diameter of the shaft, μ (Art. 6) being $\frac{3}{2}$, and G 10,000,000 lbs. *Ans.* 5.8 ins.

73. Find the horse-power which may be transmitted by a shaft 4 ins. in diameter when running at 150 revolutions per minute, if the stress due to twisting be limited to 9000 lbs. per square inch. *Ans.* 273.

74. The working stress in a steel shaft subjected to a twisting couple of 1000 in.-tons is limited to 11,200 lbs. per square inch. Find its diameter; also find the diameter of the steel shaft which will transmit 5000 H.P. at 66 revolutions per minute, μ being $\frac{3}{2}$. *Ans.* 10 ins.; 6.88 ins.

75. A wrought-iron shaft is twisted by a couple of 10 ft.-tons. Find its diameter (*a*) if the torsion is not to exceed 1° per lineal foot, (*b*) if the safe working stress is 7200 lbs. per square inch. ($G=10,000,000$ lbs.)
Ans. (*a*) 3.7 ins.; (*b*) 5.7 ins.

76. A steel shaft 2 ins. in diameter makes 100 revolutions per minute and transmits 25 H.P. Find the maximum working stress and the torsion per lineal foot, G being 10,000,000 lbs. Also find the diameter of a shaft of the same material which will transmit 100 H.P. with the same maximum working stress. *Ans.* $10,022\frac{8}{11}$ lbs.; .0574°; 3.17 ins.

77. A steel shaft 300 ft. in length makes 200 revolutions per minute and transmits 10 H.P. Determine its diameter so that the greatest stress in the material may be the same as the stress at the circumference of an iron shaft 1 in. in diameter and transmitting 500 ft.-lbs. If 10 is a factor of safety, find the coefficient of torsional rupture. *Ans.* .807 in. ($=\frac{4}{5}$ in.); 60,000.

78. A round bar of steel is 1 in. in diameter and 8 ft. in length (or $l=48$ ins.). Take $F=1500$ lbs. Show that an endlong load only sufficient of itself to produce a stress of 1910 lbs. per square inch, and a bending moment which by itself would only produce a stress of 816 lbs. per square inch, if both act together, produce a stress of 23,190 lbs. per square inch.

79. A wooden shaft of d ins. diameter and l ins. length makes n revolutions per second and "whirls." If the weight of the wood is 36 lbs. per cubic foot, the modulus of elasticity being 2,000,000 lbs. per square inch, show that $l=274\sqrt{\frac{d}{n}}$.

80. A pulley is keyed truly to a shaft which rotates with an angular velocity ω. If, when rotation takes place, the shaft bends slightly, show that the couple on the shaft is equal to $\omega^2(A-B)\frac{dy}{dx}$, in which A and B are the moments of inertia of the pulley about axes through its centre of gravity perpendicular to its plane and perpendicular to the axis of the shaft respectively, and $\frac{dy}{dx}$ is the inclination of the plane of the pulley to a plane perpendicular to the original alignment of the shaft.

81. If a thin disk weighing 10 lbs. and of 10 ins. diameter rotates at 1000 revolutions per minute about an axis through its centre, and if, instead of being perpendicular to the shaft, it is out of truth by $\frac{1}{100}$ of its radius, find the couple on the shaft in inch-pounds.

82. A wrought-iron shaft is subjected to a twisting moment of 36,000 lb.-ins. and a bending moment of 18,000 lb.-ins.; find the diameter when the maximum shear stress is 8000 lbs. per square inch. Find also the twisting moment which alone would produce a shear stress of the same numerical value.

83. A screw propeller-shaft 10 ins. in diameter is subjected to a twisting moment of 35 ft.-tons, and to a bending moment of 10 ft.-tons, due to the weight of the shaft and the pitching of the ship. What is the maximum compressive stress if the thrust of the screw is 10 tons? *Ans.* 2.9 tons.

84. Find the diameter of a wrought-iron shaft to transmit 90 H.P. at 130 revolutions per minute. If there is a bending moment equal to the twisting moment, what ought to be the diameter? *Ans.* 3.54 ins; 4.75 ins.

85. A round bar $\frac{7}{8}$ in. in diameter and 36 ins. between the supports deflects .33 in. under a load of 90 lbs. in the middle and twists through an angle of .75° when subjected to a twisting moment of 1000 in.-lbs. on a 10-in. length. Find E and G.

86. A wrought-iron shaft 3 ins. in diameter and making 140 revolutions per minute is supported on wall-brackets 16 ft. apart. There is a pulley on the shaft midway between the bearings. If the resultant side pull due to the weight of the pulley and the pull of the belt is 210 lbs., what is the greatest horse-power the shaft will transmit with safety? Safe shear stress 7800 lbs. per square inch. *Ans.* .66.

87. A shaft 12 ins. in diameter transmitting a twisting moment of 100 ft.-tons, is also subject to a bending moment of 20 ft-tons. Find the maximum stress induced. *Ans.* 4.3 tons/sq. in.

88. If a round bar 1 in. in diameter and 40 ins. between supports deflects .0936 in. under a load of 100 lbs. in the middle, and twists through an angle of .037 radian when subjected to a twisting moment of 1000 in.-lbs. throughout its length of 40 ins. find E, G, and K.

Ans. 14,510; 5,510, and 13,125 lbs./sq. in.

89. A steel shaft carries a 5-ft. pulley midway between the supports and makes six revolutions per minute, the tangential force on the pulley being 500 lbs. Taking the coefficient of working strength at 11,200 lbs. per square inch, find the diameter of the shaft and the proper distance between the bearings the stiffness of the shaft being $\frac{1}{1200}$.

90. A counter-shaft 10 ft. between bearings carries two 24-in. pulleys, the one 1 ft. and the other 5 ft. from a bearing. Assuming that the tight is twice the slack tension, determine (*a*) the equivalent twisting moment on the shaft, (*b*) the diameter of the shaft, (*c*) the angle of torsion when one pulley receives and the other transmits 50 H.P. at 80 revolutions per minute, the belts being horizontal and on opposite sides of the shaft.

Ans. (*a*) 3281.25 ft.-lbs.;
(*b*) 7.13 ins., f being 10,000 lbs.;
(*c*) 2.06 minutes, G being 12,000,000 lbs.

91. Find the greatest torque for a bar 3 ins. in diameter, if the longitudinal extension of the material is to be limited to $\frac{1}{2000}$, the modulus of rigidity being 5000 lbs./sq. in.

92. Power is taken from a shaft by means of a pulley 24 ins. in diameter which is keyed on to the shaft at a point dividing the distance between two consecutive supports into segments of 40 and 60 ins.; the tangential force at the circumference of the pulley is 6600 lbs. If the shaft is of steel, determine its diameter, taking into account the bending action to which it is subjected, the working stress being 11,200 lbs. per sq. in. *Ans.* 5.65 ins.

93. A wrought-iron shaft is subjected simultaneously to a bending moment of 8000 in.-lbs., and to a twisting moment of 15,000 in.-lbs. Find the twisting-moment equivalent to these two and the least safe diameter of the shaft, the safe shear stress being taken at 8000 lbs. per square inch.

Ans. 25,000 in.-lbs.; 2.52 ins.

94. A shaft 9 ins. in diameter and 12 ft. long is supported at its two ends and loaded at the two points which divide its length into three equal parts with 4 tons at each point; a twisting moment of 20 ft.-tons is applied to one end of the shaft, while the other is held fixed. Find the greatest intensity of the thrust, tension, and shearing stress, and the angle that the line of greatest principal stress makes with the axis of the shaft.

Ans. 3.5 and 2.1 tons/sq. in.; 25°.

95. A steel tube of 10 ins. external diameter and 12 ft. long is supported horizontally at the ends. At a point 4 ft. from one end a bracket is fixed at right angles to the axis of the tube and supports at its end a weight of $3\frac{3}{4}$ tons, the distance between the centre of the tube and the weight being 24 ins. Find the thickness of the tube so that the stress may nowhere exceed 5 tons per square inch. *Ans.* .39 in.

96. An engine-crank is 12 ins. long and 9000 lbs. is the greatest force transmitted along the connecting-rod. If the wrought iron of the shaft will safely bear a shear stress of 9000 lbs., find the diameter of the shaft, the horizontal distance of the centre of the crank-pin from the centre of the nearest bearing being 10 ins. *Ans.* 4.76 ins.

97. A shaft 8 ins. in diameter is subjected to a thrust of 100 tons uniformly distributed over its two ends, and a twisting moment of 30 ft.-tons. Find the greatest intensity of thrust and shearing stress and the angle made by the line of principal stress with the axis of the shaft.

Ans. 4.71 and 3.71 tons/sq. in.; 37°.

98. Find the diameter of a malleable-iron shaft capable of bearing a tension of 50 tons, and a twisting couple whose moment is 25 ft.-tons, the resistance of the material to tension and shearing being 5 and 4 tons per square inch respectively. *Ans.* 7.28 ins.; the diameter for maximum shear =7.06 ins.

99. A steel shaft transmits a maximum twisting moment of 70,000 in.-lbs., and is at the same time subject to a bending moment of 25,000 in.-lbs. Find the necessary diameter of the shaft if the safe stress in the material is 4 tons per square inch.

100. A shaft upon bearings 40 ft. apart carries a pulley at a point 30 ft. from one bearing. The bending effect on the shaft at this point is equivalent to that produced by a weight of 75 lbs. at the same point when the shaft is subjected to a twisting moment of 1000 ft.-lbs. Find the proper diameter of the shaft, so that the stress may nowhere exceed 12,000 lbs. per square inch.

101. An engine-crank is 16 ins. long and the distance between the centres of the pin and the bearing is 20 ins. If the force on the crank-pin centre is 5000 lbs., find the maximum intensities of thrust (or tension) and shear and also the angle between the line of greatest principal stress and the axis of the shaft. *Ans.* 6530 and 4080 lbs./sq. in.; 27°.

102. A $3\frac{1}{2}$-in. steel shaft is subjected to a twisting couple of T ft.-lbs. and to a B.M. of $\frac{3}{4}$ T ft.-lbs. Find the value of T so that the maximum shear stress may not exceed 15,000 lbs. per inch. Also find the torsion of the shaft per lineal foot of length, G being 12,000,000 lbs. *Ans.* 31,582 in.-lbs.; $\frac{27^\circ}{55}$.

103. The maximum torque is $1\frac{1}{2}$ times as great as the mean torque and there is no bending. Find the maximum shear stress.

Ans. 7536 lbs., taking $d^3N = 64$ H.P.

104. The fly-wheel of a direct-driven generator has a radius of gyration of 9 ft., weighs 100,000 lbs., and runs at 94 revolutions per minute. The crank-shaft is 22 ins. diameter at the fly-wheel seat. Suppose an accident forcibly brings the shaft to rest from full speed in five revolutions, the retardation being uniform, find the stress on the shaft due to the inertia of the fly-wheel, acting along with a torque of 1,600,000 in.-lbs. due to the engine, and a bending moment of 3,000,000 in.-lbs. due to the weight of the fly-wheel and armature. *Ans.* 29,200 lbs.

105. The main shaft of a steamship transmits 1000 H.P. at 80 revolutions per minute, and is of mild steel $9\frac{1}{2}$ ins. in diameter. It is subjected to a longi-

tudinal thrust of 25,000 lbs. Find the resultant simple tensile or compressive stress due to the combined torsion and compression. *Ans.* 4858 lbs.

106. A street-railway generator gives 3000 B.H.P. at 75 revolutions per minute. The maximum bending moment on the crank-shaft is 13,500,000 in.-lbs. If the maximum torque is 1.5 times the mean torque, find the diameter of the shaft (solid). ($f_s = 5000$.) *Ans.* 14⅛ ins.

107. The steam stop-valve of a vertical engine is worked by means of a hand-wheel 14 ins. in diameter at the bottom end of a vertical rod or shaft (of circular section) 16 ft. long. If a couple of moment 1400 in.-lbs. is applied to the hand-wheel find the diameter of the shaft (*a*) if the relative twist of the ends of the shaft is not to exceed 2 degrees; (*b*) if the maximum shear stress in the shaft is not to exceed 9000 lbs. per square inch. (Take $G = 11,000,000$.) *Ans.* (*a*) 1⅝ ins.; (*b*) $\frac{15}{16}$ in.

108. The hollow vertical sleeve of a Weston centrifugal of 3½ ins. external and 2 ins. internal diameter makes 1200 revolutions per minute and carries at its lower end a weight of 1000 lbs. Show that the centrifugal effect is 9104 times that of the weight. (Take $E = 30,000,000$ lbs. per square inch.)

109. If the critical length of a shaft 13.4 ins. in diameter, subjected to endlong thrust alone, is equal to the critical length when subjected to centrifugal force alone, show that $P = 68,500\omega$; also show that if the length is 98 ft., the critical P is 327,600, and the critical ω is nearly 4.1 radians per second, or 46 revolutions per minute.

110. A wrought-iron propeller shaft has a diameter of 22½ ins., and the pitch of the screw is 35½ ft. The indicated horse-power at 53 revolutions per minute is 10,350, and assuming *theoretically* that the whole of this is utilized, show that the end thrust is 181,530 lbs. and that the twisting couple is 2,300,000 in.-lbs. If E is 29,000,000 lbs. per square inch, compare the twisting and end-thrust effects. Also find the length of the shaft. If a shaft of this length and diameter is subject to no end thrust, to no twisting moment, nor to its own weight, show that it will break by centrifugal force if it revolves at a greater speed than 1⅓ revolutions per minute. *Ans.* 1779; 4454 ins.

111. A 4-in. steel shaft ($E = 3 \times 10^7$ lbs. per square inch) 15 ft. long, with its ends supported but not constrained as to direction, is subjected to equal and opposite endlong forces (like a strut) each of 200 lbs.; taking into account its inertia but neglecting its mere weight, what is its critical speed of rotation?

112. Taking the proof stress to be 140,000 lbs. and G to be 13,000,000 lbs., find the axial proof load, the deflection, and the resilience of a 4-in. spring of coils 48 ins. in length and 1 in. in diameter.

113. A spring is formed of steel wire; the mean diameter of the coils is 1 in.; the working stress of the wire is 50,000 lbs. per square inch; the elongation under a weight of $19\frac{9}{14}$ lbs. is 2 ins.; the coefficient of rigidity is 12,000,000 lbs. Find the diameter of the wire and the number of coils.

Ans. .1 in.; 15.28.

114. Find the weight of a helical spring which is to bear a safe load of 6 tons with a deflection of 1 in., G being 12,000,000 lbs. and f 60,000 lbs.

115. Find the time of oscillation of a spring, the normal displacement under a given load being Δ.

Ans. $\pi\sqrt{\frac{\Delta}{g}}$.

117. Find the deflection under the weight W of a conical helical spring (a) of circular section; (b) of rectangular section, the radii of the extreme coils being y_1 and y_2, and the radial distance from the axis to a point of the spring at an angular distance ϕ from the commencement of the spiral being given by the relation $\frac{y_2-R}{y_2-y_1}=\frac{\phi}{2\pi n}$. ($n$ = number of coils.)

Ans. (a) $\frac{n(y_1+y_2)(y_1^2+y_2^2)W}{Gr^4}\left(=\frac{ny^3W}{Gr^4}\text{, if } y_1=0 \text{ and } y_2=y\right)$;

(b) $1.8\pi n(y_1+y_2)(y_1^2+y_2^2)\frac{b^2+h^2}{b^3h^3}\frac{W}{G}$; b and h being the sides of the rectangular section.

118. Find the modulus of rigidity (G) and the gravitation unit for a steel spring from the following observations:

Load in Scale-pan.	Scale Reading.	No. of Vibrations.	Time in Secs.
0	3.0	—	—
5	3.28	—	—
10	3.56	—	—
15	3.84	—	—
20	4.12	100	36
25	4.40		
30	4.68		

Diam. of wire = .257 in.; length of coil = 122.59 ins.; mean diam. of coil = 3 ins.; weight of pan, etc. = 2.72 lbs. *Ans.* 11,500,000 lbs.; 32.3.

119. Find the modulus of rigidity (G) from the following results of a torsion experiment:

Load in Scale-pan in Lbs.	Right-hand Scale Reading.	Left-hand Scale Reading.	Load in Scale-pan in Lbs.	Right-hand Scale Reading.	Left-hand Scale Reading.
5	7.0	8.0	20	7.58	8.34
10	7.26	8.18	25	7.72	8.40
15	7.44	8.28	30	7.85	8.45

Diam. = .706 in.; dist. between points = 8 ins.; load lever = 25 ins.; scale-arm = 22 ins.

120. A conical spring of round wire 40 ins. long and .2 in. in diameter has coils varying from 4 to 2 ins. in diameter. If the proof stress is 60,000 lbs. per square inch, find the proof load and the shortening due to the load.

Ans. 47 lbs.; .4 in.

121. Find the length and diameter of a round wire for a spring of 4 ins. diameter, which is to shorten .4 in. under an axial load of 1800 lbs. An axial load of 4800 lbs. is the proof load and develops a proof stress in the wire of 140,000 lbs. Also G = 13,000,000 lbs. per square inch.

CHAPTER X.

BRIDGES.

1. **Classification.**—Bridges may be divided into three general classes, viz.:

A. Bridges in which the platform is carried by trusses of different types in which the chords (flanges) are either horizontal or curved or are composed of sloping members.

B. Bridges in which the platform is suspended from cables passing over high piers.

C. Bridges in which the platform rests upon arched ribs.

In the present chapter it is proposed to deal with the bridges of Class A only.

Figs. 595–614 are skeleton diagrams of the various types of truss which are commonly employed in bridge construction. The maximum depth of a truss is governed, to some extent, by local conditions, but usually varies from one fifteenth to one seventh (and even more) of the span. Girders and trusses may require to be designed to meet conditions of a specified strength, or of a specified stiffness, or of both, depending essentially on the ratio of span to depth. If, for example, this ratio should exceed twelve, deflection becomes a serious consideration, and therefore stiffness is then a most important consideration. In ordinary practice it has often been the custom to limit the maximum depth of a truss to 1½ times the width of the bridge, so that the depth would then be not more than 24 ft. for a single and 40 ft. for a double-track bridge.

Position of Platform.—The platform may be supported either at the top or bottom flanges, or in some intermediate position. In favor of the last it is claimed that the main girders may be braced together below the platform (Fig. 615), while the upper portions serve as parapets or guards, and also that the vibration communi-

FINK

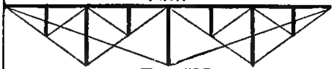


FIG. 595.

WHIPPLE

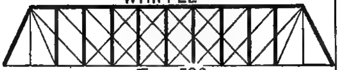


FIG. 596.

BOLLMAN

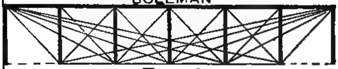


FIG. 597.

WARREN

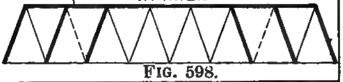


FIG. 598.

HOWE

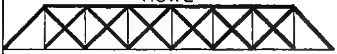


FIG. 599.

WARREN'

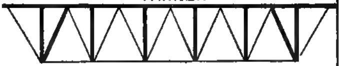


FIG. 600.

POST

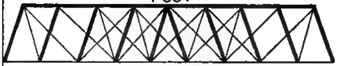


FIG. 601.

PRATT

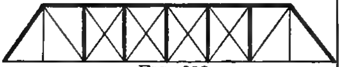


FIG. 602.

LENTICULAR

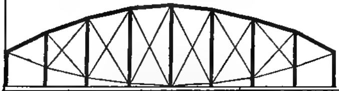


FIG. 603.

PETIT

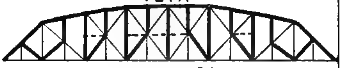


FIG. 604.

PARABOLIC

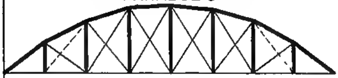


FIG. 605.

PEGRAM

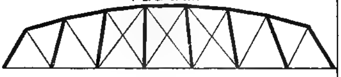


FIG. 606.

TRIANGULAR LATTICE

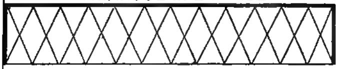


FIG. 607.

THE A

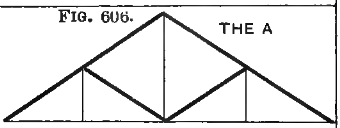


FIG. 608.

SCHWEDLER

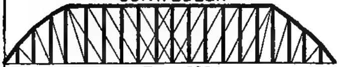


FIG. 609.

THE A

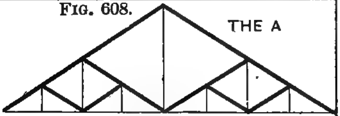


FIG. 610.

KELLOGG

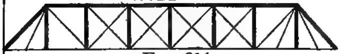


FIG. 611.

SIX PANEL RIGID

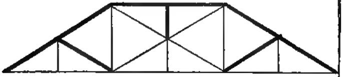


FIG. 612.

BALTIMORE

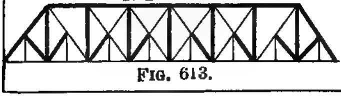


FIG. 613.

EIGHT PANEL RIGID

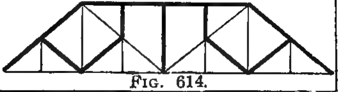


FIG. 614.

cated by a passing train is diminished. The position, however, is not conducive to rigidity, and a large amount of metal is required to form the connections.

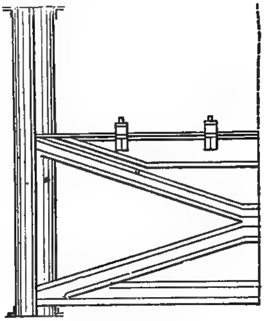


FIG. 615.

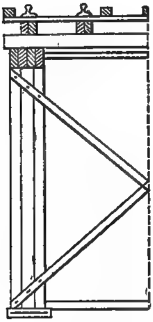


FIG. 616.

The method of supporting the platform on the top flanges (Fig. 616) renders the whole depth of the girder available for bracing, and is best adapted to girders of shallow depth. Heavy cross-girders may be entirely dispensed with in the case of a single-track bridge, and the load most effectively distributed, by laying the rails directly upon the flanges and vertically above the neutral line. Provision may be made for side spaces by employing sufficiently long cross-girders, or by means of short cantilevers fixed to the flanges, the advantage of the former arrangement being that it increases the resistance to lateral flexure and gives the platform more elasticity.

Figs. 617, 618, 619 show the cross-girders attached to the bottom flanges, and the desirability of this mode of support increases with the depth of the main girders, of which the centres of gravity should be as low as possible. If the cross-girders are suspended by hangers or bolts below the flanges (Fig. 619), the depth, and therefore the resistance to flexure, is increased.

In order to stiffen the main girders, braces and verticals, consisting of angle- or tee-iron, are introduced and connected with the cross-girders by gusset-pieces, etc.; also, for the same purpose, the cross-girders may be prolonged on each side, and the end joined to the top flanges by suitable bars.

When the depth of the main girders is more than about 5 ft.

the top flanges should be braced together. But the minimum clear headway over the rails is 16 ft., so that some other method should be adopted for the support of the platform when the depth of the main girders is more than 5 ft. and less than 16 ft.

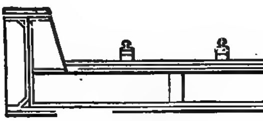


Fig. 617.

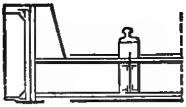


Fig. 618.

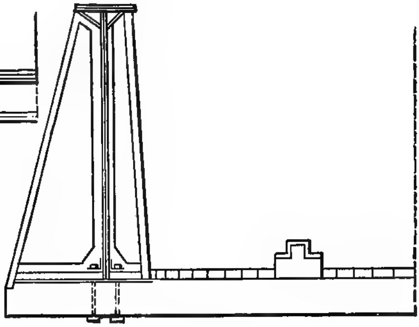


Fig. 619.

Assume that the depth of the platform below the flanges is 2 ft., and that the depth of the transverse bracing at the top is 1 ft.; the total limiting depths are 7 ft. and 19 ft., and if 1 to 8 is taken as a mean ratio of the depth to the span, the corresponding limiting spans are 56 and 152 ft.

Comparative Advantages of Two, Three, and Four Main Girders.—A bridge is generally constructed with two main girders, but if it is crossed by a double track a third is occasionally added, and sometimes each track is carried by two independent girders.

The employment of four independent girders possesses the one great advantage of facilitating the maintenance of the bridge, as one half may be closed for repairs without interrupting the traffic. On the other hand, the rails at the approaches must deviate from the main lines in order to enter the bridge, so that the width of the bridge is much increased, and far more material is required in its construction.

Few, if any, reasons can be urged in favor of the introduction of a third intermediate girder, since it presents all the objectionable features of the last system without any corresponding recommendation.

The two-girder system is to be preferred, as the rails, by such an arrangement, may be continued over the bridge without deviation at the approaches, and a large amount of material is economized, even taking into consideration the increased weight of long cross-girders.

The upper and lower chords of a bridge-truss are connected together by a web which may be *close*, i.e., may be made of plates butting the one against the other. or may be *open*, i.e., may be composed of a number of separate members in the form of verticals and diagonals. These verticals and diagonals intersect the chords in what are called panel-points, and the space between two such consecutive points is a panel. The whole of the members may be riveted together or may be connected together by means of suitably designed pins. The former method has the advantage of securing a much stiffer structure and of making the separate members interdependent and therefore also of distributing any weakness inherent in any particular member over other members with which it is rigidly connected. Thus, in the event of the failure of a tie or strut, the stress it was intended to carry is taken up by adjoining members and the bridge itself remains in working order. The safety and strength of a pin-connected bridge, on the other hand, depends upon the strength of each individual member. Its construction, however, is much simpler, and the determination of the stresses in the several members is much more definite and accurate. Although, for this purpose, it is necessary to assume that the pin-joints are frictionless, this assumption is much nearer the truth than the assumption that in a riveted structure the total shear at any vertical section is *equally* divided between all the members intersected by this section, which is equivalent to the substitution of a *mean stress* for the stresses in the several bars.

The tendency in recent American practice has been to extend the use of riveted bridges to spans as long as 200 ft. and even beyond. Above this limit, however, the connections become unduly large, if a single web system is used. Some riveted spans for heavy railway traffic have recently been built whose length is about 230 ft., but in these cases double web systems have been employed.

Dead Loads.—The dead load on a bridge consists of the entire weight of the trusses, floor, and track, less such parts as the pedestals, end floor-beams, and anchorage metal, the weight of which is borne directly by the piers, without causing any stresses in the structure as a whole. In the case of highway bridges in northern climates, a suitable allowance must be made for snow and ice. In the case of steel railway bridges the following formulæ give with sufficient accuracy the weight of structural steel. To this amount must be added from 350 to 450 lbs. per lineal foot of span for each track, to make allowance for the ties, rails, fastenings, and guard-rails.

Let W be the weight in tons (of 2000 lbs.) of one engine and tender, when the live load consists of two coupled locomotives followed by a uniform train load.

Let w = the total weight of metal in pounds per lineal foot of span;
" l = length of span in feet.

Then for deck plate-girder spans

$$w=\frac{500W+10Wl+2300l}{400-l}.$$

For half-through plate-girder spans

$$w=\frac{400W+1000l}{89-\frac{Wl}{1000}}.$$

For through riveted Pratt-truss spans

$$w=8.6(l+1.25W-112)+4.3\{388-(W+2l)\},$$

the last term being ignored if negative.

For through pin-connected Pratt-truss spans with parabolic top chords

$$w=8.63(l+1.3W-140).$$

For double-track bridges as above add 85 per cent..

For through pin-connected Petit-truss spans

$$w=10.9(l+1.1W-190).$$

For double-track bridges add the following percentages:

350 feet span	add 75 per cent.
400 " "	" 70 " "
450 " "	" 65 " "
500 " "	" 60 " "
550 " "	" 55 " "
600 " "	" 50 " "

In the case of highway bridges there is such a variety of types of flooring that general formulæ for the weights are of little use.

Live Loads.—For railway bridges the live load specified is the heaviest locomotive and train load that is likely to cross the structure, or some equivalent. As both the weight and the distribution of the loads for actual locomotives change greatly from time to time, a *typical* engine and train load is frequently adopted. The typical engine may not correspond exactly with any engine in use, but it is so chosen as to produce about the same stresses in all members as the actual engines. Convenience of calculation is also borne in mind, particularly in the wheel spacings, in which fractional numbers are avoided as far as possible. Among the systems of typical loads in general use are those of Cooper and Waddell. Cooper's Class E 40 load, as shown in Fig. 620, consists of two consolidation locomotives carrying 40,000 lbs. on each driving-axle, the engines being followed by a uniform train load of 4000 lbs. per lineal foot. Class E 30 would have 30,000 lbs. on each driving-axle and a train load of 3000 lbs. per lineal foot. For all classes the wheel spacing is the same, and the pilot and tender axle loads bear a constant ratio to the driving-axle loads, viz., 50 per cent in the former case and 65 per cent in the latter.

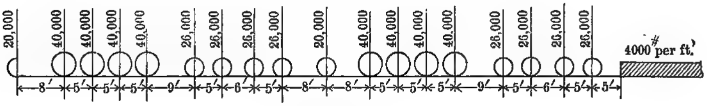


Fig. 620.

Fig. 621 shows Waddell's Class U.

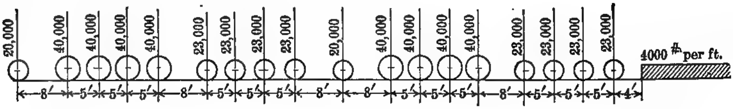


Fig. 621.

The wheel spacing is the same for all classes; and for each consecutive class above or below, the driving-axle load varies by steps of 3000 lbs., the pilot and tender axle loads by 1000 lbs., and the uniform train load by 200 lbs. per lineal foot.

When the live load has been selected, the maximum shears and bending moments caused by the actual wheel concentrations may be computed for any point in a span, as illustrated in a subsequent example. With a view to reducing the labor of computation, several conventional methods of treating the live load have been proposed. Of these the Equivalent Uniform Load method is, perhaps, most generally used. Evidently no single uniform load will produce the same shears and bending moments at all points in a structure as the actual wheel loads. But a close approximation may be arrived at by computing, in the case of plate girders, the uniform load which will cause the same bending moment at the centre of the span as the actual wheel concentrations, and by taking, in the case of trusses, the uniform load which will produce the same B.M. at the quarter-points, or by taking the average of the uniform loads which will produce the same bending moments at all the panel-points. As the span length increases, the equivalent uniform load grows smaller, since the heavy engine loads then extend over a smaller fraction of the span length.

The live load for highway bridges is taken from 40 to 120 lbs. per square foot, according to circumstances. For floor systems the concentrations caused by a road roller are sometimes specified. Loads arising from electric or other tram-cars may be treated by either of the methods indicated above for railway structures.

Ex. 1. *The live load for the Sault Ste. Marie Bridge, Fig. 623, is the loading from a standard consolidation engine with four drivers and one leading wheel, the weight concentrations being shown by Fig. 622.*

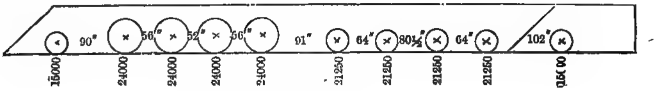


FIG. 622.

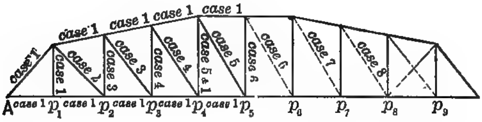


FIG. 623.

Span = 239 ft.

Length of centre verticals = 40 ft.; of end verticals = 27 ft.

For convenience of calculation assume the length of each panel to be 24 ft. (= 288 ins.). The error thus made is sufficiently small to be disregarded.

Five distributions may be considered, viz.:

When the front wheels are at a panel-point;
" " first drivers " " " "
" " second " " " " "
" " third " " " " "
" " fourth " " " " "

It may be easily shown that the stresses in the several members are greatest when either the *second* driver is at a panel-point or the *third* driver is at a panel-point, the corresponding panel loads in pounds for the whole truss being

11,900, 49,500, 38,700, 45,925, 43,750, 36,225, 36,000, 36,000, 36,000,

and

49,500, 38,700, 45,925, 43,750, 36,225, 36,000, 36,000, 36,000, 36,000.

These results may be obtained analytically or graphically.

Analytically.—For example let A, B, C, D, Fig. 624, be four consecutive panel-points, and let the *third* driver be at B. Then

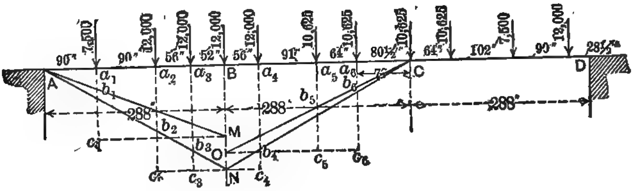


FIG. 624.

$$\text{Panel load at } A = 7500\frac{198}{288} + 12000\left(\frac{108+52}{288}\right) = 11823, \text{ say } 11{,}900 \text{ lbs.}$$

$$\text{Panel load at } B = 7500\frac{90}{288} + 12000\left(\frac{180+236+288+232}{288}\right) + 10625\ \frac{141+77}{288}$$

$$= 49387, \text{ say } 49{,}500 \text{ lbs.};$$

Panel load at C

$$= 12000\left(\frac{56+28\frac{1}{2}}{288}\right) + 10625\left(\frac{147+211+284\frac{1}{2}+220\frac{1}{2}}{288}\right) + 7500\frac{118\frac{1}{2}}{288}$$

$$= 38445, \text{ say } 38{,}700 \text{ lbs.}$$

etc. etc. etc.

Graphically. Upon the vertical through B (Fig. 624) take BM to represent 7500 lbs., and join AM. Let the vertical through a_1 meet AM in b_1, and the horizontal through M in c_1. Then a_1b_1 represents the portion of 7500 lbs. borne at B, and b_1c_1 the portion borne at A.

Also, take BN to represent 12,000 lbs.; join AN, CN. Let the verticals through a_2, a_3, a_4 meet AN, CN in b_2, b_3, b_4, and the horizontal through N in c_2, c_3, c_4. Then a_2b_2, a_3b_3, a_4b_4 represent the portions of each 12,000 lbs. borne at B, while b_2c_2, b_3c_3 represent the portions borne at A, and b_4c_4 the portion borne at C.

Finally take BO to represent 10,625 lbs., and join CO. Let the verticals through a_5, a_6 meet CO in b_5, b , and the horizontal through O in c_5, c_6. Then a_5b_5, a_6b_6 are the portions of each 10,625 lbs. borne at B, while b_5c_5, b_6c_6 are the portions borne at C. Thus the total weight at B

$$=a_1b_1+a_2b_2+a_3b_3+BN+a_4b_4+a_5b_5+a_6b_6,$$

etc.

The distributions of live load, *concentrated at the panel-points*, which will give the maximum stresses in the several members, may be tabulated as below:

Distributions.	End Reaction at A.	Load at p_1.	Load at p_2.	Load at p_3.	Load at p_4.	Load at p_5.	Load at p_6.	Load at p_7.	Load at p_8.	Load at p_9.
Case 1......	187990	49500	38700	45925	43750	36225	36000	36000	36000	36000
" 2......	162920	11900	49500	38700	45925	43750	36225	36000	36000	36000
" 3......	124230	.....	6400	47200	40200	43400	45800	37100	36000	36000
" 4......	95020	.....	.....	6400	47200	40200	43400	45800	37100	36000
" 5......	69410	.....	.....	.....	6400	47200	40200	43400	45800	37100
" 6......	47400	.....	.....	.....	.....	6400	47200	40200	43400	45800
" 7......	29100	.....	.....	.....	.....	.....	6400	47200	40200	43400
" 8......	15380	.....	.....	.....	.....	.....	.....	6400	47200	40200
Dead weight	121500	27000	27000	27000	27000	27000	27000	27000	27000	27000

The case giving the maximum stress in any member is indicated in Fig. 623.

2. **Stringers.**—Each length of stringer between consecutive floor-beams may be regarded as an independent girder resting upon supports at the ends, and should be designed to bear with safety the *absolute maximum* bending moment to which it may be subjected by the live load. If the beams are not too far apart, the absolute maximum bending moment will be at the centre when a driver is at that point. Again, in the case of the Sault Ste. Marie Bridge, Art. I, it may be easily shown that the maximum bending moment is produced when the four pairs of drivers are *between* the floor-beams.

Let y = distance of first driver from nearest point of support.

The reaction at this support

$$=\frac{12000}{288}(824-4y)=\frac{500}{3}(206-y).$$

The bending moment is evidently a maximum at the second or third driver. and at the second driver

$$=\frac{500}{3}(206-y)(56+y)-12000\times 56$$

at the third driver $=\dfrac{500}{3}(206-y)(108+y)-12000(52+108)$.

In the first case it is an *absolute maximum* when $y=75''$;
" " second " " " " " $y=49''$;
its value in each case being 2,188,166$\frac{2}{3}$ in.-lbs.

Hence the bending moment is an absolute maximum and equal to 2,188,166$\frac{2}{3}$ in.-lbs., at two points distant 75 ins. from each point of support.

Also, if I_1 is the moment of inertia of the section of the stringer at these points, c_1 the distance of the neutral axis from the outside skin, and f_1 the coefficient of strength, then

$$\tfrac{2}{3}(2188166\tfrac{2}{3}) = f_1\frac{I_1}{c_1} \text{ for the inner stringer,}$$

and

$$\tfrac{1}{3}(2188166\tfrac{2}{3}) = f_1\frac{I_1}{c_1} \text{ for the outer stringer.}$$

The continuity of the stringers adds considerably to their strength.

3. Camber.—Owing to the play at the joints, a girder or truss will deflect to a much greater extent than is indicated by theory, and the material will receive a permanent set, which, however, will not prove detrimental to the stability of the structure unless it is increased by subsequent loads. If the chords were initially made straight, they would curve downwards; and although it does not necessarily follow that the strength of the truss would be sensibly impaired, the appearance would not be pleasing.

In practice it is often specified that the girder or truss is to have such a camber or upward convexity that under ordinary loads the grade line will be true and straight; or, again, that a camber shall be given to the span by making the panel lengths of the top chord greater than those of the bottom chord by .125 in. for every 10 ft.

The lengths of the web members in a cambered truss are not the same as if the chords were horizontal, and must be carefully calculated so as to insure that the several parts will fit together.

To find an Approximate Value for the Camber, etc.

Let d be the depth of the truss.

Let s_1, s_2 be the lengths of the upper and lower chords respectively.

Let f_1, f_2 be the unit stresses in upper and lower chords respectively.

Let d_1, d_2 be the distances of the neutral axis from the upper and lower chords respectively.

Let R be the radius of curvature of the neutral axis.

Let l be the span of the truss.

Then

$$\frac{d_1}{R}=\frac{s_1-l}{l}=\frac{f_1}{E} \quad \text{and} \quad \frac{d_2}{R}=\frac{l-s_2}{l}=\frac{f_2}{E}, \text{ approximately,}$$

the chords being assumed to be circular arcs.

Hence the excess in length of the upper over the lower chord

$$=s_1-s_2=\frac{l}{E}(f_1+f_2)=l\frac{d}{R}.$$

Let x_1, x_2 be the cambers of the upper and lower chords respectively. $R+d_1$ and $R-d_2$ are the radii of the upper and lower chords respectively.

By similar triangles,

$$\left.\begin{array}{c}\text{the horizontal distance between}\\ \text{the ends of the upper chord}\end{array}\right\}=\frac{R+d_1}{R}l;$$

$$\left.\begin{array}{c}\text{the horizontal distance between}\\ \text{the ends of the lower chord}\end{array}\right\}=\frac{R-d_2}{R}l.$$

Hence $$\left(\frac{1}{2}\frac{R+d_1}{R}l\right)^2=x_1\times 2(R+d_1), \text{ approximately,}$$

and $$\left(\frac{1}{2}\frac{R-d_2}{R}l\right)^2=x_2\times 2(R-d_2), \text{ approximately.}$$

Therefore $$x_1=\frac{l^2}{8R}\left(1+\frac{d_1}{R}\right) \quad \text{and} \quad x_2=\frac{l^2}{8R}\left(1-\frac{d_2}{R}.\right)$$

4. Rivet-connection between Flanges and Web.—The web is generally riveted to angle-irons forming part of the flanges.

The *increment* of the flange stress transmitted through the web from point to point tends to make the angle-irons slide over the flange surfaces.

Denote the increment by F, and let h be the effective depth of the girder or truss.

Then, if S be the shearing force at any point,

Fh = the increment of the bending moment per unit of length

$$=\left(\frac{dM}{dx}\right)=S \text{ in the case of a close web,}$$

and Fh = the increment of the bending moment

$$=(\Delta M)=Sa \text{ in the case of an open web;}$$

a being the distance between the two consecutive apices or panel-points within which S lies.

Hence, if N be the number of rivets *per unit* of length for the close web, or the number between the two consecutive apices for the open web,

$$N\frac{\pi d^2}{4}f_s=F=\frac{S}{h} \text{ for the close web,}$$

and

$$=\frac{Sa}{h} \text{ for the open web,}$$

d being the diameter of a rivet, and f_s the safe coefficient of shearing strength.

5. Eye-bars and Pins.—Eye-bars connected with pins have been commonly employed in the construction of suspension cables,

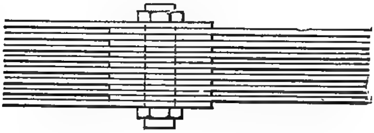


Fig. 625.

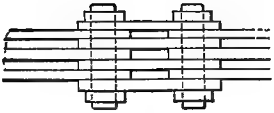


Fig. 626.

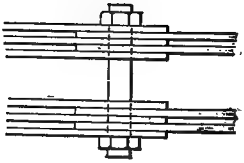


Fig. 627.

the tension chords of ordinary trusses and cantilevers, and the diagonals of web systems. The requisite sectional area is obtained by placing a number of bars side by side on the same pin, and, if necessary, by setting two or more tiers of bars one above another.

The figures represent groups of eye-bars as they often occur in practice.

If two sets of $2n$ bars pull upon the pin in opposite directions, as in Figs. 626 and 627, the bending moment on the pin will be nPp, P being the pull upon each bar, and p the distance between the centre lines of two consecutive bars. Hence

$$nPp = \frac{f}{c}I,$$

f being the stress in the material of the pin at a distance c from the neutral axis, and I the moment of inertia.

In general, the bending action upon a pin connecting a number of vertical, horizontal, and inclined bars may be determined as follows:

Consider one half of the pin only.

Let V, Fig. 626, be the resultant stress in the vertical bars. It is necessarily equal in magnitude but opposite in direction to the vertical component of the resultant of the stresses in the inclined bars. Let v be the distance between the lines of action of these two resultants. The corresponding bending action upon the pin is that due to a couple of which the moment is Vv.

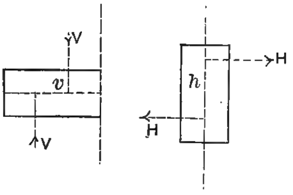


FIG. 628.

Let h be the distance between the lines of action of the equal resultants H of the horizontal stresses upon each side of the pin. The corresponding bending action upon the pin is that due to a couple of which the moment is Hh.

Hence the maximum bending action is that due to a couple of which the moment is the resultant of the two moments Vv and Hh, viz.,

$$\sqrt{(Vv)^2 + (Hh)^2}.$$

6. Determination of Stresses.—Stresses are developed in the several members of a bridge-truss by (*a*) The *dead load*, i.e., the weight of the bridge-trusses and platform;

(*b*) The *live load*, i.e., the weight of a passing train and also the pressure of the wind;

(*c*) Changes of temperature.

For the present it will be assumed that the dead and live loads are uniformly distributed and are equivalent to d and l respectively at each panel-point.

In any panel, if a shear (s) developes a stress (d) in a sloping member inclined at θ to the vertical, then evidently

$$d \cos \theta = s, \quad \text{or} \quad d = s \sec \theta.$$

Also, the corresponding stress induced in a horizontal chord

$$= d \sin \theta = s \tan \theta.$$

Again, in the case of a riveted bridge-truss with horizontal chords it is assumed that the total shear in any panel is divided equally between all the members intersected by a vertical section in that panel, which is equivalent to the assumption that the mean stress, in each sloping member of the panel in question, is the same.

Take the length of panel to be p.

Ex. 2. *A through bridge-truss of the Warren type, of nine panels (in which the sloping members are inclined at* 30° *to the vertical, so that the truss is made up of equilateral triangles). In such a truss it is evident that the maximum stresses are tensions in the members sloping down towards the centre and compressions in the members which slope up towards the centre.*

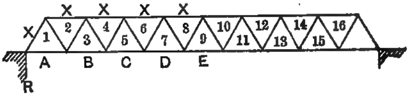


FIG. 629.

A. Dead-load stresses in sloping members.

Let R be the reaction at the left support. Then

$$R \times 9p = 8d \times 4\tfrac{1}{2}p, \quad \text{or} \quad R = \frac{d}{9}36.$$

Hence the shears in the 1st, 2d, 3d, 4th, etc., panels are $R-d$, $R-2d$, $R-3d$, etc., respectively; i.e., $\frac{d}{9}36$, $\frac{d}{9}27$, $\frac{d}{9}18$, $\frac{d}{9}9$, etc.

Thus the shear in any panel is the product of a constant quantity $\frac{d}{9}$ $\left(=\frac{\text{panel dead load}}{\text{no. of panels}}\right)$ and a multiplier, which is 36 in the first panel, and which in each succeeding panel is diminished by the total number of panels.

The corresponding stress in the sloping member in any panel is the shear in that panel × sec θ, θ in the present case being 30°, i.e., is the product of a constant quantity $\frac{d}{9}\sec\theta$ $\left(=\frac{\text{panel dead load}}{\text{no. of panels}}\times\sec\theta\right)$ and a multiplier which is 36 in the 1st panel, and which in each succeeding panel is diminished by the number of panels viz., 9. These *diagonal* stresses may be tabulated as follows:

Col. I. Member.	Col. II. Multiplier.	Col. III. $\frac{d}{9}\sec\theta=D$.	Col. IV. Multiplier.	Col. V. $\frac{l}{9}\sec\theta=L$.	Col. VI Total Maximum Diagonal Stress.
X, 1 = 1, 2	36	$36D$	$8.4\frac{1}{2}=36$	$36L$	$36D+36L$
2, 3 = 3, 4	27	$27D$	$7.4=28$	$28L$	$27D+28L$
4, 5 = 5, 6	18	$18D$	$6.3\frac{1}{2}=21$	$21L$	$18D+21L$
6, 7 = 7 8	9	$9D$	$5.3=15$	$15L$	$9D+15L$
8, 9 = 9,10	0	0	$4.2\frac{1}{2}=10$	$10L$	$10L$
10,11 = 11,12	−9	$-9D$	$3.2=6$	$6L$	$-9D+6L$
12,13 = 13,14	−18	$-18D$	$2.1\frac{1}{2}=3$	$3L$	$-18D+3L$
14,15 = 15,16	−27	$-27D$	$1.1=1$	L	$-27D+L$

The first column indicates the various sloping members. Column 3 gives the stresses in the several members due to the dead loads, and these are the product of a constant quantity $\frac{d}{9}\sec\theta$ at the head of Column 3, by the corresponding multiplier in Column 2.

B. Live-load Stresses in the Sloping Members.—The maximum live-load stresses of the *same kind* as those due to the dead load are produced when the greater segment of the truss on one side of any given panel is loaded. Thus the maximum live-load stresses in the sloping members of the first panel are due to the concentration of l at each of the panel-points from the first to the eighth; in the second panel, from the second to the eighth; in the third panel, from the third to the eighth; etc. When all the panel-points are loaded the reaction at the left support $=\frac{l}{9}\times8\times4\frac{1}{2}$. For each succeeding panel one load leaves the truss and the centre of gravity of the remaining loads move one half panel towards the right. Thus the reactions at the left support are $\frac{l}{9}\times7\times4$; $\frac{l}{9}\times6\times3\frac{1}{2}$; $\frac{l}{9}\times5\times3$; $\frac{l}{9}\times4\times2\frac{1}{2}$; $\frac{l}{9}\times3\times2$; $\frac{l}{9}\times2\times1\frac{1}{2}$; $\frac{l}{9}\times1\times1$. These are the shears which develope the maximum live-load stresses in the sloping members of the 1st, 2d, 3d, etc., panels respectively.

Until the middle of the truss is reached the stresses are evidently of the same kind as those due to the dead load, but as soon as the centre is passed the live load covers less than one half of the bridge and the stresses developed immediately in front of the load are of the opposite kind to those due to the dead load. As long, however, as these stresses are less than those due to the dead load they will not affect the design of the bridge, but if they are greater than those due to the dead load, the members must be strengthened so as safely to bear stresses of the opposite kind, or else additional members, called counter-braces, must be introduced to take up the stresses in question.

As before, the stresses in the sloping members are the product of a constant quantity $\frac{l}{9}\times\sec\theta$ and a multiplier which is 8.4½ in the first panel, and 7.4, 6.3½, 5.3, etc., in the 2d, 3d, 4th, etc., panels.

These results are tabulated in the above table. Thus the total maximum stress in any given sloping member is equal to the algebraic sum of the two corresponding stresses in Cols. 3 and 5. If the total maximum stresses in the 6th, 7th, and 8th panels are negative, no strengthening or counter-braces are required, as the stresses are less in amount but of the *same* kind as those for which the ordinary bracing is designed. If, however, any of these stresses are *positive*, then counter-braces must be introduced.

C. Chord Stresses.—The maximum stresses in a chord occur when the live load covers the whole bridge, and in the present case this is equivalent to assuming that a load $d+l$ is concentrated at each panel-point. Take $P=d+l$. Then

the shears are $\frac{P}{9}36$, $\frac{P}{9}27$, $\frac{P}{9}18$, $\frac{P}{9}9$, $\frac{P}{9}0$, in the 1st, 2d, 3d 4th, and 5th panels respectively.

Therefore $A1=\frac{P}{9}\tan\theta\times36;$

$$\begin{aligned}B3&=A1+\text{stress transmitted through 2d diagonal}\\&\quad+\text{\quad ``\quad\quad ``\quad\quad ``\quad 3d\quad ``}\\&=A1+\frac{P}{9}\tan\theta(36+27);\end{aligned}$$

$$\begin{aligned}C5&=B3+\text{stress transmitted through 4th diagonal}\\&\quad+\text{\quad ``\quad\quad ``\quad\quad ``\quad 5th\quad ``}\\&=B3+\frac{P}{9}\tan\theta(27+18).\end{aligned}$$

Similarly, $D7=C5+\frac{P}{9}\tan\theta(18+9)$, and $E9=D7+\frac{P}{9}\tan\theta(9+0)$.

So again, $X2=\frac{P}{9}\tan\theta(36+36)$, $X4=X2+\frac{P}{9}\tan\theta(27+27)$,

$X6=X4+\frac{P}{9}\tan\theta(18+18)$, $X8=X6+\frac{P}{9}\tan\theta(9+9)$.

These results may be tabulated as follows:

Col. I. Member.	Col. II. Multiplier.	Col. III. $\frac{P}{9}\tan\theta = F$.	Col. IV. Total Maximum Chord Stress.
$A1$	36	$36F$	$36F$
$B3$	$36+27=63$	$63F$	$99F$
$C5$	$27+18=45$	$45F$	$144F$
$D7$	$18+9=27$	$27F$	$171F$
$E9$	$9+0=9$	$9F$	$180F$
$X2$	$36+36=72$	$72F$	$72F$
$X4$	$27+27=54$	$54F$	$126F$
$X6$	$18+18=36$	$36F$	$162F$
$X8$	$9+9=18$	$18F$	$180F$

Col. III gives the stress transmitted to the chord through the sloping members and is the product of the constant quantity $\frac{P}{9}\tan\theta$ and the multiplier in Col. II. The total maximum chord stress is given by Col. IV and is obtained by adding to the stress in the preceding panel the stresses transmitted through the sloping members at a panel-point.

Ex. 3. *An eight-panel deck of the Pratt type, with web members sloping at θ to the vertical. These members are designed to be in tension, the verticals being in compression.*

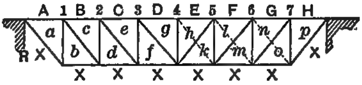


FIG. 630.

A. Dead-load Stresses. Let R be the reaction at the left support. Then

$$R8p = 7d\cdot 4p, \quad \text{or} \quad R = \frac{d}{8}28.$$

Thus the dead-load shears in the 1st, 2d, 3d, etc., panels are

$$R-d, \quad R-2d, \quad R-3d, \quad \ldots, \text{ etc., i.e., } \frac{d}{8}28, \quad \frac{d}{8}20, \quad \frac{d}{8}12, \quad \ldots, \text{ etc.,}$$

the stresses in the corresponding diagonals being

$$\frac{d}{8}\sec\theta\times 28, \quad \frac{d}{8}\sec\theta\times 20, \quad \frac{d}{8}\sec\theta\times 12, \quad \ldots, \quad \text{etc.}$$

Hence the dead-load stress in a sloping member is the product of a constant quantity $\frac{d}{8}\sec\theta\left(=\frac{\text{panel dead load}}{\text{no. of panels}}\times\sec\theta\right)$ and a factor which is 28 for the first panel and is diminished by the number of panels, viz., 8 for each succeeding panel. These stresses may therefore at once be tabulated as follows:

Col. I. Member.	Col. II. Multiplier.	Col. III. $\frac{d}{8}\sec\theta=D$.	Col. IV. Multiplier.	Col. V. $\frac{l}{8}\sec\theta=L$.	Col. VI. Total Maximum Diagonal Stress.
Xa	28	$28D$	$7\times4\ =28$	$28L$	$28D+28L$
bc	20	$20D$	$6\times3\frac{1}{2}=21$	$21L$	$20D+21L$
de	12	$12D$	$5\times3\ =15$	$15L$	$12D+15L$
fg	4	$4D$	$4\times2\frac{1}{2}=10$	$10L$	$4D+10L$
hk	-4	$-4D$	$3\times2\ =\ 6$	$6L$	$-4D+6L$
lm	-12	$-12D$	$2\times1\frac{1}{2}=\ 3$	$3L$	$-12D+3L$
no	-20	$-20D$	$1\times1\ =\ 1$	L	$-20D+L$

Col. I indicates the member; Col. III gives the stress in the member, and is the product of the constant quantity at the head of Col. III by the corresponding multiplier in Col. II.

B. Live-load stresses. The maximum live-load shear in any panel, of the *same kind* as that due to the dead-load shear, occurs when the live load covers the greater segment of the bridge on one side of the panel in question. These maximum live-load shears for the 1st, 2d, 3d, and 4th panels occur when L is concentrated at the panel-points from 1 to 7, 2 to 7, 3 to 7, 4 to 7, respectively.

On passing the centre of the bridge, that is, when the live load covers less than one half of the bridge, the live-load shears are of the *opposite kind* to those due to the dead load and therefore develope stresses of the opposite kind to those in the sloping members, for which provision must therefore be made either by strengthening these members or by introducing counter-braces as shown by the dotted lines.

There is a different end reaction for the maximum live-load shear in each panel, and if R_1, R_2, R_3, etc., are the end reactions when l is concentrated at the panel-points 1 to 7, 2 to 7, 3 to 7, etc., respectively, then

$$R_1\cdot 8p=7l\cdot 4p, \quad \text{or} \quad R_1=\frac{l}{8}7\times4;$$

$$R_2\cdot 8p=6l\cdot 3\tfrac{1}{2}p, \quad \text{or} \quad R_2=\frac{l}{8}6\times3\tfrac{1}{2};$$

$$R_3\cdot 8p=5l\cdot 3p, \quad \text{or} \quad R_3=\frac{l}{8}5\times3;$$

$$\text{etc.} \qquad\qquad \text{etc.}$$

Thus the maximum live-load shears in the 1st, 2d, 3d, etc., panels are R_1, R_2, R_3, etc., and the maximum live-load stresses in the corresponding sloping members are $R_1\sec\theta$, $R_2\sec\theta$, $R_3\sec\theta$, etc., respectively, or

$$\frac{l}{8}\sec\theta\times7\times4, \quad \frac{l}{8}\sec\theta\times6\times3\tfrac{1}{2}, \quad \frac{l}{8}\sec\theta\times5\times3, \quad \ldots, \quad \text{etc.}$$

Hence the maximum live-load stresses in the 1st, 2d, 3d, etc., panels are the product of a constant quantity $\frac{l}{8}\sec\theta\left(=\frac{\text{panel live load}}{\text{number of panels}}\times\sec\theta\right)$ and a multiplier composed of two factors, one of which is the number of loads on the truss, while the other is the distance of the centre of gravity of these loads in number of panels from the right support.

These results can be at once tabulated as already shown, Col. V giving the maximum live-load stresses in the sloping members, and these stresses are obtained by multiplying the constant quantity at the head of Col. V by the corresponding multiplier in Col. IV.

It will be noted that for each succeeding panel one live load leaves the truss, so that the centre of gravity of the remaining live loads moves one half panel nearer to the right support.

C. Dead-load Stresses in Verticals. The dead-load stresses in the 1st, 2d, 3d, etc., verticals are evidently the dead-load shears in the 1st, 2d, 3d, 4th, etc., panels, respectively, these shears being transmitted through the sloping members Xa, bc, de, fg, etc. The values of these stresses, therefore, which are evidently *compressions*, are $\frac{d}{8}28$, $\frac{d}{8}20$, $\frac{d}{8}12$, $\frac{d}{8}4$, . . . , etc., and the compression on the middle vertical is d, the dead weight concentrated at its head.

The maximum live-load stresses in the same verticals evidently occur when l is concentrated at the panel-points 1 to 7, 2 to 7, 3 to 7, 4 to 7, etc., the values being $\frac{l}{8}7\times4$, $\frac{l}{8}6\times3\frac{1}{2}$ $\frac{l}{8}5\times3$, $\frac{l}{8}4\times2\frac{1}{2}$ etc., respectively.

The max. live-load stress upon the central vertical is a compression due to the weight l concentrated at its head. The total maximum stress in any vertical, say ef, is evidently the sum of the corresponding dead- and live-load stresses, that is, $\frac{d}{8}12+\frac{l}{8}4\times2\frac{1}{2}$.

When l is concentrated at 6 and 7 the corresponding stress in kl is a tension and $=\frac{l}{8}2\times1\frac{1}{2}$, and the tension in mn due to l concentrated at 7 is $\frac{l}{8}$.

Hence the total resultant stress in kl $=-\frac{d}{8}12+\frac{l}{8}3$

and " " " " " $mn=-\frac{d}{8}20+\frac{l}{8}$

Chord Stresses.—The stresses in the chords are greatest when the live load covers the whole bridge, so that there is a panel load of $d+l$ $(=P)$ at each panel-point. Then, remembering that the stress in the chord is due to the shear transmitted through the sloping members and is equal to the product of this shear by $\tan\theta$, and also remembering that the shears in the 1st, 2d, 3d, and 4th panels, etc., are

$$\frac{P}{8}28,\quad \frac{P}{8}20,\quad \frac{P}{8}12,\quad \frac{P}{8}4,\quad \ldots,\quad \text{etc., respectively}$$

we have

$$Aa=\frac{P}{8}\tan\theta\times 28,\quad BC=Aa+\frac{P}{8}\tan\theta\times 20,$$

$$Ce=Bc+\frac{P}{8}\tan\theta\times 12,\quad Dg=Ce+\frac{P}{8}\tan\theta\times 4,$$

and $$Xb=\frac{P}{8}\tan\theta\times 28,\quad Xd=Xb+\frac{P}{8}\tan\theta\times 20,\quad Xf=Xd+\frac{P}{8}\tan\theta\times 12,$$

These results may be tabulated as follows:

Col. I. Member.	Col. II. Multiplier.	Co. III. $\frac{P}{8}\tan\theta=F.$	Col. IV. Total Maximum Chord Stress.
Aa	28	$28F$	$28F$
Bc	20	$20F$	$48F$
Ce	12	$12F$	$60F$
Dg	4	$4F$	$64F$
Xb	28	$28F$	$28F$
Xd	20	$20F$	$48F$
Xf	12	$12F$	$60F$

Note.—If this truss is inverted, it becomes a truss of the Howe type, the sloping members being now in compression and the verticals in tension. The magnitude of the stresses remains the same as above.

Graphical Method.—The stresses obtained in the above tables may be determined in a very simple manner graphically. Fig. 631 gives the stress diagram for the dead loads on the truss, in which XA, the reaction at the left support, $=3\frac{1}{2}d$, and $AB=BC=CD=$etc., $=d$, the dead load concentrated at each panel-point.

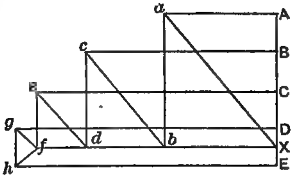


FIG. 631.

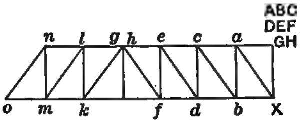


FIG. 632.

From the same diagram can be obtained the maximum live-load stresses in the chords by multiplying the corresponding dead-load stresses by the ratio $\frac{l}{d}$, as the stresses are greatest when the live load covers the whole bridge.

Again, it has already been shown that there is a different end reaction corresponding to the maximum stress in each diagonal. Suppose that the only force acting upon the truss is a vertical reaction of 1000 units at the left support, Fig. 632 is the corresponding stress dagram and it shows that

(*a*) the stress in each diagonal *due to the assumed reaction* $=1000 \sec\theta$;
(*b*) " " " " vertical " " " " " $=1000$.

Now the actual reaction at the left support is

$\frac{l}{8}7\times4$ when l is at panel-points 1 to 7;
$\frac{l}{8}6\times3\frac{1}{2}$ " " " " 2 to 7;
$\frac{l}{8}5\times3$ " " " " 3 to 7;
$\frac{l}{8}4\times2\frac{1}{2}$ " " " " 4 to 7;
$\frac{l}{8}3\times2$ " " " " 5 to 7;
$\frac{l}{8}2\times1\frac{1}{2}$ " " " " 6 to 7;
$\frac{l}{8}$ " " " panel-point 7.

It is also evident that

$$\frac{\text{the actual stress in a member}}{\text{the stress due to assumed reaction}}=\frac{\text{actual reaction}}{\text{assumed reaction}}.$$

Hence

the maximum stress in $Xa=\frac{1000 \sec\theta}{1000}\frac{l}{8}28=\frac{l}{8}\sec\theta\times28$,

" " " " $bc=\frac{1000 \sec\theta}{1000}\frac{l}{8}21=\frac{l}{8}\sec\theta\times21$,

" " " " $de=\frac{1000 \sec\theta}{1000}\frac{l}{8}15=\frac{l}{8}\sec\theta\times15$,

etc., etc.

" " " " $ab=\frac{1000}{1000}\frac{l}{8}28=\frac{l}{8}28$,

" " " " $cd=\frac{1000}{1000}\frac{l}{8}20=\frac{l}{8}20$,

" " " " $ef=\frac{1000}{1000}\frac{l}{8}12=\frac{l}{8}12$,

etc., etc.

Generally speaking, in the case of trusses with horizontal chords, unless the panels are of unequal length or unless the bridges are skew or are otherwise specially designed, the stresses are more easily and rapidly obtained by tabulating the results as previously described.

Ex. 4. *A ten-panel double-intersection lattice (trellis) deck-bridge, with members sloping at θ to the vertical.*

First, assume that the members are riveted together and therefore that in any panel the shear is equally divided between the *two* diagonals met by a vertical section.

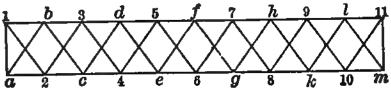


FIG. 633.

A. Dead-load Stresses in Web. If R is the reaction at the left support,

$$R\cdot 10p = 9d\cdot 5p, \quad \text{or} \quad R = \frac{d}{10}45,$$

and the dead-load shears in the 1st, 2d, 3d, . . . , panels are

$$R,\ R_1-d_1,\quad R_1-2d,\quad R_1-3d,\quad \ldots, \text{ respectively, or}$$

$$\frac{d}{10}45,\quad \frac{d}{10}35,\quad \frac{d}{10}25,\quad \ldots,$$

the corresponding stresses in *each* of the diagonals in the 1st, 2d, 3d, . . . , panels being

$$\frac{1}{2}\left(\frac{d}{10}\sec\theta\times 45\right),\quad \frac{1}{2}\left(\frac{d}{10}\sec\theta\times 35\right),\quad \frac{1}{2}\left(\frac{d}{10}\sec\theta\times 25\right),\quad \ldots$$

Thus the dead-load stress in a diagonal is the product of a constant quantity $\frac{1}{2}\frac{d}{10}\sec\theta\left(=\frac{1}{2}\frac{\text{panel dead load}}{\text{number of panels}}\times\sec\theta\right)$ and a multiplier which is 45 in the 1st panel and is diminished by the number of panels in each succeeding panel.

These results may be tabulated as follows:

Col. I Member	Col II Multiplier	Col. III. $\frac{1}{2}\frac{d}{10}\sec\theta = D.$	Col. IV. Multiplier	Col. V $\frac{2}{1}\frac{l}{10}\sec\theta = L.$	Col VI. Total Maximum Diagonal Stress.
$12 = ab$	45	$45D$	$9\ 5 = 45$	$45L$	$45D + 45L$
$23 = bc$	35	$35D$	$8.4\frac{1}{2} = 36$	$36L$	$35D + 36L$
$34 = cd$	25	$25D$	$7.4 = 28$	$28L$	$25D + 28L$
$45 = de$	15	$15D$	$6.3\frac{1}{2} = 21$	$21L$	$15D + 21L$
$56 = ef$	5	$5D$	$5.3 = 15$	$15L$	$5D + 15L$
$67 = fg$	-5	$-5D$	$4.2\frac{1}{2} = 10$	$10L$	$-5D + 10L$
$78 = gh$	-15	$-15D$	$3.2 = 6$	$6L$	$-15D + 6L$
$89 = hk$	-25	$-25D$	$2.1\frac{1}{2} = 3$	$3L$	$-25D + 3L$
$9\ 10 = kl$	-35	$-35D$	$1.1 = 1$	L	$-35D + L$

Col. I indicates the member; Col. III is the dead-load stress on the member and is obtained by multiplying the constant quantity at the head of Col. III by the corresponding number in Col. IV.

It may be noted that the stresses in the inclined members in any panel are necessarily of the same magnitude but are opposite in character, being tensions when the members slope in one direction and compressions when they slope in the opposite direction. It may also be noted that the riveting divides the compression members into two equal lengths, so that the ratio of the length of the strut to its least radius of gyration is diminished one half and the rigidity is therefore increased (Art. 7, Chap. VIII).

B. Live-load Stresses in Web. The maximum live-load shear in a panel occurs when the live load covers the greater segment of the bridge on one side of the panel. Thus for the maximum stresses in the diagonals in a panel there is a separate end reaction equal to the maximum live-load shear in the panel in question.

These reactions are evidently equal to the live-load shears just in front of the live load.

Let R_1, R_2, R_3, R_4, etc., be the end reaction when the live load l is concentrated on each panel-point from b to l, 3 to l, d to l, 5 to l, and f to l, respectively. Then

$$R_1 \cdot 10p = 9l \cdot 5p, \quad \text{or} \quad R_1 = \frac{l}{10} 9 \times 5;$$

$$R_2 \cdot 10p = 8l \cdot 4\tfrac{1}{2}p, \quad \text{or} \quad R_2 = \frac{l}{10} 8 \times 4\tfrac{1}{2};$$

$$R_3 \cdot 10p = 7l \cdot 4p, \quad \text{or} \quad R_3 = \frac{l}{10} 7 \times 4;$$

$$\text{etc.} \qquad \qquad \text{etc.}$$

Again, on passing the middle of the bridge, that is, when the live load covers a smaller segment of the bridge, the live-load shears are of the opposite kind to those due to the dead load and therefore develope stresses in the sloping members of an opposite kind to those due to the dead load.

Thus the maximum live-load stresses in each of the diagonals in the 1st, 2d, 3d, . . . , panels are

$$\tfrac{1}{2}R_1 \sec\theta, \quad \tfrac{1}{2}R_2 \sec\theta, \quad \tfrac{1}{2}R_3 \sec\theta, \ldots \;.$$

respectively, and therefore the maximum live-load stress in any diagonal is the product of a constant quantity $\frac{1}{2}\frac{l}{10}\sec\theta \left(=\frac{1}{2}\frac{\text{panel live load}}{\text{number of panels}} \times \sec\theta\right)$ and a multiplier composed of two factors, the one being the number of loads on the bridge and the other the distance in number of panels of the centre of gravity of these loads from the right support.

It may be observed that for each succeeding panel one live load leaves the truss and the centre of gravity therefore moves one half panel nearer the right support.

Col. V of the preceding table gives the maximum live-load stress in any specified diagonal. It is the product of the constant quantity at the head of Col. V by the corresponding multiplier in Col. IV. The total maximum stress in any diagonal is the algebraic sum of the corresponding stresses due to the dead and live loads. For example, the total maximum stress in $5 = 65D + 15L$. The total maximum stresses in 67, 78, 89, and 9,10 are $-5D + 10L$, $-15D + 6L$, $-25D + L$, and $-35D + L$, respectively.

If any one of these last four stresses is positive, it indicates that the live load developes a stress in the member of an *opposite kind* to that due to the dead load and the member must therefore be designed to meet this change of stress. In some cases the members of the truss receive a greater sectional area for this purpose, while in others any additional member, called a counter-brace, is introduced to take up the stress in question. If any of the stresses are negative, they are less in amount but of the *same kind* as those for which the members are designed.

C *Chord Stresses.* These stresses are greatest when the live load covers the whole bridge, so that at each panel-point there is concentrated a load $d + l = P$.

The shears in the 1st, 2d, 3d, 4th, etc., panels are $45\frac{P}{10}$, $35\frac{P}{10}$, $25\frac{P}{10}$, $15\frac{P}{10}$, etc., respectively, and in any given panel one half of the shear is transmitted to the chord through each of the sloping members. The corresponding increment of chord stress is the product of a constant quantity $\frac{1}{2}\frac{P}{10}\tan\theta$ and a multiplier which is 45 for the first panel, and for each succeeding panel is diminished by the number of panels.

Hence

$$1b\ (= -a2) = \text{stress transmitted through 12 (or } ab) = \frac{1}{2}\frac{P}{10}\tan\theta \times 45;$$

$$\begin{aligned} b3\ (= -2c) &= 1b + \text{stress transmitted through } ab \\ & + \text{''} \qquad \text{''} \qquad \text{''} \quad bc \end{aligned}$$

$$= 1b + \frac{1}{2}\frac{P}{10}\tan\theta(45 + 35);$$

$$\begin{aligned} 3d\ (= -c4) &= b3 + \text{stress transmitted through } 23 \\ & + \text{''} \qquad \text{''} \qquad \text{''} \quad 34 \end{aligned}$$

$$= b3 + \frac{1}{2}\frac{P}{10}\tan\theta(35 + 25).$$

So,

$$d5\ (= -4e) = 3d + \frac{1}{2}\frac{P}{10}\tan\theta(25 + 15);$$

etc. etc.

These results may be tabulated as follows:

Col. I. Member.	Col. II. Multiplier.	Col. III. $\frac{1}{2}\frac{P}{10}\tan\theta=F$.	Co. IV. Total Maximum Chord Stress.
$1b=-a2$	45	$45F$	$45F$
$b3=-2c$	$45+35=80$	$80F$	$125F$
$3d=-c4$	$35+25=60$	$60F$	$185F$
$d5=-4e$	$25+15=40$	$40F$	$225F$
$5f=-e6$	$15+5=20$	$20F$	$245F$

It is evident that the panel lengths in the upper chord are in compression and those in the lower chord in tension.

Second. Let the truss be of the pin-connected type. *Each system, abcd .. and 1234 . . . , must be regarded as being entirely independent of the other and as being strained only by the loads at the panel-points belonging to the particular system under consideration.*

A. Dead-load Stresses in Web Members. Consider, *first,* the system *abcd . . . lm.* A load l is concentrated at each of the panel-points b, d, f, h, l, and the corresponding reaction R at the left support is given by

$$R'\cdot 10p=5d\cdot 5p, \quad \text{or} \quad R'=\frac{d}{10}25.$$

The corresponding shears from 1 to b, b to d, d to f, f to h, h to l, and l to 11 are

$$\frac{d}{10}25, \quad \frac{d}{10}15, \quad \frac{d}{10}5, \quad -\frac{d}{10}5, \quad -\frac{d}{10}15,$$

so that the dead-load stresses are $\frac{d}{10}\sec\theta\times 25$ in ab, $\frac{d}{10}\sec\theta\times 15$ in bc and cd, $\frac{d}{10}\sec\theta\times 5$ in de and ef, $-\frac{d}{10}\sec\theta\times 5$ in fg and gh, $-\frac{d}{10}\sec\theta\times 15$ in hk and kl.

Next, consider the system 1234 . . . 10,11. The load l is now concentrated at each of the panel-points 3, 5, 7, 9, and the corresponding reaction R'' at the left support is given by

$$R''\cdot 10p=4d\cdot 5p, \quad \text{or} \quad R''=\frac{d}{10}\cdot 20.$$

The corresponding shears from 1 to 3, 3 to 5, 5 to 7, 7 to 9, and 9 to 11 are $\frac{d}{10}20$, $\frac{d}{10}10$, $\frac{d}{10}0$, $-\frac{d}{10}10$, respectively, so that the dead-load stresses are $\frac{d}{10}\sec\theta\times 20$ in 12 and 23, $\frac{d}{10}\sec\theta\times 10$ in 34 and 45, 0 in 56 and 67, $-\frac{d}{10}\sec\theta\times 10$ in 78 and 89.

These results may be tabulated as follows:

Col. I. Member.	Col. II. Multiplier.	Col. III. $\frac{d}{10}\sec\theta = D$.	Col. IV. Multiplier.	Col. V. $\frac{l}{10}\sec\theta = L$.	Col. VI. Total Maximum Diagonal Stress.
ab	25	$25D$	$5.5=25$	$25L$	$25D+25L$
$12=23$	20	$20D$	$4.5=20$	$20L$	$20D+20L$
$bc=cd$	15	$15D$	$4.4=16$	$16L$	$15D+16L$
$34=45$	10	$10D$	$3.4=12$	$12L$	$10D+12L$
$de=ef$	5	$5D$	$3.3=\ 9$	$9L$	$5D+\ 9L$
$56=67$	0	0	$2.3=\ 6$	$6L$	$6L$
$fg=gh$	-5	$-5D$	$2.2=\ 4$	$4L$	$-5D+\ 4L$
$78=89$	-10	$-10D$	$1.2=\ 2$	$2L$	$-10D+\ 2L$
$hk=kl$	-15	$-15D$	$1.1=\ 1$	L	$-15D+\ \ L$

Col. I designates the member and Col. III gives the corresponding dead-load stress. It is the product of the constant quantity at the head of Col. III by the multiplier in Col. II.

B. Live-load Stresses in Web Members. Consider, *first,* the system $abcd\,..\,lm$. The maximum live-load shears from 1 to b, b to d, d to f, f to h, and h to l occur when l is concentrated at the panel-points (of the system in question) b to l, d to l, f to l, h to l, and at l, respectively. If R_1, R_2, R_3, R_4, R_5 are the reactions at the left support for these several concentrations, then

$$R_1\cdot 10p=5l\cdot 5p,\quad\text{or}\quad R_1=\frac{l}{10}5\times 5;$$

$$R_2\cdot 10p=4l\cdot 4p\quad\text{or}\quad R_2=\frac{l}{10}4\times 4;$$

$$R\ \cdot 10p=3l\cdot 3p,\quad\text{or}\quad R_3=\frac{l}{10}3\times 3;$$

$$R_4\cdot 10p=2l\cdot 2p,\quad\text{or}\quad R_4=\frac{l}{10}2\times 2;$$

$$R_5\cdot 10p=\ l\cdot p,\quad\text{or}\quad R_5=\frac{l}{10}1\times 1.$$

Similarly if R_1', R_2', R_3', R_4' are the reactions at the left support for the system 1234 . . . 10,11,

$$R_1'=\frac{l}{10}4\times 5,\ R_2'=\frac{l}{10}3\times 4,\ R_3'=\frac{l}{10}2\times 3,\ \text{and}\quad R_4'=\frac{l}{10}1\times 2.$$

Hence the maximum live-load stresses are

$\frac{l}{10}\sec\theta\times 5\times 5$ in ab, $\frac{l}{10}\sec\theta\times 4\times 4$ in bc and cd, $\frac{l}{10}\sec\theta\times 3\times 3$ in de and ef,

$\frac{l}{10}$ sec $\theta\times2\times2$ in *fg* and *gh* and $\frac{l}{10}$ sec $\theta\times1\times1$ in *hk* and *kl*;

$\frac{l}{10}$ sec $\theta\times4\times5$ in 12 and 23, $\frac{l}{10}$ sec $\theta\times3\times4$ in 34 and 45,

$\frac{l}{10}$ sec $\theta\times2\times3$ in 56 and 67, and $\frac{l}{10}$ sec $\theta\times1\times2$ in 78 and 89.

These results are shown in the preceding table, and Col. V gives the maximum live-load stresses in the diagonals. This stress is the product of the constant quantity $\frac{l}{10}$ sec θ at the head of Col. V and a multiplier composed of two factors of which one is the number of loads on the truss, while the other is the distance of the centre of gravity of the loads in panels from the right support.

The total maximum stress in any member is given in Col. VI, and is the *algebraic* sum of the corresponding stresses in Cols. III and V. For example, the total maximum stress in $de=5D+9L$.

Again, the total maximum stresses in *fg* (=*gh*), 78 (=89), *hk* (=*kl*) are

$$-5D+4L, \quad -10D+2L, \quad \text{and} \quad -15D+L, \quad \text{respectively.}$$

If any one of these results is positive, it indicates that the stress of the opposite kind to that of the dead load is developed and that therefore provision must be made for this in designing the members affected. If the results are positive, it shows that the members are simply subjected to the same kind of stress, but less in amount, as those for which they are usually designed. Again, in the two systems, members sloping in one direction are in tension and those sloping in the opposite direction are in compression.

C. Chord Stresses.—The stresses in the chord lengths are greater when the live load covers the whole girder, so that a load of $d+l=P$ is concentrated at each panel-point. Then, calling $t_1, t_2, t_3, \ldots$ the tensions in *a*2, 2*c*, *c*4, etc., and $c_1, c_2, c_3, \ldots$ the compressions in 1*b*, *b*3, 3*d*, . . .

t_1 = stress transmitted through $ab=\frac{P}{10}\tan\theta\times25$,
$t_2=t_1+$ “ “ “ 12
$+$ “ “ “ $23=\frac{P}{10}\tan\theta(20+20)$,
$t_3=t_2+$ “ “ “ *bc*
$+$ “ “ “ $cd=\frac{P}{10}\tan\theta(15+15)$,
$t_4=t_3+$ “ “ “ 34
$+$ “ “ “ $45=\frac{P}{10}\tan\theta\ 10+10)$,
$t_5=t_4+$ “ “ “ *de*
$+$ “ “ “ $ef=\frac{P}{10}\tan\theta(5+5)$,

and $c_1 =$ stress transmitted through $12 = \frac{P}{10} \tan \theta \times 20$;

$c_2 = c_1 +$ " " " ab
$+$ " " " $bc = \frac{P}{10} \tan \theta (25+15)$,
$c_3 = c_2 +$ " " " 23
$+$ " " " $34 = \frac{P}{10} \tan \theta (20+10)$,
$c_4 = c_3 +$ " " " cd
$+$ " " " $de = \frac{P}{10} \tan \theta (15+5)$,
$c_5 = c_4 +$ " " " 45
$+$ " " " $56 = \frac{P}{10} \tan \theta (10+0)$.

These results may be tabulated as follows:

Col. I. Member.	Col. II. Multiplier.	Col. III. $\frac{P}{10} \tan \theta = F$.	Col. IV. Total Maximum Chord Stress.
$a2$	25	$25F$	$25F$
$2c$	$20+20=40$	$40F$	$65F$
$c4$	$15+15=30$	$30F$	$95F$
$4e$	$10+10=20$	$20F$	$115F$
$e6$	$5+5=10$	$10F$	$125F$
$1b$	20	$20F$	$20F$
$b3$	$25+15=40$	$40F$	$60F$
$3d$	$20+10=30$	$30F$	$90F$
$d5$	$15+5=20$	$20F$	$110F$
$5f$	$10+0=10$	$10F$	$120F$

Col. I. indicates the chord member; Col. III indicates the stress transmitted to any given member through the sloping members, and is the product of a constant quantity at the head of Col. III by the corresponding multipliers in Col. II. Col. IV gives the total maximum chord stress in any given panel, and its value is obtained by adding to the stress in the preceding panel the stresses transmitted through the diagonals meeting at the common panel-point.

Ex. 5. *An eight-panel through lattice truss with horizontal chords and two series of diagonals inclined in opposite directions at angle θ to the vertical.*

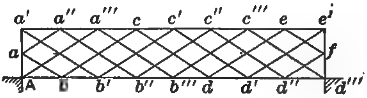


FIG. 634.

An objection to this class of girder is the number of the joints.

First, consider the members to be riveted together.

Precisely the same method of analysis for the determination of the stresses in the several *web members* is to be adopted as in the preceding example. In veery panel a vertical section now intersects *four* members, and therefore in the following table the constant quantity is $\frac{1}{4}\left(\frac{d}{8}\text{ see }\theta\right)=D$ at the head of Col. III for the dead load and $\frac{1}{4}\left(\frac{l}{8}\sec\theta\right)=L$ at the head of Col. V for the live load. The total maximum diagonal stresses are given by Col. VI as indicated:

Col. I. Member.	Col. II. Multiplier.	Col. III. *D*.	Col. IV. Multiplier.	Col. V. *L*.	Col. VI. Total Maximum Diagonal Stress.
$ab=aa''$	28	$28D$	$7.4=28$	$28L$	$28D+28L$
$Aa'''=a'b'$	20	$20D$	$6.3\frac{1}{2}=21$	$21L$	$20D+21L$
$bc=a''b''$	12	$12D$	$5.3=15$	$15L$	$12D+15L$
$b'c'=a'''b'''$	4	$4D$	$4.2\frac{1}{2}=10$	$10L$	$4D+10L$
$b''c''=cd$	-4	$-4D$	$3.2=6$	$6L$	$-4D+6L$
$b'''c'''=c'd'$	-12	$-12D$	$2.1\frac{1}{2}=3$	$3L$	$-12D+3L$
$de=c''d''$	-20	$-20D$	$1.1=1$	L	$-20D+L$

The total stress in a diagonal is the *algebraic* sum of the corresponding stresses due to the dead and live loads. For example,

$$\text{the total maximum stress in } bc\ (=a''b'')=12D+15L.$$

Again, the total maximum diagonal stresses in $b''c''$ $(=cd)$, $b'''c'''$ $(=c'd')$, and de $(=c''d'')$ are $-4D+6L$, $-12D+3L$, and $-20D+L$, respectively.

If either of these is negative, the result indicates that the live load produces a stress in the web member under consideration of a kind opposite to that due to the dead load, and therefore the web member must be designed to carry this additional stress. If, however, the result is positive, it indicates that the stress is of the *same* kind but *less* in amount than that due to the dead load, and the ordinary bracing is therefore quite sufficient.

It is also evident that the stresses in the members in any panel sloping in opposite directions are opposite in kind.

For the greatest stresses in the *chord* panel lengths the live load l is concentrated at each panel-point, and the truss now carries what is a uniformly distributed load, the load at each panel-point being $d+l$. Thus the constant quantity at the head of Col. III for the chords is now $\frac{1}{4}\frac{d+l}{8}\tan\theta=P$. Hence the table for the chord stresses is as follows:

Col. I. Member.	Col. II. Multiplier.	Col. III. *F*.	Col. IV. Total Maximum Chord Stress.
$Ab=-a'a''$	28	$28F$	$28F$
$bb'=-a''a'''$	$28+20=48$	$48F$	$76F$
$b'b''=-a'''c$	$20+12=32$	$32F$	$108F$
$b''b'''=-cc'$	$12+4=16$	$16F$	$124F$

Second, let the truss be pin-connected. There are now four systems of bracing, viz., $abcdef$, $a'b'c'd'e'$, $aa''b''c''d''f$, $Aa'''b'''c'''d'''$.

The following table gives the maximum stresses in the diagonals, $\frac{d}{8}\sec\theta = D$ being the constant quantity for the dead load at the head of Col. III and $\frac{l}{8}\sec\theta = L$ being the constant quantity for the live load at the head of Col. V:

Col. I. Member.	Col. II. Multiplier.	Col. III. D.	Col. IV. Multiplier.	Col. V. L.	Col. VI. Total Maximum Diagonal Stress.
ab	10	$10D$	10	$10L$	$10D+10L$
$bc=cd$	2	$2D$	3	$3L$	$2D+3L$
$de=ef$	-6	$-6D$	0	0	$-6D$
$a'b'$	8	$8D$	8	$8L$	$8D+8L$
$b'c'=c'd'$	0	0	2	$2L$	$2L$
$d'e'$	-8	$-8D$	0	0	$-8D$
$aa''=a''b''$	6	$6D$	6	$6L$	$6D+6L$
$b''c''=c''d''$	-2	$-2D$	1	L	$-2D+L$
$d''f$	-10	$-10D$	0	0	$-10D$
$Aa'''=a'''b'''$	4	$4D$	4	$4L$	$4D+4L$
$b'''c'''=c'''d'''$	-4	$-4D$	0	0	$-4D$

The following table gives the maximum chord stresses, the constant quantity for the combined dead and live load at the head of Col. III being $\frac{d+l}{8}\tan\theta = F$:

Col. I. Member.	Col. II. Multiplier.	Col. III. F.	Col. IV. Total Maximum Chord Stress.
Ab	4	$4F$	$4F$
bb'	$10+2=12$	$12F$	$16F$
$b'b''$	$8+0=8$	$8F$	$24F$
$b''b'''$	$6-2=4$	$4F$	$28F$
$a'a''$	8	$8F$	$8F$
$a''a'''$	$6-2=4$	$4F$	$12F$
$a'''c$	$4-4=0$	0	$12F$
cc'	$10+2=12$	$12F$	$24F$

Ex. 6. *The bracing of a lattice girder consists of a single system of triangles in which one of the sides is a strut and the other a tie inclined to the horizontal at angles of α and β respectively; in order to give the strut sufficient rigidity its section is made k times that indicated by theory, the coefficient k being $>$ unity. Show that the amount of material in the struts and ties is a minimum when $\tan\alpha = k\tan\beta$.*

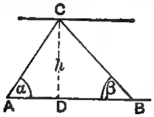


FIG. 635.

Let h = depth of truss. Also, let S be the shear between the two consecutive panel-points A and B. Take AC as the strut and BC as the tie. Then

$$\text{stress in } AC = S\operatorname{cosec}\alpha \quad \text{and} \quad \text{in } BC = S\operatorname{cosec}\beta.$$

Therefore, total amount of material in AC and BC

$$=\frac{kS\operatorname{cosec}\alpha}{f}AC+\frac{S\operatorname{cosec}\beta}{f}BC=\frac{S}{f}(k\operatorname{cosec}\alpha\cdot h\operatorname{cosec}\alpha+\operatorname{cosec}\beta\cdot h\operatorname{cosec}\beta)$$

$$=\frac{Sh}{f}(k\operatorname{cosec}^2\alpha+\operatorname{cosec}^2\beta)=\text{ minimum,}$$

so that

$$\frac{k\cos\alpha}{\sin^3\alpha}d\alpha+\frac{\cos\beta}{\sin^3\beta}d\beta=0. \qquad \text{(I)}$$

Again, $AB=h(\cot\alpha+\cot\beta)$ = a constant.

Therefore

$$\frac{d\alpha}{\sin^2\alpha}+\frac{d\beta}{\sin^2\beta}=0. \qquad \text{(II)}$$

Hence, by (I) and (II),

$$\tan\alpha=k\tan\beta.$$

Ex. 7. *Determine the maximum stresses in the members of a through lattice truss of 40 ft. span and 4 ft. depth, with two systems of triangles (base = 8 ft.), (a) when riveted together; (b) when pin-connected. Dead load = ¼ ton per lineal foot, live load = ½ ton per lineal foot.*

(*a*) *Riveted Truss.*—Panel dead load = 1 ton; panel live load = 2 tons.

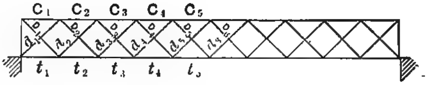


FIG. 636.

Diagonal Stresses. Constant for dead load $=\frac{1}{2}\left(\frac{1}{10}\sec 45°\right)=\frac{\sqrt{2}}{20}$;

" " live load $=\frac{1}{2}\left(\frac{2}{10}\sec 45°\right)=\frac{\sqrt{2}}{10}$.

Chord Stresses. Constant for combined dead and live loads

$$=\frac{1}{2}\left(\frac{3}{10}\tan 45°\right)=\frac{3}{20}.$$

Table of Max. Diagonal Stresses.

Col. I. Member.	Col. II. Multiplier.	Col. III. $\frac{\sqrt{2}}{20}$.	Col. IV. Multiplier.	Col. V. $\frac{\sqrt{2}}{10}$.	Col. VI. Total Maximum Diagonal Stress.
$d_1=-D_1$	45	$2.25\sqrt{2}$	$9\times 5=45$	$4.5\sqrt{2}$	$6.75\sqrt{2}$
$d_2=-D_2$	35	$1.75\sqrt{2}$	$8\times 4\frac{1}{2}=36$	$3.6\sqrt{2}$	$5.35\sqrt{2}$
$d_3=-D_3$	25	$1.25\sqrt{2}$	$7\times 4=28$	$2.8\sqrt{2}$	$4.05\sqrt{2}$
$d_4=-D_4$	15	$.75\sqrt{2}$	$6\times 3\frac{1}{2}=21$	$2.1\sqrt{2}$	$2.85\sqrt{2}$
$d_5=-D_5$	5	$.25\sqrt{2}$	$5\times 3=15$	$1.5\sqrt{2}$	$1.75\sqrt{2}$
$d_6=-D_6$	−5	$-.25\sqrt{2}$	$4\times 2\frac{1}{2}=10$	$\sqrt{2}$	$.75\sqrt{2}$

The stresses in the remaining diagonals are negative.

Table of Max. Chord Stresses.

Col. I. Member.	Col. II. Multiplier.	Col. III. $\frac{3}{20}$	Col. IV. Total Maximum Chord Stress.
$t_1=-c_1$	45	6.75	6.75
$t_2=-c_2$	45+35=80	12.00	18.75
$t_3=-c_3$	35+25=60	9.00	27.75
$t_4=-c_4$	25+15=40	6.00	33.75
$t_5=-c_5$	15+ 5=20	3.00	36.75

(*b*) *Pin-connected Truss.—*

Diagonal Stresses. Constant for dead load $=\frac{\sqrt{2}}{10}$;

" " live load $=\frac{2\sqrt{2}}{10}$.

Chord Stresses. Constant for combined dead and live loads

$$=\frac{3}{10}\tan 45^\circ=\frac{3}{10}.$$

Table of Max. Diagonal Stresses.

Col. I. Member.	Col. II. Multiplier.	Col. III. $\frac{\sqrt{2}}{10}$.	Col. IV. Multiplier.	Col. V. $\frac{2\sqrt{2}}{10}$.	Col. VI. Total Maximum Chord Stress.
D_1	25	$2.5\sqrt{2}$	$5\times5=25$	$5\sqrt{2}$	$7.5\sqrt{2}$
$d_1=-D_2$	20	$2\sqrt{2}$	$4\times5=20$	$4\sqrt{2}$	$6\sqrt{2}$
$d_2=-D_3$	15	$1.5\sqrt{2}$	$4\times4=16$	$3.2\sqrt{2}$	$4.7\sqrt{2}$
$d_3=-D_4$	10	$\sqrt{2}$	$3\times4=12$	$2.4\sqrt{2}$	$3.4\sqrt{2}$
$d_4=-D_5$	5	$.5\sqrt{2}$	$3\times3=\ 9$	$1.8\sqrt{2}$	$2.3\sqrt{2}$
$d_5=-D_6$	0	0	$2\times3=\ 6$	$1.2\sqrt{2}$	$1.2\sqrt{2}$
$d_6=-D_7$	-5	$-.5\sqrt{2}$	$2\times2=\ 4$	$.8\sqrt{2}$	$.3\sqrt{2}$

The stresses in the remaining diagonals are negative.

Table of Max. Chord Stresses.

Col. I. Member.	Col. II. Multiplier.	Col. III. $\frac{3}{10}$	Col. IV. Total Maximum Chord Stress.
t_1	20	6	6
t_2	25+15=40	12	18
t_3	20+10=30	9	27
t_4	15+ 5=20	6	33
t_5	10+ 0=10	3	36
c_1	25	7.5	7.5
c_2	20+20=40	12.0	19.5
c_3	15+15=30	9.0	28.5
c_4	10+10=20	6.0	34.5
c_5	5+ 5=10	3.0	37.5

Ex. 8. *Determine the stresses in the several members of a deck-truss for a double-track bridge of* 342 *ft. span,* 33 *ft. depth, and with eighteen panels. The panel engine, live and dead loads are* 121,000, 65,000, and 40,000 *lbs., respectively, per truss.* (Single intersection.)

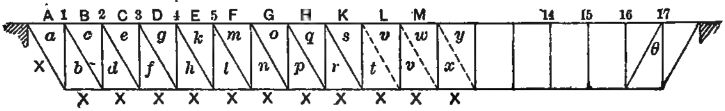


FIG. 637.

$$\sec\theta = 1.13393; \quad \tan\theta = \frac{19}{33}; \quad \theta° = 29° 56'.$$

Diag. Stresses. Constant for engine load $= \dfrac{121000}{18} \sec\theta = 7757;$

$$\text{for train load} = \frac{65000}{18} \sec\theta = 4167;$$

$$\text{for dead load} = \frac{40000}{18} \sec\theta = 2565.$$

Chord Stresses. Constant for combined engine and dead loads

$$= \frac{121000 + 40000}{18} \tan\theta = 5150;$$

for combined dead and train loads

$$= \frac{40000 + 65000}{18} \tan\theta = 3359.$$

DIAGONAL STRESSES.

Member.	Multiplier.	7757.	Multiplier.	4167.	Multiplier.	2565.	Total Maximum Diagonal Stress in Pounds.
Xa	17	131,869	136	566,712	153	392,445	1,091,026
bc	16	124,112	120	500,040	135	346,275	970,427
de	15	116,355	105	437,535	117	300,105	853,995
fg	14	108,598	91	379,197	99	253,935	741,730
hk	13	100,841	78	325,026	81	207,765	633,632
lm	12	93,084	66	275,022	63	161,595	529,701
no	11	85,327	55	229,185	45	115,425	429,837
pq	10	77,570	45	187,515	27	69 255	334,340
rs	9	69,813	36	150,012	9	23,085	242,910
tu	8	62,056	28	116,676	−9	−23,085	155,647
vw	7	54,299	21	87,507	−27	−69,255	72,551

The stress in the next diagonal *xy* is negative, and therefore counterbraces (or additional strengthening) are required in the two centre panels only, although it is usual in practice, for the purpose of stiffening the truss, to introduce them into other panels.

Again, *the maximum stresses in the verticals ab, cd, ef, gh, kl, mn, op, qr,* and *st* are the vertical components of the maximum stresses in the diagonals *Xa, bc, de, fg, hk, lm, no, pq,* and *rs,* respectively. Hence the maximum stress

$ab = 1{,}091{,}026 \cos\theta = 945{,}486$ lbs.; $cd = 970{,}247 \cos\theta = 840{,}820$ lbs.;
$ef = 853{,}895 \cos\theta = 740{,}000$ lbs.; $gh = 741{,}730 \cos\theta = 642{,}800$ lbs.;
$kl = 633{,}632 \cos\theta = 549{,}100$ lbs.; $mn = 529{,}701 \cos\theta = 459{,}050$ lbs.;
$op = 429{,}837 \cos\theta = 372{,}500$ lbs.; $qr = 334{,}340 \cos\theta = 289{,}740$ lbs.;
$st = 242{,}910 \cos\theta = 210{,}500$ lbs.; $uv = 155{,}647 \cos\theta = 134{,}890$ lbs.;
$wx = 72{,}551 \cos\theta = 62{,}874$ lbs.

CHORD STRESSES.

Member.	Multiplier.	5150.	Multiplier.	3359.	Sum of 3d and 5th Columns.	Total Maximum Chord Stress in Lbs.
$Aa=Xb$	17	87,550	136	456,824	544,374	544,374
$Bc=Xd$	−1	−5,150	136	456,824	451,674	996,048
$Ce=Xf$	−1	−5,150	118	396,362	391,212	1,387,260
$Dg=Xh$	−1	−5,150	100	335,900	330,750	1,718,010
$Ek=Xl$	−1	−5,150	82	275,438	270,288	1,988,298
$Fm=Xn$	−1	−5,150	64	214,976	209,826	2,198,124
$Go=Xp$	−1	−5,150	46	154,514	149,364	2,347,488
$Hq=Xr$	−1	−5,150	28	94,052	88,902	2,436,390
Ks	−1	−5,150	10	33,590	28,440	2,464,830

These results can easily be checked by the method of moments.

For instance, the reaction R at the right support

$$=\frac{1}{18}(56000)+\frac{17\times105000}{2}.$$

Taking moments about the foot of vertical *st*,

$$Ks\times33=\left(\frac{56000}{18}+\frac{17\times105000}{2}\right)9\times19-9\times105000\times4\times19=81{,}329{,}500$$

and $$Ks=2{,}464{,}530 \text{ lbs.}$$

As already stated at the end of Ex. 2, the stresses in the several members can be more easily and readily obtained by tabulating them in the manner just described, but the same results may be found *graphically* as follows:

Fig. 638 gives the stresses in the several members for a *dead load* of 40,000 lbs. concentrated at every panel-point. The greatest stresses in the chord panel lengths occur when the live load covers the whole bridge, so that the total load on the bridge is then equivalent to a load of 161,000 lbs. at the 1st panel-point and 105,000 lbs. at every other panel-point. Then, at the left support,

$$\text{the reaction } R=\frac{17}{18}161000+\frac{1}{342}16\times105000\times8\tfrac{1}{2}\times19=945{,}389 \text{ lbs.}$$

Taking $XA = 945{,}389$ lbs., $AB = 161{,}000$ lbs., and $BC = CD = \ldots = 105{,}000$ lbs., Fig. 639 gives the *total maximum chord stresses.*

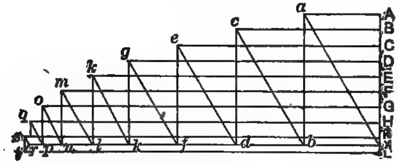


FIG. 638

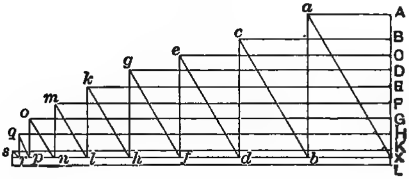


FIG. 639.

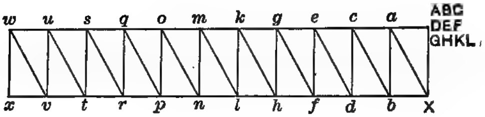


FIG. 640.

For the *maximum diagonal stresses* assume that the only force acting upon the truss is a vertical reaction of 1000 units at the left support. Then Fig. 640, in which $XA = 1000$ units, is the corresponding stress diagram and shows that

(*a*) the stress in each diagonal *due to the assumed reaction*

$$= 1000 \sec \theta = 11539.3 \text{ units;}$$

(*b*) the stress in each vertical *due to the assumed reaction*

$$= 1000 \text{ units.}$$

The *total actual reactions* at the left support are

945,389 lbs. when the live load covers 1 to 17;
840,889 " " " " " " 2 to 17;
740,000 " " " " " " 3 to 17;
642,723 " " " " " " 4 to 17;
549,055 " " " " " " 5 to 17;
459,000 " " " " " " 6 to 17;
372,556 " " " " " " 7 to 17;
289,723 " " " " " " 8 to 17;
210,500 " " " " " " 9 to 17;
134,889 " " " " " " 10 to 17;
62,889 " " " " " " 11 to 17.

Hence
$$\frac{\text{the actual stress in any diagonal, say } on}{\text{the stress due to the assumed reaction}}=\frac{372556}{1000},$$

or
$$\text{actual stress } on = 372556\,\frac{1000 \sec\theta}{1000} = 429{,}902 \text{ lbs.};$$

$$\frac{\text{the actual stress in any diagonal, say } op}{\text{the stress due to assumed reaction}}=\frac{289723}{1000},$$

or
$$\text{actual stress } op = 289{,}723 \text{ lbs.}$$

Ex. 9. *The Schwedler truss in which the minimum stress in every diagonal is to be nil.*

Consider the eight-panel truss, Fig. 641, and let D and L be the panel dead and live loads respectively. Let a be the panel length. Let y_1, y_2, y_3, y_4 be the

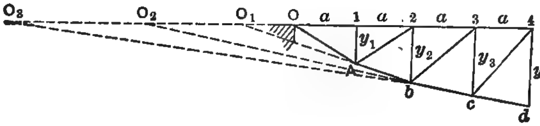


FIG. 641.

lengths of the verticals. Let the straight portions of the bow meet the horizontal through O in O_1, O_2, O_3, and take $OO_1 = x_1$, $OO_2 = x_2$, $OO_3 = x_3$. For minimum stress in $2A$, L is at 1 and the reaction at O is then

$$3\tfrac{1}{2}D + \tfrac{7}{8}L.$$

The stress in $2A$ is to be *nil*, and therefore, taking moments about O_1,

$$-(3\tfrac{1}{2}D + \tfrac{7}{8}L)x_1 + (D+L)(x_1+a) = 0, \quad \text{or} \quad \frac{x_1}{a} = \frac{8(D+L)}{20D-L}.$$

Hence
$$\frac{y_2}{y_1} = \frac{x_1+2a}{x_1+a} = \frac{6(8D+L)}{7(4D+L)}.$$

For minimum stress in $3b$, L is at 1 and 2, and the reaction at O is then

$$3\tfrac{1}{2}D + \tfrac{18}{8}L.$$

The stress in $3b$ is to be *nil*, and therefore, taking moments about O_2,

$$-(3\tfrac{1}{2}D + \tfrac{18}{8}L)x_2 + 2(D+L)(x_2 + \tfrac{3}{2}a) = 0, \quad \text{or} \quad \frac{x_2}{a} = \frac{8(D+L)}{4D-L}.$$

Hence
$$\frac{y_3}{y_2} = \frac{x_2+3a}{x_2+2a} = \frac{5(4D+L)}{2(8D+3L)}.$$

For minimum stress in $4c$, L is at 1, 2, and 3. and the reaction at O is then

$$3\tfrac{1}{2}D + \tfrac{9}{4}L.$$

The stress in $4c$ is to be *nil*, and therefore, taking moments about O_3,

$$-(3\tfrac{1}{2}D+{}^{4}L)x_3+3(D+L)(x_3+2a)=0, \quad \text{or} \quad \frac{x_3}{a}=\frac{24(D+L)}{2D-3L}.$$

Hence
$$\frac{y_4}{y_3}=\frac{x_3+4a}{x_3+3a}=\frac{4(8D+3L)}{15(2D+L)}.$$

Case a. If $L=\tfrac{2}{3}D$, $x_3=\infty$, and therefore the two central chord panel lengths of the bow are horizontal.

Also,
$$\frac{y_4}{y_3}=1, \quad \frac{y_3}{y_2}=\frac{7}{6}, \quad \frac{y_2}{y_1}=\frac{78}{49}.$$

If $L>\tfrac{2}{3}D$, x_3 is negative and the depth of the truss might be diminished from 3 towards the centre.

So if $L>4D$, x_2 is negative and the depth might diminish from 2 towards the centre.

If $L>20D$, x_3 is negative and the depth might diminish from 1 towards the centre.

It is inadvisable, however, to construct a truss in such a manner.

Case b. If $L=D$, $x_3=-\frac{a}{24}$, and the three central verticals should be each made of the same length, viz., y_3. Also,

$$\frac{y_3}{y_2}=\frac{25}{22} \quad \text{and} \quad \frac{y_2}{y_1}=\frac{54}{35}.$$

The calculations will be the same if the truss is inverted, and also the depths y_1, y_2, . . . may be plotted one half above and one half below the horizontal through O.

If the diagonals in Fig. 641 are made to slope in the opposite direction, they are necessarily in tension.

Ex. 10. *Pauli's truss in which each panel length of the bow is to be subjected to the same stress.*

Consider an eight-panel truss, Fig. 642, each panel length being a. Let p_1, p_2, p_3, p_4 be the perpendiculars from 2, 3, 4, 5 upon the opposite chord length, inclined to the vertical at angles $\alpha, \beta, \gamma, \delta$. The chord stresses are greatest when the live load covers the whole truss.

Let P $(=D+L)$ be the panel load; and let F be the constant force in each panel length of the bow. The reaction at $1=3\tfrac{1}{2}P$. Then

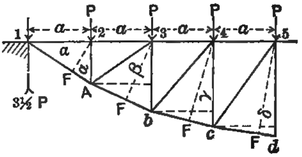


FIG. 642.

$$3\tfrac{1}{2}P\frac{a}{p_1}=F=\frac{3\tfrac{1}{2}P\cdot 2a-P\cdot a}{p_2}=\frac{3\tfrac{1}{2}P\cdot 3a-2P\cdot\tfrac{3}{2}a}{p_3}=\frac{3\tfrac{1}{2}P\cdot 4a-3P\cdot 2a}{p_4},$$

or
$$\frac{3\frac{1}{2}}{p_1}=\frac{6}{p_2}=\frac{7\frac{1}{2}}{p_3}=\frac{8}{p_4}=\frac{F}{Pa},$$

defining the values of p_1, p_2, p_3, p_4.

Construction.—Through 1 draw the tangent $1A$ to the circle having 2 as centre and p_1 as radius to meet the vertical through 2 in A.

Through A draw the tangent Ab to the circle with 2 as centre and p_2 as radius to meet the vertical through 3 in b.

Through b draw the tangent bc to the circle with 4 as centre and p_3 as radius to meet the vertical through 4 in c.

Through c draw the tangent cd to the circle with 5 as centre and p_4 as radius to meet the vertical through 5 in d.

Again, if y_1, y_2, y_3, and y_4 are the lengths of the verticals,

$$\frac{y_1}{a}=\tan\alpha=\frac{p_1}{\sqrt{a^2-p_1^2}};\quad \frac{h_2-h_1}{a}=\tan\beta=\frac{\sqrt{h_2^2-p_2^2}}{p_2};$$

$$\frac{h_3-h_2}{a}=\tan\gamma=\frac{\sqrt{h_3^2-p_3^2}}{p_3};\quad \frac{h_4-h_3}{a}=\tan\delta=\frac{\sqrt{h_4^2-p_4^2}}{p_4}.$$

If in any panel, e.g., the 3d, $p_3=a$, then

$$\frac{h_3}{a}=\frac{1}{2}\left(\frac{a}{h_2}+\frac{h_2}{a}\right).$$

If the perpendiculars are increased m times, then F is diminished m times, but remains unchanged if P is increased m times.

7. Single- and Double-intersection Trusses.—Fig. 643 represents the simplest form of single-intersection (or Pratt) truss; i.e., a truss in which a diagonal crosses *one* panel only. It may be constructed entirely of iron or steel, or may have the chords and verticals of wood. The verticals are in compression and the diagonals in tension. The angle-blocks are therefore placed above the top and below the bottom chord. Counter-braces, shown by the dotted diagonals, are introduced to withstand the effect of a live load.

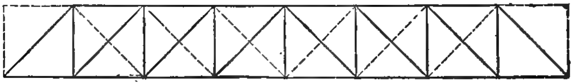


FIG. 643.

Fig. 644 shows a 16-panel deck truss of the Baltimore type, having straight chords and subdivided panels. The Petit truss is similar, the only difference being that one of its chords is inclined. A

14-panel through truss of the Petit type is represented by Fig. 645. Both classes of truss are standard forms for bridges of long span.

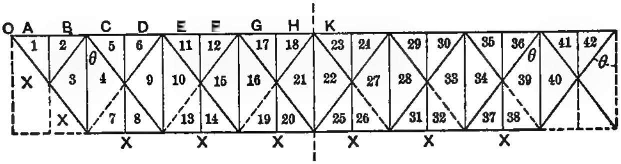


FIG. 644.

Ex. 11. *The stresses in a Baltimore or Petit truss may be easily obtained graphically. Take, for example, the truss represented by Fig.* 644, *and let d and l be the panel dead and live loads respectively.*

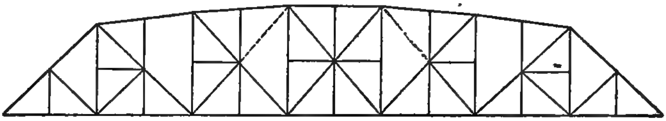


FIG. 645.

Chord Stresses.—The stresses in the chords are greatest when the live load l is concentrated at each of the panel-points. Taking $p=d+l$, Fig. 646,

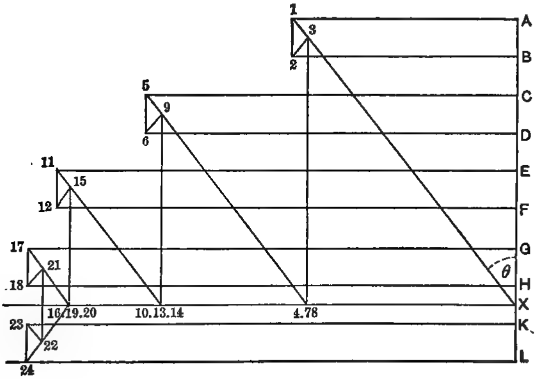


FIG. 646.

is the corresponding stress diagram, in which $XA=7\frac{1}{2}p$, and $AB=BC=\alpha c=p$. Then

$$A1=B2=7\tfrac{1}{2}p\tan\theta;\quad C5=D6=12\tfrac{1}{2}p\tan\theta;\quad E,11=F,12=15\tfrac{1}{2}p\tan\theta;$$

$$G,17=H,18=16\tfrac{1}{2}p\tan\theta=K,23=L,24,\text{ for compression chord;}$$

and $X7 = X8 = 7p \tan\theta$; $X,13 = X,14 = 12p \tan\theta$; $X,19 = X,20 = 15p \tan\theta$ for tension chord.

The corresponding chord stresses due to the dead and live loads separately may be obtained by multiplying these results by the ratio $\frac{d}{p}$ and $\frac{l}{p}$ respectively.

Diagonal Stresses.—(*a*) Stresses due to dead load.

$X1 = 7\tfrac{1}{2}d \sec\theta$; $X3 = 7d \sec\theta$; $23 = \tfrac{1}{2}d \sec\theta = 69 = 12,15 = 18,21 = 22,23$;
$45 = 5\tfrac{1}{2}d \sec\theta$; $89 = 5d \sec\theta$; $10,11 = 3\tfrac{1}{2}d \sec\theta$; $14,15 = 3d \sec\theta$;

$$16,17 = 1\tfrac{1}{2}d \sec\theta; \quad 20,21 = d \sec\theta.$$

(*b*) Stresses due to live load. Fig. 647 is the stress diagram for the truss

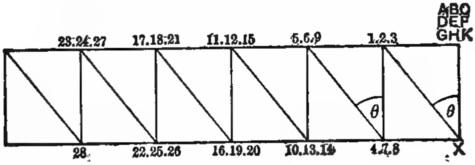


FIG. 647.

on the assumption that it is acted upon by an arbitrary vertical reaction at O only Call this reaction R, so that $XA = R$.

On the same scale the length of each sloping member in the diagram is $R \sec\theta$, and R is the length of each vertical.

For the live load l concentrated at *all* the panel-points, at all the panel-points except the first, at all the panel-points except the first two, three, four, etc., the corresponding reactions at O are:

$$7\tfrac{1}{2}l,\ \tfrac{105}{16}l,\ \tfrac{91}{16}l,\ \tfrac{78}{16}l,\ \tfrac{66}{16}l,\ \tfrac{55}{16}l,\ \tfrac{45}{16}l,\ \tfrac{36}{16}l,\ \tfrac{28}{16}l,\ \tfrac{21}{16}l,\ \tfrac{15}{16}l,\ \tfrac{10}{16}l,\ \tfrac{6}{16}l,\ \tfrac{3}{16}l, \text{ and } \tfrac{1}{16}l.$$

Hence the maximum live-load tensile stresses in the *main diagonals* are:

$X1 = 7\tfrac{1}{2}l \sec\theta$; $X3 = \tfrac{105}{16}l \sec\theta$; $45 = \tfrac{91}{16}l \sec\theta$; $89 = \tfrac{78}{16}l \sec\theta$;
$10,11 = \tfrac{66}{16}l \sec\theta$; $14,15 = \tfrac{55}{16}l \sec\theta$; $16,17 = \tfrac{45}{16}l \sec\theta$; $20,21 = \tfrac{36}{16}l \sec\theta$;

and for the *counters:*

$$10,13 = \tfrac{10}{16}l \sec\theta; \quad 16,19 = \tfrac{21}{16}l \sec\theta; \quad 12,15 = \tfrac{15}{16}l \sec\theta; \quad 18,21 = \tfrac{28}{16}l \sec\theta.$$

Also, $4,7 = \tfrac{3}{16}l \sec\theta$; $6,9 = \tfrac{6}{16}l \sec\theta$.

Vertical Stresses.—(*a*) Stresses due to dead load. It is assumed that two thirds of the dead load is concentrated at the top panel and one third at the bottom. Then, for the *subverticals*,

$12 = \tfrac{2}{3}d = 56 = 11,12 = 17,18$, and are compressions;
$78 = \tfrac{1}{3}d = 13,14 = 19,20$, and are tensions.

For the *main* verticals,

$34=6\frac{2}{3}d$; $9,10=4\frac{2}{3}d$; $15,16=2\frac{2}{3}d$; $21,22=1\frac{2}{3}d$; and are all compressions.

(*b*) Stresses due to live load. For the *subverticals*,

$12=l=56=11,12=17,18$, and are all compressions.
$78=0=13,14=19,20$.

For the *main* verticals,

$34=\frac{105}{16}l$; $9,10=\frac{78}{16}l$; $15,16=\frac{55}{16}l$; $21,22=\frac{36}{16}l$, and are all compressions.

Again, when the live load covers less than half the bridge,

the total resultant stress in $22,23=(l\frac{28}{16}-\frac{1}{2}d)\sec\theta$;
" " " " " $26,27=(\frac{21}{16}l-1\frac{1}{2}d)\sec\theta$;
" " " " " $28,29=(\frac{15}{16}l-2\frac{1}{2}d)\sec\theta$;
" " " " " $32,23=(\frac{10}{16}l-3\frac{1}{2}d)\sec\theta$;

The last term in these resultant stresses is due to the positive dead-load shear and must be neutralized by the negative shear due to the live load before the counters begin to act.

8. Bowstring Truss.—The frame represented by Fig. 648 rests upon supports at A and B and consists simply of a curved member ACB with its ends tied together by a horizontal chord AB. There is no intermediate bracing and it is assumed that deformation is prevented by the stiffness of the bow.

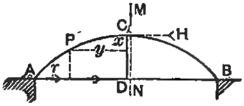


FIG. 648.

Let $AB=l$, and let the chord AB carry a uniformly distributed load of intensity w. Let k be the central depth of the frame.

Consider the equilibrium of one half of the frame between A and a vertical section MN immediately on the right of the crown C.

Let H be the horizontal thrust at C, and T the horizontal tension in the tie. Then ACD is kept in equilibrium by H, T, the vertical reaction $\frac{wl}{2}$ at A, and the load is $\frac{wl}{2}$ on AD.

Taking moments about A,

$$Hk=\frac{wl^2}{8}.$$

Taking moments about C,

$$Tk=\frac{wl^2}{8}=Hk.$$

Thus,

$$H=\frac{wl^2}{8k}=T,$$

and it is evident that the horizontal component of the thrust at every point of the bow is a constant quantity.

Again, if x, y are the co-ordinates of any point P with respect to C, the bending moment at $P=\frac{wl}{2}\left(\frac{l}{2}-y\right)-\frac{w}{2}\left(\frac{l}{2}-y\right)^2-T(k-x)$

$$=\frac{wl^2}{8}\left(\frac{x}{k}-\frac{4y^2}{l^2}\right),$$

which is *nil* if $\frac{x}{k}=\frac{4y^2}{l^2}$, i.e., if the axis of the bow is a parabola.

Fig. 649 represents an N-panel bowstring truss of span l and central depth k, the axis of the curved member being a parabola.

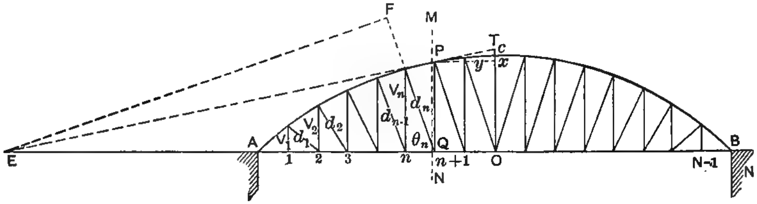


FIG. 649.

Let L be the panel live load and take $a\left(=\frac{L}{N}\right)$ to be the length of a panel.

The maximum stress in the nth diagonal is a tension $(=d_n)$ when L is concentrated at each panel-point from $n+1$ to $N-1$, and a compression (d_n') when L is concentrated at each panel-point from 1 to n.

For the maximum tension consider the equilibrium of the portion of the truss between A and a vertical section immediately on the left of $n+1$ (or Q), and let the tangent at P, the head of the vertical PQ, meet the horizontal through A in E and the axis of the parabola in T. Let x, y be the coordinates of P with respect to the vertex C. Then, since

$$\frac{x}{h}=\frac{4y^2}{l^2},$$

$$EQ=QP\tan PTC=(k-x)\frac{y}{2x}=\frac{\frac{l^2}{4}-y^2}{2y}.$$

Again, $$AE=EQ-\left(\frac{l}{2}-y\right)=\frac{\left(\frac{l}{2}-y\right)^2}{2y},$$

$$BE=AE+l=\frac{\left(\frac{l}{2}+y\right)^2}{2y}.$$

Therefore $$\frac{AE}{EQ}=\frac{\frac{l}{2}-y}{\frac{l}{2}+y}=\frac{n+1}{N-n-1},$$

and $$\frac{BE}{EQ}=\frac{\frac{l}{2}+y}{\frac{l}{2}-y}=\frac{N-n-1}{n+1}.$$

The reaction R_A at A is given by

$$R_A l=(N-n-1)L\frac{N-n}{2}a=R_A Na,$$

or $$R_A=\frac{L}{2}\frac{(N-n-1)(N-n)}{N}.$$

Draw EF at right angles to d_n, and take moments about E. Then

$$d_n EF=\frac{L}{2}\frac{(N-n-1)(N-n)}{N}AE,$$

or $$d_n=\frac{L}{2}\frac{(N-n-1)(N-n)}{N}\frac{AE}{EF}$$

$$=\frac{L}{2}\frac{(N-n-1)(N-n)}{N}\frac{AE}{EQ}\operatorname{cosec}\theta_n=\frac{L}{2}\frac{(N-n)(n+1)}{N}\operatorname{cosec}\theta_n,$$

θ_n being the inclination of the nth diagonal to the horizontal.

For the maximum compression (d_n') in the nth diagonal, consider the equilibrium of the portion of the truss between MN and B, and take moments again about E.

The reaction R_B at B is given by

$$R_B Na = nL\frac{n+1}{2}a \qquad \text{or} \qquad R_B = \frac{L}{2}\frac{n(n+1)}{N}.$$

Therefore
$$d_n'EQ' \sin\theta_n = \frac{L}{2}\frac{n(n+1)}{N}BE,$$

or
$$d_n' = \frac{L}{2}\frac{n(n+1)}{N}\frac{BE}{EQ'}\operatorname{cosec}\theta_n = \frac{L}{2}\frac{(N-n-1)n}{N}\operatorname{cosec}\theta_n.$$

Hence, if every panel-point is loaded, the resultant stress in the nth diagonal

$$= d_n - d_n' = \frac{L}{2}\frac{\operatorname{cosec}\theta_n}{N}\{(N-n)(n+1)-(N-n-1)n\} = \frac{L}{2}\operatorname{cosec}\theta_n.$$

The vertical component of this stress $=\frac{L}{2}$ and it follows that when a truss of this type carries a uniformly distributed load, the vertical component of the stress in any diagonal is a constant quantity equal to *one half of the panel load*, and is a tension.

Let D be the panel dead load. Then the maximum tensile stress in the nth diagonal

$$=\left\{D + L\frac{(N-n)(n+1)}{N}\right\}\frac{\operatorname{cosec}\theta_n}{2},$$

and the maximum compressive stress in the nth diagonal

$$=\left\{-D + L\frac{(N-n-1)n}{N}\right\}\frac{\operatorname{cosec}\theta_n}{2}.$$

If this last result is negative, it indicates a stress of the same kind as that due to the dead load, but is less in amount and is consequently provided for by the ordinary bracing.

If, however, the result is positive, it indicates that the resultant stress is of an opposite kind to that due to the dead load and must be provided for by giving the members affected an additional sectional area or by the introduction of counter-braces shown by the dotted lines, which will evidently be in tension and thus every diagonal becomes a tie. In precisely the same manner as above it may be easily shown that the stress d_n'' in the nth "dotted" diagonal is given by

$$d_n''=\frac{L}{2}\frac{(N-n)(n+1)}{N}\operatorname{cosec}\theta_n',$$

θ_n' being the inclination of the diagonal to the horizontal.

Again, the maximum live-load stress, V_n, in the nth vertical, is a compression and is developed when L is concentrated at each panel-point from $n+1$ to $N-1$. It is evidently the vertical component of the corresponding stress in the $(n-1)$th diagonal. This stress can be determined by considering the equilibrium of the portion of the truss between A and a vertical section immediately on the left of n. It is at once found that

$$d_{n-1}=\frac{L}{2}\frac{(N-n-1)(N-n)}{N}\frac{n}{N-n}\operatorname{cosec}\theta_{n-1}=\frac{L}{2}\frac{n(N-n-1)}{N}\operatorname{cosec}\theta_{n-1}.$$

Hence $$v_n=d_{n-1}\sin\theta_{n-1}=\frac{L}{2}\frac{n(N-n-1)}{N}.$$

The vertical component of the dead-load stress in the $(n-1)$th diagonal is a tension $\frac{D}{2}$, and therefore the total maximum stress in the nth vertical due to both dead and live loads $=v_n+\frac{D}{2}-D=v_n-\frac{D}{2}$.

The stress is consequently a compression or tension according as $v_n>$ or $<\frac{D}{2}$.

If the truss now under consideration is inverted, the stresses are reversed in kind but remain the same in magnitude. The *lenticular* truss (Fig. 650) is a combination of the two forms, and the most important example in practice of such a combination is the bowstring suspension bridge erected at Saltash (Eng.). The bow is a wrought-iron tube of elliptical section, and is strengthened at intervals by diaphragms. The tie consists of a pair of chains. The mathematical analysis of the stresses in the several members is precisely similar to that of the simple bowstring truss.

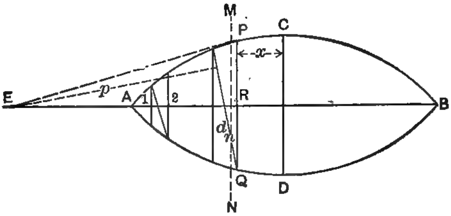


FIG. 650.

First assume that there is no intermediate bracing and that deformation is prevented by giving the bow sufficient rigidity. If the load is uniformly distributed and of intensity w, and if k and k' are the central depths of the bow and tie respectively, then the half of the truss between A and a vertical section MN immediately on the right of the crown C is kept in equilibrium by the horizontal thrust H at C, the horizontal tension T at D, the vertical reaction $\frac{wl}{2}$ at A, and the uniformly distributed load $\frac{wl}{2}$ between A and MN.

Taking moments about C and D,

$$T(k+k')=\frac{wl^2}{8}=H(k+k'),$$

and therefore
$$T=\frac{wl^2}{8(k+k')}=H.$$

If $k=k'$,
$$T=\frac{wl^2}{16k}=H.$$

If intermediate bracing is introduced, as shown in Fig. 650, the maximum tension (d_n) in the nth diagonal is produced when the panel live load L is concentrated at each panel-point from $n+1$ to $N-1$, while the maximum compressive stress (d_n') in the same diagonal is due to a panel load L concentrated at each panel-point from 1 to n.

To find these stresses, consider as before the equilibrium of each of the two portions of the truss made by a vertical plane MN immediately on the left of the $(n+1)$th vertical PQ. The two tangents at P and Q meet in a point E in the horizontal through A, the distance ER being $\dfrac{l^2-4x^2}{8x}$, where x is the horizontal distance of PQ from the vertex of each parabola.

Let p = perpendicular from E upon nth diagonal.

For maximum tension in nth diagonal, consider equilibrium of portion of truss between A and MN, and let R be the reaction at A. Then

$$RNa=(N-n-1)La\frac{N-n}{2}a, \quad \text{or} \quad R=\frac{La}{2}\frac{(N-n)(N-n-1)}{N}.$$

Also,

$$AE=\frac{\left(\frac{l}{2}-x\right)^2}{2x}=\frac{(n+1)^2a}{N-2n-2},$$

and

$$d_np=\frac{La}{2}\frac{(N-n)(N-n-1)}{N}\times AE=\frac{La}{2}\frac{(N-n)(N-n-1)(n+1)^2}{N(N-2n-2)}.$$

For maximum compression in nth diagonal, consider the equilibrium of the portion of the truss between B and MN, and let R' be the reaction at B.

Then

$$R'Na=\frac{n(n+1)}{2}La, \quad \text{or} \quad R'=\frac{L}{2}\frac{n(n+1)}{N},$$

and

$$BE=\frac{\left(\frac{l}{2}+x\right)^2}{2x}=a\frac{(N-n-1)^2}{N-2n-2}.$$

Therefore

$$d_n'p=R'\cdot BE=\frac{La}{2}\frac{n(n+1)(N-n-1)^2}{N(N-2n-2)}.$$

Hence

$$p(d_n-d_n')=\frac{La}{2}\frac{(N-n-1)(n+1)}{N-2n-2}.$$

Let θ_n be inclination of nth diagonal to the tangent at Q, and let α_n be the angle between EQ and the vertical.

Then
$$p = EQ \sin\theta_n = ER \sin\alpha_n \sin\theta_n.$$

But
$$ER = AE + (n+1)a = a\frac{(n+1)(N-n-1)}{N-2n-2}.$$

Hence
$$d_n - d_n' = \frac{La}{2} \operatorname{cosec} \alpha_n \operatorname{cosec} \theta_n.$$

Thus the vertical component of the diagonal stress

$$= \frac{La}{2}\frac{\cos(\alpha_n - \theta_n)}{\sin\alpha_n \sin\theta_n} = \frac{La}{2}(\cot\alpha_n \cot\theta_n + 1).$$

If $\alpha_n = 90°$, this component $= \frac{La}{2}$, as before.

Bowstring trusses of the type just described are now rarely used, but it is a common practice to specify that the panel-points in one of the chords are to lie in the arc of a parabola, the portions of the chord between consecutive panel-points being straight. The determination of the stresses in this case is very simple and is best made by the graphical method. With a truss of this type the diagonals are *unstrained* under a uniformly distributed load, and only come into play when a live load crosses the bridge. Take, for example, a ten-panel truss, Fig. 651, with panel dead and live loads of D and L respectively. The upper ends of the verticals are in the arc of a parabola with its vertex at the highest panel-point.

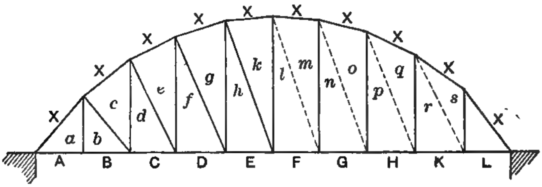


FIG. 651.

Fig. 652 is the dead-load stress diagram and the diagonal stresses are evidently *nil.* The maximum stresses in the chords are also produced when the live load covers the whole bridge, i.e., is uniformly distributed, and the magnitude of these live-load stresses may be found from the dead-load-stress diagram by multiplying the corresponding chord stress by the ratio $\frac{L}{D}$. The total maximum chord stresses may be found by using the ratio $\frac{L+D}{D}$ as the multiplier.

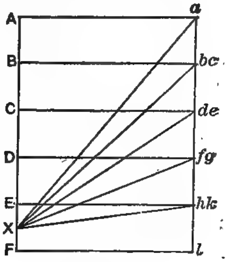


FIG. 652.

For each diagonal and vertical (excepting the vertical next a support) there is a different end reaction. Assume that a constant reaction of 1000 units at the left support is the only load on the truss, then the corresponding stress diagram is that shown by Fig. 653, and the stress in the diagonals and verticals due to the assumed reactions can be easily scaled off.

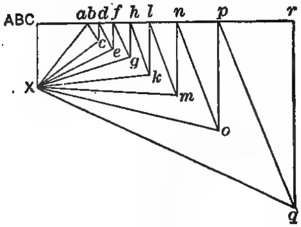


FIG. 653.

Again, the reactions at the left support are

$$\frac{L}{10}9\times5,\ \frac{L}{10}8\times4\tfrac{1}{2},\ \frac{L}{10}7\times4,\ \frac{L}{10}6\times3\tfrac{1}{2},\ \frac{L}{10}5\times3,$$

$$\frac{L}{10}4\times2\tfrac{1}{2},\ \frac{L}{10}3\times2,\ \frac{L}{10}2\times1\tfrac{1}{2},\ \frac{L}{10}1\times1,$$

when L is concentrated at the panel-points 2 to 9, 3 to 9, 4 to 9, 5 to 9, 6 to 9, 7 to 9, 8 to 9, and 9, respectively. Then

$$\frac{\text{the actual stress in a member}}{\text{the stress due to the assumed reaction}}=\frac{\text{the actual reaction}}{\text{the assumed reaction}}.$$

Again, it can be easily shown analytically that the diagonal stress in an N-panel truss under a uniformly distributed panel load is *nil.*

Let p, q be the lengths of the nth and $(n+1)$th verticals, and let x be the horizontal distance of the latter from the vertex.

Let the sloping chord member in the nth panel meet the horizontal through A in E, and take $AE=z$.

Then $$x=\left(\frac{N}{2}-n-1\right)a,$$

and therefore $$q=k\left(1-\frac{4x^2}{l^2}\right)=\frac{4k}{N^2}(n+1)(N-n-1),$$

also $$p=k\left(1-\frac{4\overline{x+a}^2}{l^2}\right)=\frac{4k}{N^2}n(N-n).$$

Hence $$\frac{z+(n+1)a}{z+na}=\frac{a}{p}=\frac{(n+1)(N-n-1)}{n(N-n)},$$

and $$z=a\frac{n(n+1)}{N-2n-1}.$$

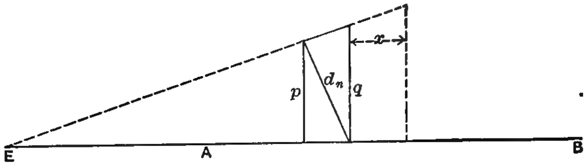


FIG. 654.

If p is the perpendicular from E upon d_n,

$$d_np=Rz-nP\left(\frac{n+1}{2}a+z\right),$$

where P is the panel load and R is the reaction at $A=\frac{N-1}{2}P$.

Hence

$$d_np=P\left\{\frac{N-1}{2}\frac{n(n+1)}{N-2n-1}a-n\left(\frac{n+1}{2}a+a\frac{n\cdot\overline{n+1}}{N-2n-1}\right)\right\}=0,$$

and therefore $$d_n=0.$$

9. Bowstring Girder with Isosceles Bracing.

Diagonal stresses due to the dead load.—Under a dead load the parabolic bow is equilibrated and the tie is subjected to a uniform tensile stress equal in amount to the horizontal thrust at the crown. The braces merely serve to transmit the load to the bow and are all ties.

Let T_1, T_2 be the tensile stresses in the two diagonals meeting at any panel-point n. Let θ_n, θ_n' be the inclinations of the diagonals to the horizontal.

Let D be the dead weight concentrated at each panel-point. Since the stress in the tie is the same from end to end, T_1, T_2 and D are in equilibrium.

Therefore $$T_1=\frac{D\cos\theta_n'}{\sin(\theta_n+\theta_n')} \quad\text{and}\quad T_2=\frac{D\cos\theta_n}{\sin(\theta_n+\theta_n')}.$$

Diagonal stress due to the live load (L).—Let d_n, d_n' be the diagonal stresses at the nth panel-point. Let x be the horizontal distance of p from the vertex C, and let the tangent at p meet the horizontal AB in E. Then

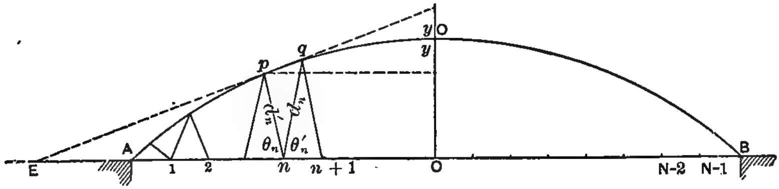


FIG. 655.

$$AE=\frac{(l-2x)^2}{8x} \quad\text{and}\quad BE=\frac{(l+2x)^2}{8x}.$$

Also, $$x=\frac{l}{2}-(n-\tfrac{1}{2})\frac{l}{N} \quad\text{and}\quad x-\frac{l}{N}=\frac{l}{2}-(n+\tfrac{1}{2})\frac{l}{N};$$

d_n is the *maximum tensile stress* when L is concentrated at all the panel-points from n to $N-1$.

The corresponding reaction at $A=\frac{L}{2}\frac{(N-n)(N-n+1)}{N}$.

Consider the equilibrium of the portion of the truss between A and a vertical section immediately on the right of p. Taking moments about E,

$$d_n\cdot En\sin\theta_n=\frac{L}{2}\frac{(N-n)(N-n+1)}{N}-AE.$$

Therefore $$d_n=\frac{L}{2}\frac{(N-n)(N-n+1)}{N}\frac{AE\operatorname{cosec}\theta_n}{AE+n\dfrac{L}{N}},$$

and the angle θ_n is given by

$$\tan\theta_n = \frac{2Nk}{l}\left(1 - \frac{4x^2}{l^2}\right).$$

d_n is the *maximum compressive stress* when L is concentrated at all the panel-points from 1 to $n-1$.

The corresponding reaction at B is now $= \frac{L}{2}\frac{n(n-1)}{N}$.

Consider the equilibrium of the portion of the truss between B and a vertical section immediately on the right of p. Taking moments about E,

$$d_n \cdot En \sin\theta_n = \frac{L}{2}\frac{n(n-1)}{N}BE.$$

Therefore
$$d_n = \frac{L}{2}\frac{n(n-1)}{N}\frac{BE \operatorname{cosec}\theta_n}{AE + n\frac{L}{N}}.$$

Again, d_n is the *maximum tensile stress* when L is concentrated at every panel-point from 1 to n.

The corresponding reaction at $B = \frac{L}{2}\frac{n(n+1)}{N}$.

Consider the equilibrium of the portion of the truss between B and a vertical section immediately on the left of q. Let the tangent at q meet the horizontal AB in E'. Taking moments about E',

$$d_n' = \frac{L}{2}\frac{n(n+1)}{N}\frac{BE' \operatorname{cosec}\theta_n'}{AE' + n\frac{L}{N}},$$

where
$$AE' = \frac{(l-2x)^2}{8x}, \qquad BE' = \frac{(l+2x)^2}{8x},$$

and
$$x = \frac{l}{2} - (n + \tfrac{1}{2})\frac{l}{N}.$$

The angle θ_n' is given by $\tan\theta_n' = \frac{2Nk}{l}\left(1 - \frac{4x^2}{l^2}\right)$.

d_n' is the *maximum compressive stress* when L is concentrated at all the panel-points from $n+1$ to $N-1$.

The corresponding reaction at $A=\frac{L}{2}\,\frac{(N-n)(N-n-1)}{N}$.

Consider the equilibrium of the portion of the truss between A and a vertical section immediately on the left of q. Taking moments about E',

$$d_n'=\frac{L}{2}\,\frac{(N-n)(N-n-1)}{N}\;\frac{AE'\operatorname{cosec}\theta_n'}{AE'+n\frac{L}{N}}.$$

It is a common practice to require that the panel-points in the curved chord lie in a parabola and that the portions of the chord between consecutive panel-points are straight. The maximum stresses in the several members can then be easily determined *graphically*. Fig. 657 is the stress diagram for the dead weight upon the truss represented by Fig. 656. The chord stresses are greatest when the live load covers the whole truss, and therefore the same diagram gives the maximum chord stresses due to the live load or

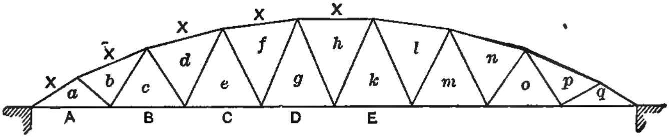


FIG. 656.

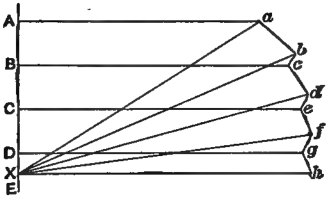


FIG. 657.

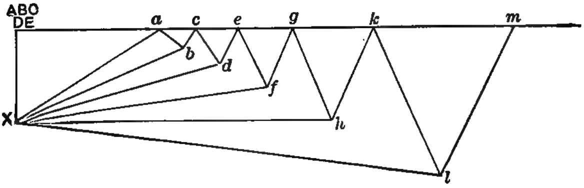


FIG. 658.

the maximum chord stresses due to the combined dead and live load by multiplying the corresponding stresses obtained from the dead load diagram by $\frac{L}{D}$ or $\frac{L+D}{D}$, respectively.

Again, assuming that a single vertical reaction XA, at the lef support, is the only force acting upon the truss, Fig. 658 is th stress diagram from which the maximum stresses in the diagonal can at once be found. Thus

$$\textit{actual stress } ef = \text{actual corresponding end reaction} \times \frac{\text{length of } ef}{\text{length of } XA}$$

Ex. 12. *Given a through eight-panel truss of* 80 *ft. span and* 10 *ft. rise, wit a parabolic compression chord, the panel live and dead loads being* 10 *and* 5 *ton respectively and the bracing isosceles as in Fig.* 656; *find the maximum stresse in the 3d and 4th diagonals.*

$$L=10 \text{ tons}; \quad N=8; \quad n=2; \quad l=80 \text{ ft.}; \quad k=10 \text{ ft.}$$

d_2 is a maximum tension when $x=25$ ft. Then, Fig. 655,

$$AE=4\tfrac{1}{2} \text{ ft.} \quad \text{and} \quad BE=84\tfrac{1}{2} \text{ ft.};$$

$$\tan\theta_2=\tfrac{39}{32}; \quad \theta_2=50° \, 38'; \quad \operatorname{cosec}\theta_2=1.2935.$$

Therefore

$$d_2(\textit{max. tens.})=\frac{10}{2}\,\frac{6\times7}{8}\,\frac{4\frac{1}{2}\times1.2935}{4\frac{1}{2}+.20}=6.24 \text{ tons}$$

and $$d_2(\textit{max. comp.})=\frac{10}{2}\,\frac{2\times1}{8}\,\frac{84\frac{1}{2}\times1.2935}{24\frac{1}{2}}=5.58 \text{ tons.}$$

d_2' is a maximum tension when $x=15$ ft. Then

$$AE'=\tfrac{250}{12} \text{ ft.}, \quad BE'=\tfrac{1210}{12} \text{ ft.};$$

$$\tan\theta_2'=\tfrac{55}{32}; \quad \theta_2'=59° \, 48'; \quad \operatorname{cosec}\theta_2'=1.157.$$

Therefore

$$d_n'(\textit{max. tens.})=\frac{10}{2}\,\frac{2\times3}{8}\,\frac{\frac{1210}{12}\times1.157}{\frac{250}{12}+20}=10.72 \text{ tons}$$

and $$d_n'(\textit{max. comp.})=\frac{10}{2}\,\frac{6\times5}{8}\,\frac{\frac{250}{12}\times1.157}{\frac{250}{12}+20}=11.07 \text{ tons.}$$

For the dead load stresses

$$T_1=\frac{5\cos 59^\circ\,48'}{\sin 110^\circ\,26'}\quad\text{and}\quad T_2=\frac{5\cos 50^\circ\,38'}{\sin 110^\circ\,26'}.$$

Therefore $T_1=2.684$ tons and $T_2=3.38$ tons.

10. Three-hinged Bridge.—Fig. 659 represents two bridge-trusses hinged to the abutments at A and B, and also hinged at an intermediate point C. An objection to this type of truss is the large deflection due to the hinge at C, but on the other hand it has the advantage of eliminating temperature stresses. If the trusses are inverted, the bridge is then of the suspension type. The stresses remain the same in magnitude, but are reversed in kind.

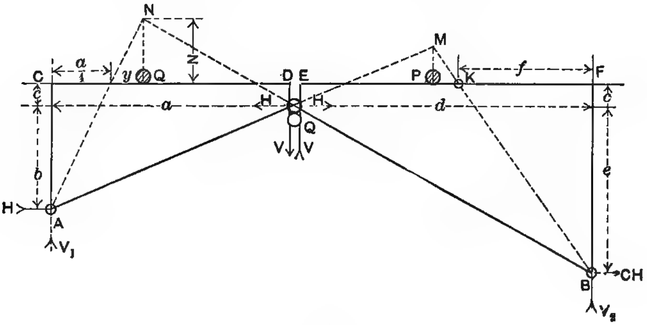


Fig. 659.

Any weight P concentrated as in Fig. 659 produces a reaction at A which, to prevent rotation at the hinges A and O, must pass along AO, intersecting the direction of P in M. Then, for equilibrium, the reaction at B must pass through M. Hence—

First. Every load on the horizontal member EF induces a stress in the sloping member AO, but has no effect upon the horizontal member CD or upon the bracing connecting CD with AO.

So every load on the horizontal member CD affects only the sloping member BO.

Second. If P is the resultant of a number of loads on EF, then BM is the direction of the resultant reaction at B, and the loads on the left of K tend to produce rotation from left to right, while those on the right of K tend to produced rotation from right to left, K being evidently the point of *no moment.*

For equilibrium of left portion, taking moments about A,

$$Va-Hb=0.$$

For equilibrium of right portion, taking moments about B,

$$Vd+He-Pf=0.$$

Also, $\qquad V_1-V=0$ and $V_2+V=P.$

These four equations give H, V, V_1, and V_2.

Note.—H and V are horizontal and vertical components of reaction at O; H and V_1 are horizontal and vertical components of reaction at A; H and V_2 are horizontal and vertical components of reaction at B.

Ex. 13. *To find point in CD at which a weight Q must be concentrated to make the B.M. at* $\frac{a}{4}$ *from C equal to nil.*

The resultant reaction at A must necessarily pass through the point in question, and its direction intersects the reaction along BO, arising from Q, in a point N. Then Q must be vertically below N. Also,

$$z=\frac{b+c}{\frac{1}{4}a}y \quad \text{and} \quad \frac{3}{4}a-y=\frac{d}{e}(z+c)=\frac{d}{e}\left\{\frac{4(b+c)}{a}y+c\right\},$$

an equation giving y.

Ex. 14. *Three-hinged bridge-trusses with horizontal top chords, straight sloping lower chords, of the dimensions and loaded as in Fig.* 660.

$$\tan\theta=\tfrac{1}{3};\quad \cot\alpha=\tfrac{6}{7};\quad \cot\beta=\tfrac{6}{5};\quad \cot\gamma=2;\quad \cot\delta=6.$$

Dead-load Stresses.—Since the dead load is symmetrically distributed, the

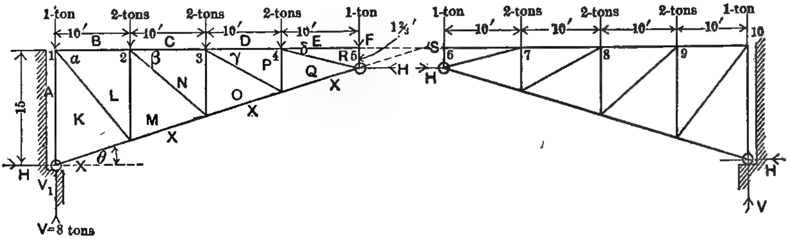


FIG. 660.

vertical reaction V at each side support is 8 tons, and there is no vertical shear at the central hinge.

Calling H the horizontal reaction at the hinges and taking moments about a side hinge,

$$-H\times 13\tfrac{1}{3}+8\times 20=0, \quad \text{or} \quad H=12 \text{ tons.}$$

Fig. 661, the stress diagram for the dead-load stresses, can now be drawn. The live load is three times the dead load, and hence, when the live load covers the whole bridge, the corresponding stresses in the several members are three times the stresses due to the dead load.

Method of Sections.—For the *chords and diagonals* consider, successively, the equilibrium of portions of the truss between the central hinge and vertical sections in the 1st, 2d, 3d, and 4th panels.

For the *verticals* consider, successively, the equilibrium of portions of the truss between the central hinge and sloping sections between the 1st and 2d, 2d and 3d, 3d and 4th diagonals.

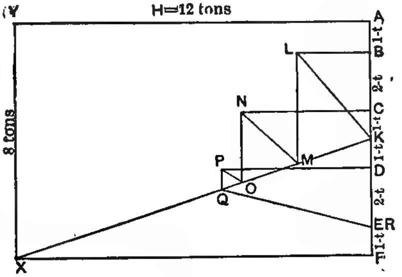


FIG. 661.

For horizontal chord panel stresses take moments about the foot of a vertical.
" sloping " " " " " " " top " "
" stresses in diagonals and verticals take moments about the point S, the point of intersection of the two chords produced. If x is the horizontal distance of the point from the central hinge,

$$\frac{x+40}{x}=\frac{15}{1\tfrac{2}{3}}=9 \quad \text{and} \quad x=5 \text{ ft.}$$

Then, denoting tensions by T and compressions by C, the stresses in tons are given by

$$\begin{array}{lrll}
BL\cdot 11\tfrac{2}{3}=H\cdot 10-2(10+20)-1\times 30 & =30 & \text{and} & BL=2\tfrac{4}{7}(C);\\
KX\cdot 15\cos\theta=H\cdot 1\tfrac{2}{3}+3\cdot 2\cdot 20+1\times 40 & =180 & \text{``} & XK=12\sec\theta=4\sqrt{10}(C);\\
KL\cdot 45\sin\alpha=-H\cdot 1\tfrac{2}{3}+3\cdot 2\cdot 25+1\times 5 & =135 & \text{``} & KL=3\operatorname{cosec}\alpha=\sqrt{85}(T);\\
CN\cdot 8\tfrac{1}{3}=H\cdot 6\tfrac{2}{3}+2\times 10-1\times 20 & =40 & \text{``} & CN=4\tfrac{4}{5}(C);\\
XM\cdot 11\tfrac{2}{3}\cos\theta=H\cdot 1\tfrac{2}{3}+2\cdot 2\cdot 15+1\times 30 & =110 & \text{``} & XM=\tfrac{66}{7}\sec\theta=\tfrac{22}{7}\sqrt{10}(C);\\
MN\cdot 35\sin\beta=-H\cdot 1\tfrac{2}{3}+2\cdot 2\cdot 20+1\times 5 & =65 & \text{``} & MN=\tfrac{13}{7}\operatorname{cosec}\beta=\tfrac{13}{35}\sqrt{61}(T);\\
DP\cdot 5=H\cdot 3\tfrac{1}{3}-1\times 10 & =30 & \text{``} & DP=6(C);\\
XO\cdot 8\tfrac{1}{3}\cdot\cos\theta=H\cdot 1\tfrac{2}{3}+2\times 10+1\times 20 & =60 & \text{``} & XO=\tfrac{36}{5}\sec\theta=\tfrac{12}{5}\sqrt{10}(C);\\
OP\cdot 25\sin\gamma=-H\cdot 1\tfrac{2}{3}+2\times 15+1\times 5 & =15 & \text{``} & OP=\tfrac{3}{5}\operatorname{cosec}\gamma=\tfrac{3}{5}\sqrt{5}(T);\\
ER\cdot 1\tfrac{2}{3}=0 & & \text{``} & ER=0;\\
XQ\cdot 5\cos\theta=H\cdot 1\tfrac{2}{3}+1\times 10 & =30 & \text{``} & XQ=6\sec\theta=2\sqrt{10}(C);\\
QR\cdot 15\sin\partial=-H\cdot 1\tfrac{2}{3}+1\times 5 & =-15 & \text{``} & QR=-\operatorname{cosec}\partial=-\sqrt{37}(C);\\
ML\cdot 35=-H.1\tfrac{2}{3}+3\cdot 2\cdot 25+1\times 5 & =135 & \text{``} & ML=3\tfrac{6}{7}(C);\\
ON\cdot 25=-H\cdot 1\tfrac{2}{3}+2\cdot 2\cdot 20+1\times 5 & =65 & \text{``} & ON=2\tfrac{3}{5}(C);\\
QP\cdot 15=-H\cdot 1\tfrac{2}{3}+2\times 15+1\times 5 & =15 & \text{``} & QP=1(C);\\
RF\cdot 5=1\times 5 & =5 & \text{``} & RF=1(C).
\end{array}$$

II. *Live Loads.*—Every load on the right truss induces a stress along the sloping chord of the left truss and therefore affects the members of this chord only. For all other members of the left truss the stresses remain the same whether the right truss is loaded or unloaded.

The panel live load is 6 tons, or three times the panel dead load. The values of V and H for any distribution of the load may be easily found by the method already described.

Maximum Stresses in Sloping Chords.—The moments of the stresses in XK, XM, XO, and XQ, with reference to the points 1, 2, 3, and 4 respectively, are severally equal in magnitude to the moments with reference to the same points of the resultant reactions through a. If a reaction passes through a panel-point in the upper chord, its moment with respect to this point is *nil*, and therefore the resulting stress induced in the corresponding member of the sloping chord is also *nil*.

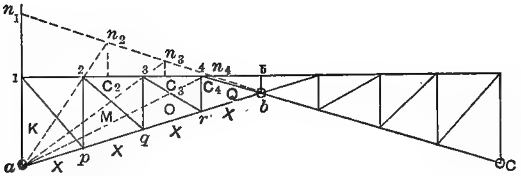


FIG. 662.

Let cb produced meet $a1$, $a2$, $a3$, $a4$ produced in the points n_1, n_2, n_3, n_4. The reaction at c_2 due to a load vertically below n_2 evidently passes through 2 and has no moment about 2, so that the corresponding stress in XM is *nil*. Similarly loads at 1, c_3, and c_4, respectively, produce no corresponding stresses in XK, XO, and XQ. Hence

all loads on the *right* of 1 tend to produce rotation from left to right and induce *compression* in XK;

all loads on the *right* of c_2 tend to produce rotation from left to right and induce *compression* in XM;

all loads on the *right* of c_3 tend to produce rotation from left to right and induce *compression* in XO;

all loads on the *right* of c_4 tend to produce rotation from left to right and induce *compression* in XQ;

all loads on the *left* of 1 tend to produce rotation from left to right and induce *tension* in XK;

all loads on the *left* of c_2 tend to produce rotation from left to right and induce *tension* in XM;

all loads on the *left* of c_3 tend to produce rotation from left to right and induce *tension* in XO;

all loads on the *left* of c_4 tend to produce rotation from left to right and induce *tension* in XQ.

TABLE OF MAXIMUM STRESSES IN SLOPING CHORD.

Member.	Panel-points Loaded for Max. Stress.	Maximum Stresses.
XK	1 to 10	$V=0;\quad H=36;$ $XK.15\cos\theta=H.1\frac{2}{3}+4.6.20$ and $XK=12\sqrt{10}$ tons = max. *comp.*
XK	0	$V=0;\quad H=0;$ no tension in XK
XM	3 to 10	$V=\frac{3}{4};\quad H=33\frac{3}{4};$ $XM.11\frac{2}{3}\cos\theta=H.1\frac{2}{3}+V.30+2.6.15+3\times30$ and $XM=\frac{279}{28}\sqrt{10}$ tons = max. *comp.*
XM	1 to 2	$V=-\frac{3}{4};\quad H=2\frac{1}{4};$ $XM.11\frac{2}{3}\cos\theta=-H.1\frac{2}{3}-V.30$ and $XM=\frac{15}{28}\sqrt{10}$ tons = max. *tens.*
XO	4 to 10	$V=2\frac{1}{4};\quad H=29\frac{1}{4};$ $XO.8\frac{1}{3}\cos\theta=H.1\frac{2}{3}+V.20+6\times10+3\times20$ and $XO=\frac{171}{20}\sqrt{10}$ tons = max. *comp.*
XO	1 to 3	$V=-2\frac{1}{4};\quad H=6\frac{3}{4};$ $XO.8\frac{1}{3}\cos\theta=-H.1\frac{2}{3}-V.20$ and $XO=\frac{27}{10}\sqrt{10}$ tons = max. *tens.*
XQ	5 to 10	$V=4\frac{1}{2};\quad H=22\frac{1}{2};$ $XQ.5\cos\theta=H.1\frac{2}{3}+V.10+3\times10$ and $XQ=\frac{15}{2}\sqrt{10}$ tons = max. *comp.*
XQ	1 to 4	$V=-4\frac{1}{2};\quad H=13\frac{1}{2};$ $XQ.5\cos\theta=-H.1\frac{2}{3}-V.10$ and $XQ=\frac{3}{2}\sqrt{10}$ tons = max. *tens.*

Hence

total maximum stress $XK=4\sqrt{10}+12\sqrt{10}=16\sqrt{10}$ tons;
" " " $XM=\frac{22}{7}\sqrt{10}+\frac{279}{28}\sqrt{10}=\frac{367}{28}\sqrt{10}$ "
" " " $XO=\frac{12}{5}\sqrt{10}+\frac{171}{20}\sqrt{10}=\frac{219}{20}\sqrt{10}$ "
" " " $XQ=2\sqrt{10}+\frac{15}{2}\sqrt{10}=\frac{19}{2}\sqrt{10}$ "

These stresses are all compressions and there are never any tensile stresses in any member of the sloping chords.

Maximum Live-load Stresses in Members of Horizontal Chord.—The loads at 2, 3, 4 combined, severally, with the corresponding reactions along *cb*, tend to cause a rotation from right to left with respect to the points *p*, *q*, *r*, respectively, and this tendency must be equilibrated by a tendency of the stresses in the members 12, 23, 34 to cause rotation from left to right, with respect to the same points. The stresses in these members are therefore always compressive and are greatest when the live load covers the left truss.

These stresses, again, are unaffected, as already shown, by loads on the right truss which produce reactions along *ab* only. It may therefore be assumed that the truss is loaded or unloaded. Assuming that the live load covers the whole bridge, the maximum compressive stresses in 12, 23, 34 are three times the corresponding stresses due to the *dead* load. Thus

max. $BL=3\times2\frac{4}{7}=7\frac{5}{7}$ tons;
" $CN=3\times4\frac{4}{5}=14\frac{2}{5}$ "
" $DP=3\times6=18$ "
" $ER=0$

As a verification, find CN on the assumption that the right truss is unloaded. Then

$$V\cdot 40+H\cdot 13\tfrac{1}{3}=4\cdot 6\cdot 20=480, \quad\text{or}\quad 3V+H=36;$$

$$V\cdot 40-H\cdot 13\tfrac{1}{3}=0, \qquad\qquad \text{or}\quad 3V-H=0.$$

Therefore $V=6$ tons and $H=18$ tons.

Hence $CN\cdot 8\tfrac{1}{3}=V\cdot 20+H\cdot 6\tfrac{2}{3}-6\cdot 10-3\cdot 20=120,$

and $CN=\frac{360}{25}=14.4$ tons.

Maximum Live-load Stresses in Diagonals.—Assume right truss unloaded, as the loads on this truss produce reactions along ab only and do not affect the diagonal stresses.

Let the horizontal and sloping chords be produced to meet in the point S, Fig. 660, which is 5 ft. from the panel-point 5. Then the stresses in the several diagonals are obtained by taking moments about S. These stresses are tabulated as follows:

TABLE OF MAXIMUM LIVE-LOAD STRESSES IN DIAGONALS.

Member.	Panel-points Loaded for Max. Stress.		Maximum Stresses.
KL	2 to 5	 and	$V=6;\quad H=18;$ $KL.45\sin\alpha=3.6.25+3\times 5-H.1\tfrac{2}{3}-V.5$ $KL=\frac{9}{7}\sqrt{85}$ tons = max. *tens.*
KL	1		$V=0;\quad H=0:$ no compression in KL.
MN	3 to 5	 and	$V=5\tfrac{1}{4};\quad H=15\tfrac{3}{4};$ $MN.35\sin\beta=2.6.20+3\times 5-H.1\tfrac{2}{3}-V.5$ $MN=\frac{81}{70}\sqrt{61}$ tons = max. *tens.*
MN	2	 and	$V=\tfrac{3}{4};\quad H=2\tfrac{1}{4}:$ $MN.35\sin\beta=H.1\tfrac{2}{3}+V.5$ $MN=\frac{3}{70}\sqrt{61}$ tons = max. *comp.*
OP	4 to 5	 and	$V=3\tfrac{3}{4};\quad H=11\tfrac{1}{4};$ $OP.25\sin\gamma=6.15+3\times 5-H.1\tfrac{2}{3}-V.5$ $OP=\frac{27}{10}\sqrt{5}$ tons = max. *tens.*
OP	2 to 3	 and	$V=2\tfrac{1}{4};\quad H=6\tfrac{3}{4}:$ $OP.25\sin\gamma=H.1\tfrac{2}{3}+V.5$ $OP=\frac{9}{10}\sqrt{5}$ tons = max. *comp.*
QR	5	 and	$V=1\tfrac{1}{2};\quad H=4\tfrac{1}{2}:$ $QR.15\sin\delta=3.5-H.1\tfrac{2}{3}-V.5$ $QR=0.$
QR	2 to 4	 and	$V=4\tfrac{1}{2};\quad H=13\tfrac{1}{2};$ $QR.15\sin\delta=H.1\tfrac{2}{3}+V.5$ $QR=3\sqrt{37}$ tons = max. *comp.*

Hence *the total maximum stresses in* tons are

$$KL=\tfrac{9}{7}\sqrt{85}+\tfrac{3}{7}\sqrt{85}=\tfrac{12}{7}\sqrt{85}, \text{ a tension;}$$
$$MN=\tfrac{81}{70}\sqrt{61}+\tfrac{13}{35}\sqrt{61}=\tfrac{107}{70}\sqrt{61}, \text{ a tension;}$$
$$OP=\tfrac{27}{10}\sqrt{5}+\tfrac{3}{5}\sqrt{5}=\tfrac{33}{10}\sqrt{5}, \text{ a tension;}$$

and $OP=\frac{9}{10}\sqrt{5}-\frac{3}{5}\sqrt{5}=\frac{3}{10}\sqrt{5}$, a compression;
$QR=3\sqrt{37}-(-\sqrt{37})=4\sqrt{37}$, a compression.

Maximum Live-load Stresses in Verticals.—It is assumed that there are no loads on the right truss, as such loads produce reactions along *ab* only and do not therefore affect the stresses in the verticals. Thus, from the right truss, $V\cdot 40-H\cdot 13\frac{1}{3}=0$, or $H=3V$.

The sloping planes (Method of Sections, p. 733) divide the loads on the left truss into *two* groups. The group on the right combined with the corresponding reaction along *cb* tends to produce, *with respect to the point S*, a rotation from *right to left* which must be equilibrated by an equal tendency to rotation from *left to right* due to a *compressive* stress in the vertical cut by the sloping plane. So the stress in this vertical developed by the group *on the left* must be a *tension.*

TABLE OF MAXIMUM VERTICAL STRESSES.

Member.	Panel-points Loaded for Max. Stress.	Maximum Stresses.
AK	1 to 5	$V=6;\quad H=18;$ $AK.45=4.6.25-H.1\frac{2}{3}$ and $AK=12$ tons $=$ max. *comp.*
AK	0	0
LM	2 to 5	$V=6;\quad H=18;$ $LM.35=3.6.25+3.5-H.1\frac{2}{3}-V.5$ and $LM=11\frac{4}{7}$ tons $=$ max. *comp.*
LM	1	0
NO	3 to 5	$V=5\frac{1}{4};\quad H=15\frac{3}{4};$ $NO.25=2.6.20+3.5-H.1\frac{2}{3}-V.5$ and $NO=8\frac{1}{10}$ tons $=$ max *comp.*
NO	2	$V=\frac{3}{4};\quad H=2\frac{1}{4};$ $NO.25=H.1\frac{2}{3}+V.5$ and $NO=\frac{3}{10}$ tons $=$ max *tens.*
PQ	4 to 5	$V=3\frac{3}{4};\quad H=11\frac{1}{4};$ $PQ.15=6.15+3.5-H.1\frac{2}{3}-V.5$ and $PQ=4\frac{1}{2}$ tons $=$ max. *comp.*
PQ	2 to 3	$V=2\frac{1}{2};\quad H=6\frac{3}{4};$ $PQ.15=H.1\frac{2}{3}+V.5$ and $PQ=1\frac{1}{2}$ tons $=$ max. *tens.*
RF	5	$V=1\frac{1}{2};\quad H=4\frac{1}{2};$ $RS.5=3.5$ and $RS=3$ tons $=$ max. *comp.*
RF	2 to 4	$V=4\frac{1}{2};\quad H=13\frac{1}{2};$ $PQ.5=0$ and $PQ=0.$

Hence the total maximum stresses are

$AK=4+12=16$ tons, a compression;
$LM=3\frac{6}{7}+11\frac{4}{7}=15\frac{3}{7}$ " "
$NO=2\frac{3}{5}+8\frac{1}{10}=10\frac{7}{10}$ " "
$PQ=1+4\frac{1}{2}=5\frac{1}{2}$ " "
and $PQ=1-1\frac{1}{2}=-\frac{1}{2}$ " a tension;
$RF=1+3=4$ "

Graphical Method.—It is much better, however, to use the graphical method in the determination of the stresses in trusses of this type. The following

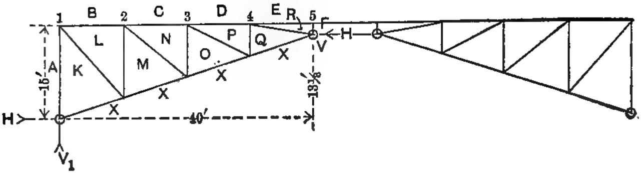


FIG. 663.

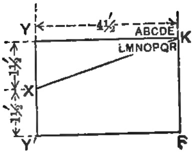


FIG. 664.

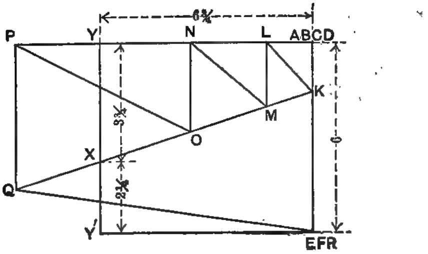


FIG. 665.

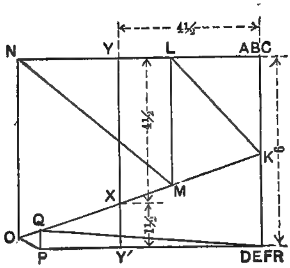


FIG. 666.

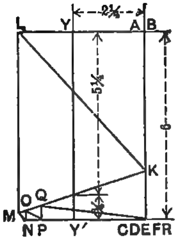


FIG. 667

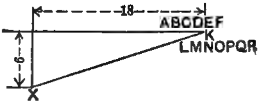


FIG. 668.

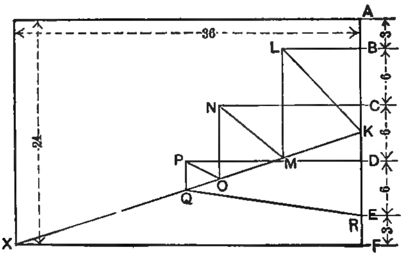


FIG. 669.

table and diagrams give the stresses corresponding to the different distributions of load:

Member.	Fig. 664.	Fig. 665.	Fig. 666.	Fig. 667.	Fig. 660.	Fig. 668.	Fig. 669.
	$3t$ at 5.	$6t$ at 4.	$6t$ at 3.	$6t$ at 2.	$3t$ at 1.	Loads 6 to 10.	Loads 1 to 10.
	$V=1\frac{1}{2}$, $H=4\frac{1}{2}$.	$V=2\frac{1}{4}$, $H=6\frac{3}{4}$.	$V=1\frac{1}{2}$, $H=4\frac{1}{2}$.	$V=\frac{3}{4}$, $H=2\frac{1}{4}$.	$V=0$, $H=0$.	$V=6$, $H=18$.	$V=24$, $H=36$.
XK	$-\frac{3}{2}\sqrt{10}$	$-\frac{9}{4}\sqrt{10}$	$-\frac{3}{2}\sqrt{10}$	$-\frac{3}{4}\sqrt{10}$	0	$-6\sqrt{10}$	$-12\sqrt{10}$
XM	$-\frac{3}{2}\sqrt{10}$	$-\frac{51}{28}\sqrt{10}$	$-\frac{9}{14}\sqrt{10}$	$\frac{15}{28}\sqrt{10}$	0	$-6\sqrt{10}$	$-\frac{66}{7}\sqrt{10}$
XO	$-\frac{3}{2}\sqrt{10}$	$-\frac{21}{10}\sqrt{10}$	$\frac{9}{10}\sqrt{10}$	$\frac{9}{20}\sqrt{10}$	0	$-6\sqrt{10}$	$-\frac{36}{5}\sqrt{10}$
XQ	$-\frac{3}{2}\sqrt{10}$	$\frac{3}{4}\sqrt{10}$	$\frac{1}{2}\sqrt{10}$	$\frac{1}{4}\sqrt{10}$	0	$-6\sqrt{10}$	$-6\sqrt{10}$
BL	0	$-1\frac{2}{7}$	$-2\frac{4}{7}$	$-3\frac{6}{7}$	0	0	$-7\frac{5}{7}$
CN	0	$-3\frac{3}{5}$	$-7\frac{1}{5}$	$-3\frac{3}{5}$	0	0	$-14\frac{2}{5}$
DP	0	-9	-6	-3	0	0	-18
ER	0	0	0	0	0	0	0
KL	0	$\frac{3}{14}\sqrt{85}$	$\frac{3}{7}\sqrt{85}$	$\frac{9}{14}\sqrt{85}$	0	0	$\frac{9}{7}\sqrt{85}$
MN	0	$\frac{27}{70}\sqrt{61}$	$\frac{27}{35}\sqrt{61}$	$-\frac{3}{70}\sqrt{61}$	0	0	$\frac{39}{35}\sqrt{61}$
OP	0	$\frac{27}{10}\sqrt{5}$	$-\frac{3}{5}\sqrt{5}$	$-\frac{3}{10}\sqrt{5}$	0	0	$\frac{9}{5}\sqrt{5}$
QR	0	$-\frac{3}{2}\sqrt{37}$	$-\frac{3}{2}\sqrt{37}$	$-\frac{1}{2}\sqrt{37}$	0	0	$-3\sqrt{37}$
AK	0	$-1\frac{1}{2}$	-3	$-4\frac{1}{2}$	-3	0	-12
LM	0	$-1\frac{13}{14}$	$-3\frac{6}{7}$	$-5\frac{11}{14}$	0	0	$-11\frac{4}{7}$
NO	0	$-2\frac{7}{10}$	$-5\frac{2}{5}$	$+\frac{3}{10}$	0	0	$-7\frac{4}{5}$
PQ	0	$-4\frac{1}{2}$	$+1$	$+\frac{1}{2}$	0	0	-3
RF	-3	0	0	0	0	0	-3

11. **Three-hinged Bridge** (arched or suspension) with the panel-points of the curved chord in the arc of a parabola having its vertex at the central hinge.

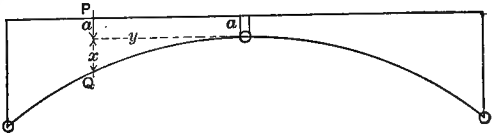


FIG. 670.

Let the whole bridge be covered with a uniformly distributed load of intensity w. There is no shear at the central hinge, but merely a horizontal reaction H.

Let h be the stress at any point P in the horizontal chord, vertically above the point Q in the bow, whose vertical and horizontal distances from the central hinge are x and y respectively, a being the central depth of the bridge.

Take moments about Q. Then

$$h(a+x)=Hx-\frac{wy^2}{2}.$$

But $$y^2 = \frac{l^2}{k}x \quad \text{and} \quad Hk = \frac{wl^2}{2}.$$

Therefore $$h(a+x) = \frac{wl^2}{2k}x - \frac{w}{2}\frac{l^2}{k}x = 0.$$

If the trusses are inverted, the bridge is one of the ordinary suspension type.

Thus *under a uniformly distributed load h is nil and there is no stress in the horizontal chord.*

Ex. 14. *A three-hinged eight-panel truss of* 80 *ft. span,* 15 *ft. deep at the supports,* $1\frac{2}{3}$ *ft. deep at the centre, and with the lower-chord panel-points lying in a parabola having its vertex at the central hinge. The panel dead and live loads are* 2 *and* 6 *tons respectively, Fig.* 671.

Then $5b = 1\frac{2}{3}$ft., $4r = 5$ft., $3q = 2\frac{1}{2}$ft., $2p = 9\frac{1}{6}$ft., and $1a = 15$ft.

Also, $\cot\alpha = \frac{7}{12}$, $\cot\beta = \frac{5}{12}$, $\cot\gamma = \frac{1}{4}$, $\cot\delta = \frac{1}{12}$.

Stress diagram for a uniformly distributed load.

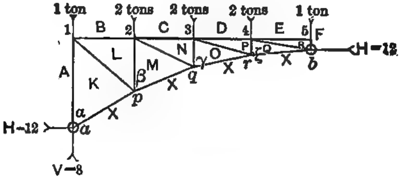


FIG. 671.

Fig. 672 is the stress diagram for a uniformly distributed panel dead load of 2 tons. The stresses in similar members for a panel live load of 6 tons are, of course, three times the corresponding stresses in the diagram, if the live load is uniformly distributed.

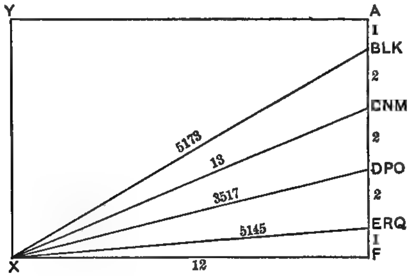


FIG. 672.

The diagram also shows that for uniformly distributed loads the stresses in the members of the horizontal chord and in the diagonals are always *nil.*

Method of Sections.—Maximum live-load stresses in horizontal chord. Loads at the points m_1, m_2, m_3 which are vertically below the points n_1, n_2, n_3 in which the members dp, pq, qr, produced, intersect the line of reaction cb, develope no stress in the members BL, CN, DP, respectively. Thus

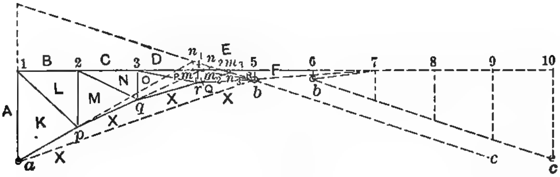


FIG. 673.

For the left truss every load on the left of m_1, combined with the corresponding reaction along cb, tends to produce a rotation from *right to left*, with respect to the point p, and must be equilibrated by an equal tendency to rotation from *left to right*, with respect to the same point p, due to a *compression* in BL, and this compression is greatest when the live load is concentrated on the panel-points 1, 2, 3. So, every load from m_1 to 5, combined with the corresponding reaction along cb, tends to produce a rotation from *left to right* with respect to the point p, and developes a tension in BL.

For the right truss every load from 6 to 10 causes a reaction from b to a and tends to produce a rotation from *left to right* with respect to the point p. A *tension* is therefore developed in BL which is greatest when every panel-point from 4 to 10 is loaded.

Again, the compression in CN is greatest when the live load is concentrated on panel-points from 1 to 4.

TABLE OF STRESSES DEVELOPED BY A UNIFORMLY DISTRIBUTED LOAD.

Member.	Panel-points Loaded for Max. Stress.	Maximum Stresses.
BL	1 to 3	$V=2\frac{1}{4}$; $H=6\frac{3}{4}$; $BL.9\frac{1}{6}=V.30+H.7\frac{1}{2}-6\times 10$ and $BL=6\frac{15}{44}$ tons = max. *comp.*
BL	4 to 10	$V=2\frac{1}{4}$; $H=29\frac{1}{4}$; $BL.9\frac{1}{6}=V.30-H.7\frac{1}{2}+6.30+3\times 30$ and $BL=6\frac{15}{44}$ tons = max. *tens.*
CN	1 to 4	$V=4\frac{1}{2}$; $H=13\frac{1}{2}$; $CN.5=V.20+H.3\frac{1}{3}-6\times 10$ and $CN=15$ tons = max. *comp.*
CN	5 to 10	$V=4\frac{1}{2}$; $H=22\frac{1}{2}$; $CN.5=V.20-H.3\frac{1}{3}+3\times 20$ and $CN=15$ tons = max. *tens.*
DP	1 to 4	$V=4\frac{1}{2}$; $H=13\frac{1}{2}$; $DP.2\frac{1}{2}=V.10+H.\frac{5}{8}$ and $DP=22\frac{1}{2}$ tons = max. *comp.*
DP	5 to 10	$V=4\frac{1}{2}$; $H=22\frac{1}{2}$; $DP.2\frac{1}{2}=V.10-H.\frac{5}{8}+3\times 10$ and $DP=22\frac{1}{2}$ tons = max. *tens.*

Maximum live-load stresses in curved chord.

$$\cot\alpha=\tfrac{7}{12},\quad \operatorname{cosec}\alpha=\tfrac{1}{12}\sqrt{193};$$
$$\cot\beta=\tfrac{5}{12},\quad \operatorname{cosec}\beta=\tfrac{13}{12};$$
$$\cot\gamma=\tfrac{1}{4},\quad \operatorname{cosec}\gamma=\tfrac{1}{4}\sqrt{17};$$
$$\cot\delta=\tfrac{1}{12}\quad \operatorname{cosec}\delta=\tfrac{1}{12}\sqrt{145}.$$

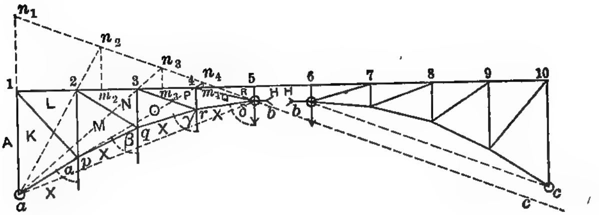


FIG. 674.

For the *right truss* every load from 6 to 10 causes a reaction from b to a which tends to produce a *left-handed* rotation with respect to the points 1, 2, 3, 4, and 5. This must be equilibrated by a *right-handed* rotation due to *compressive* forces developed in the members XK, XM, XO, and XQ.

Again, n_1, n_2, n_3, n_4 are the points in which cb produced is intersected by $a1$, $a2$, $a3$, $a4$ produced, and loads at 1, m_2, m_3, m_4, vertically below the points, produce no stress in the members XK, XM, XO, and XQ, respectively. Hence for the left truss loads on the right of 1 develope compressive stresses in XK, and the stresses in XM, XO, XQ are compressive or tensile according as the loads are on the right or left of m_2, m_3, m_4, respectively.

TABLE OF MAXIMUM LIVE-LOAD STRESSES IN MEMBERS OF CURVED CHORD.

Member.	Panel-points Loaded for Max. Stress.	Maximum Stresses.
XK	2 to 10	$V=0;\quad H=36;$ $XK.15\sin\alpha=4.6.20+H.1\frac{2}{3}$ and $XK=3\sqrt{193}$ tons = max. *comp.*
XK	0	0
XM	3 to 10	$V=\frac{3}{4};\quad H=33\frac{3}{4};$ $XM.9\frac{1}{6}\sin\beta=2.6.15+3\times30+H.1\frac{2}{3}+V.30$ and $XM=41\frac{19}{88}$ tons = max. *comp.*
XM	1 to 2	$V=\frac{3}{4};\quad H=2\frac{1}{4};$ $XM.9\frac{1}{6}\sin\beta=V.30-H.1\frac{2}{3}$ and $XM=2\frac{19}{88}$ tons = max. *tens.*
XO	4 to 10	$V=2\frac{1}{4};\quad H=29\frac{1}{4};$ $XO.5\sin\gamma=6\times10+3\times20+H.1\frac{2}{3}+V.20$ and $XO=\frac{171}{16}\sqrt{17}$ tons = max. *comp.*
XO	1 to 3	$V=2\frac{1}{4};\quad H=6\frac{3}{4};$ $XO.5\sin\gamma=V.20-H.1\frac{2}{3}$ and $XO=\frac{27}{16}\sqrt{17}$ tons = max. *tens.*
XQ	5 to 10	$V=4\frac{1}{2};\quad H=22\frac{1}{2};$ $XQ.2\frac{1}{2}\sin\delta=3\times10+H.1\frac{2}{3}+V.10$ and $XQ=\frac{15}{4}\sqrt{145}$ tons = max. *comp.*
XQ	1 to 4	$V=4\frac{1}{2};\quad H=13\frac{1}{2};$ $XQ.2\frac{1}{2}\sin\delta=V.10-H.1\frac{2}{3}$ and $XQ=\frac{3}{4}\sqrt{145}$ tons = max. *tens.*

Maximum live-load stresses in diagonals. Consider the equilibrium of a portion of the left truss between vertical sections y_1y_1 and yy in the first panel and immediately on the right of the central hinge. The diagonal stress KL is found by taking moments about the point m_1 in which ap produced intersects the horizontal chord.

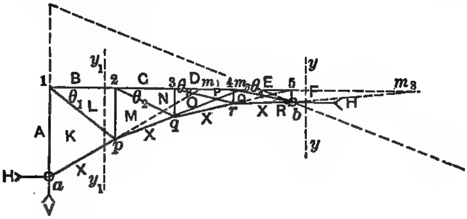


FIG. 675.

Every load on the left of m_1 combined with the corresponding reaction along cb gives a resultant which lies *above* the point m_1 and which therefore tends to produce a rotation from *left to right* with respect to m_1. This must be equilibrated by an equal tendency to rotation from *right to left*, with respect to the same point, and must therefore be produced by a tension in KL.

Every load from m_1 to 5, combined with the corresponding reaction along cb, gives a resultant which lies *below* m_1 and therefore tends to produce, with respect to this point, a rotation from *right to left*, developing a *compression* in KL.

Also, every load on the right truss causes a reaction from b to a which lies *below* the point m_1, and the resulting stress in KL is therefore a compression.

Again, let pq, qr, rb be produced to meet the horizontal chord in the points m_2, 5, and m_3, and let x_1, x_2, x_5, and x_3 be the distances of m_1, m_2, 5, and m_3, respectively, from 1. Then

$$\frac{x_1-10}{x_1}=\frac{9\frac{1}{6}}{15},\quad \frac{x_2-20}{x_1-10}=\frac{5}{9\frac{1}{6}},\quad \frac{x_5-30}{x_5-20}=\frac{2\frac{1}{2}}{5},\quad \frac{x_3-40}{x_3-30}=\frac{1\frac{2}{3}}{2\frac{1}{2}},$$

and $$x_1=25\tfrac{5}{7}\text{ ft.},\qquad x_2=32\text{ft.},\qquad x_5=40\text{ ft.},\qquad x_3=60\text{ ft.}$$

Consider the equilibrium of portions of the truss between yy and vertical sections y_2y_2, y_3y_3, y_4y_4 in the 2d, 3d, and 4th panels, respectively.

Every load on the right truss produces a reaction along ba which falls *below* the points m_2 and 5, and above the point m_3, so that the tendency to rotation with respect to the points m_2 and 5 is from *left to right*, while with respect to m_3 it is from *right to left.* Thus the corresponding stresses developed in MN and QP must be *compressions*, while the stress in GR must be a *tension.*

Again, every load on the left of the points m_2, 5, m_3, combined respectively with the corresponding reactions along cb, give resultants which fall above the points m_2, 5, and m_3, and therefore the tendency to rotation with respect to these points is from right to left and the corresponding stresses developed in MN, OP, and QR are compressions.

So, also, the stresses developed in the diagonals MN, OP, and QR, respectively, by loads between m_2 and 5 are tensile.

Finally, when the whole bridge is loaded equably, the diagonal stresses are *nil.* Hence, in any diagonal, the maximum live-load compression must be equal in magnitude to the maximum tensile load.

Graphical Method.—The stresses in the several members may be obtained more easily by drawing the stress diagram for each distribution of the load and then superposing the results as follows:

Member.	Fig. 677.	Fig. 678.	Fig. 679.	Fig. 680.	Fig. 681.	Fig. 0.	Fig. 682.
	$3t$ at 5.	$6t$ at 4.	$6t$ at 3.	$6t$ at 2.	Loads 6 to 10.	$3t$ at 1.	Loads 1 to 10.
	$V=1\frac{1}{2}$, $H=4\frac{1}{2}$.	$V=2\frac{1}{4}$, $H=6\frac{3}{4}$.	$V=1\frac{1}{2}$, $H=4\frac{1}{2}$.	$V=\frac{3}{4}$, $H=2\frac{1}{4}$.	$V=6$, $H=18$.	$V=0$, $H=0$.	$V=18$, $H=36$.
XK	$-\frac{3}{8}\sqrt{193}$	$-\frac{9}{16}\sqrt{193}$	$-\frac{3}{8}\sqrt{193}$	$-\frac{3}{16}\sqrt{193}$	$-\frac{3}{2}\sqrt{193}$	0	$-3\sqrt{193}$
XM	$-6\frac{9}{44}$	$-7\frac{47}{88}$	$-2\frac{29}{44}$	$2\frac{19}{88}$	$-24\frac{9}{11}$	0	-39
XO	$-\frac{15}{8}\sqrt{17}$	$-\frac{71}{16}\sqrt{17}$	$\frac{9}{8}\sqrt{17}$	$\frac{9}{16}\sqrt{17}$	$-\frac{15}{2}\sqrt{17}$	0	$-9\sqrt{17}$
XQ	$-\frac{3}{4}\sqrt{145}$	$\frac{3}{8}\sqrt{145}$	$\frac{1}{4}\sqrt{145}$	$\frac{1}{8}\sqrt{145}$	$-3\sqrt{145}$	0	$-3\sqrt{145}$
BL	$1\frac{5}{22}$	$\frac{9}{44}$	$-2\frac{1}{22}$	$-4\frac{13}{44}$	$4\frac{10}{11}$	0	0
CN	3	$-1\frac{1}{2}$	-9	$-4\frac{1}{2}$	12	0	0
DP	$4\frac{1}{2}$	$-11\frac{1}{4}$	$-7\frac{1}{2}$	$-3\frac{3}{4}$	18	0	0
ER	0	0	0	0	0	0	0
KL	$-\frac{9}{88}\sqrt{265}$	$-\frac{3}{176}\sqrt{265}$	$\frac{15}{88}\sqrt{265}$	$\frac{63}{176}\sqrt{265}$	$-\frac{9}{22}\sqrt{265}$	0	0
MN	$-\frac{39}{44}\sqrt{5}$	$\frac{75}{88}\sqrt{5}$	$\frac{153}{44}\sqrt{5}$	$\frac{9}{88}\sqrt{5}$	$-\frac{39}{11}\sqrt{5}$	0	0
OP	$-\frac{3}{8}\sqrt{17}$	$\frac{39}{16}\sqrt{17}$	$-\frac{3}{8}\sqrt{17}$	$-\frac{3}{16}\sqrt{17}$	$-\frac{3}{2}\sqrt{17}$	0	0
QR	$\frac{3}{4}\sqrt{37}$	$-\frac{15}{8}\sqrt{17}$	$-\frac{5}{8}\sqrt{17}$	$-\frac{5}{8}\sqrt{37}$	$3\sqrt{37}$	0	0
AK	$1\frac{1}{8}$	$\frac{3}{16}$	$-1\frac{7}{8}$	$-3\frac{15}{16}$	$4\frac{1}{2}$	-3	0
LM	$\frac{39}{44}$	$-\frac{75}{88}$	$-3\frac{21}{44}$	$-6\frac{9}{88}$	$3\frac{6}{11}$	0	0
NO	$\frac{3}{8}$	$-2\frac{7}{16}$	$-5\frac{5}{8}$	$\frac{3}{16}$	$1\frac{1}{2}$	0	0
PQ	$-\frac{3}{4}$	$-4\frac{1}{8}$	$1\frac{1}{4}$	$\frac{5}{8}$	-3	0	0
RF	-3	0	0	0	0	0	0

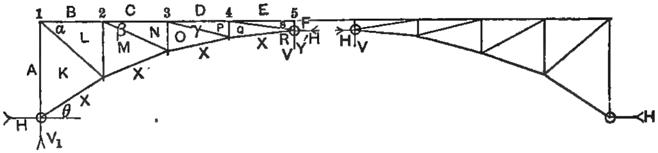


FIG. 676.

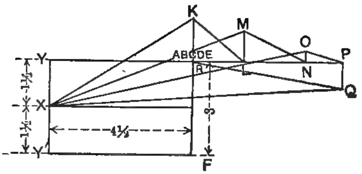


FIG. 677.

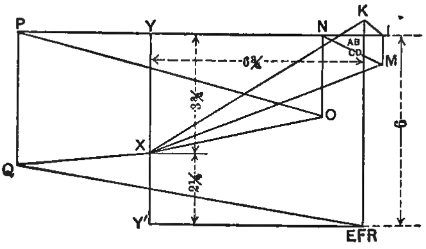


FIG. 678.

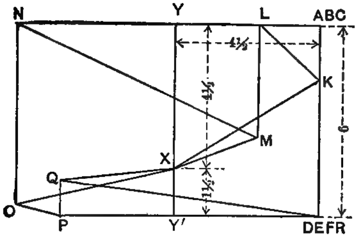


FIG. 679.

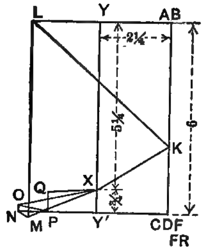


FIG. 680.

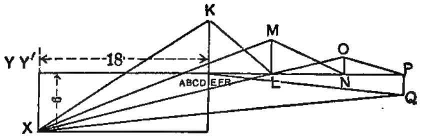


FIG. 681.

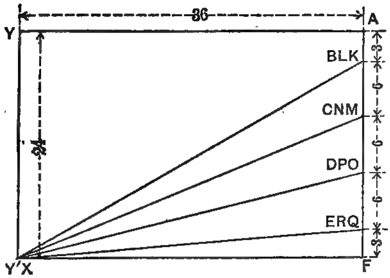


FIG. 682.

Ex. 16. *Show how to determine the maximum stresses in the verticals and diagonals of the trusses for the Sault Ste. Marie Bridge, Fig.* 624, *Ex.* 1.

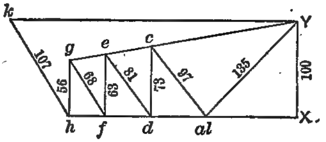


FIG. 683.

As already stated, the maximum stresses in the several members are best obtained graphically. Fig. 683 is the stress diagram when the only force on the truss is a vertical reaction at A of 100,000 lbs. $=XY$. Taking $XY=100$, so that each unit is 1000 lbs., the stresses in the other members are indicated on the figure.

The load distribution for which the stress cd is greatest is Case 3, the *actual reaction* at A being 124,230 lbs. Then

actual stress $cd=\frac{78}{100}(124{,}230)=90{,}688$ lbs.,
and " " $de=\frac{81}{100}(124{,}230)=100{,}627$ lbs.
So " " $fg=\frac{68}{100}(95{,}020)=64{,}614$ lbs.,
etc., etc.

12. Wind-pressure.—Numerous experiments to determine the pressure and velocity of the wind have been made by means of feathers, cloud-shadows, anemometers of various kinds, wind-gauges, pendulum, tube, and spring instruments. The results, either through errors of observation, errors of construction, or for other occult reasons, are almost wholly unreliable and give the engineer no accurate information upon which to base his calculations as to the effect of wind upon a structure. Theoretical investigations on the subject are equally unsatisfactory. The formulæ expressing the relations between the speed of the anemometer, the velocity of the wind and its pressure, are of a purely empirical character, and are only applicable to a specific series of recorded observations.

Smeaton inferred from Rouse's experiments that the average pressure in pounds per square foot = (velocity in miles per hour)^2 ÷ 200, or

$$P=\frac{V^2}{200}.$$

According to Dines the formula should be

$$P=\frac{7V^2}{2000}.$$

The Wind-pressure Commission (Eng.) recommended the formula

$$P=\frac{V^2}{100},$$

as giving with tolerable accuracy the *maximum* pressure.

Stokes considered that the *actual* wind velocities should be about four fifths of the values recorded by anemometers, so that a velocity of 64 miles per hour recorded as corresponding to a *maximum* pressure of 40.6 lbs. per square foot (the average of *five* observed pressures) would be reduced to 51.2 miles per hour. The *average* pressure corresponding to 51.2 miles per hour would be 13.1 lbs. per square foot according to Smeaton's rule and only 9.18 lbs. according to Dines.

Again, certain experiments at Greenwich indicated that the pressure was increased by the stiffness of the copper wire connecting the recording pencil with the pressure plate, and a flexible brass chain was therefore substituted for the wire. Thus modified, a pressure of 29 lbs. per square foot was registered as corresponding to a velocity of 64 miles per hour, whereas with the copper wire a pressure of 49½ lbs. per square foot had been registered with a velocity of only 53 miles per hour.

These facts tend to show that the *actual* pressure is much less than that given by a recording instrument, and that the very high pressures, as, e.g., 80 lbs. per square foot and even more, must be due to gusts or squalls having a purely local effect. This opinion seems to be confirmed by Sir B. Baker's experiments at the Forth Bridge, which also indicate that the pressure per square foot diminishes as the area acted upon increases. No engineering structure could withstand a pressure to 80 lbs. per square foot of surface, and a pressure of 28 lbs. to 32 lbs. would overturn carriages, drive trains from the track, and stop all traffic.

It is, of course, well known that wind-forces sufficiently powerful to uproot huge trees and to demolish the strongest buildings are occasionally developed by whirlwinds, tornadoes, and cyclones, but these must be classed as *acta Dei* and can scarcely be considered by an engineer in his calculations.

Numerous observations as to the effect of wind upon structures in different localities must yet be made before any useful and reliable rules can be enunciated. In the case of existing bridges the elongation of the wind-braces during a storm can easily be measured within $\frac{1}{1000}$ of an inch. Investigations should be made as to the action of the wind upon surfaces of different forms and upon sheltered surfaces, as, e.g., upon the surfaces behind the windward face in bridge-trusses. Again, it is quite possible, if not probable, that many of the recorded upsets have been due to a *combined* lifting and side action, requiring a much less flank pressure than would be necessary if there were no upward force, and hence further light should be obtained on this point.

Under any circumstances, the wind-stresses should be as small as possible, compatible with safety, seeing how largely they influence the sections of the several members, especially in bridges of long span.*

13. Empirical Regulations.

Wind-pressure Commission Rules.—For railway bridges and viaducts assume a maximum pressure of 56 lbs. per square foot upon an area to be estimated as follows:

A. In *close*-girder bridges or viaducts the area
=area of windward face of girder
+area of train surface *above* the top of the same girder.

---

* For most recent researches see paper by Stanton, Proc. Inst. of C. E. (Eng.) Vol. 156.

B. In *open*-girder bridges or viaducts the area for the *windward* girder
$=$area of windward face, *assumed close*, between rails and top of train
$+$*calculated* area of windward surface *above* the top of the train
$+$*calculated* area of windward surface *below* the rails.

For the leeward girder or girders the area
$=$calculated area of surface of *one* girder above the top of the train and below the level of the rails, the pressure being 28, 42, or 56 lbs. per square foot, according as this area $<\frac{2}{3}S$, $>\frac{2}{3}S$ and $<\frac{3}{4}S$, or $>\frac{3}{4}S$, where S is the total area within the outline of the girder. The assumed factor of safety is to be 4.

American Specifications.—(*a*) The lateral bracing *in the plane of the roadway* is to be designed so as to bear a pressure of 30 lbs. per square foot upon the vertical surface of one truss and upon the surface of a train averaging 12 sq. ft. per lineal foot, i.e., 360 lbs. per lineal foot; this latter is to be regarded as a *live* load. The lateral bracing *in the plane of the other chord* is to be designed so as to bear a pressure of 50 lbs. per square foot upon *twice* the vertical surface of one truss.

(*b*) The portal, vertical, and horizontal bracing is to be proportioned for a pressure of 30 lbs. per square foot upon *twice* the vertical surface of one truss and upon the surface of a train averaging 10 sq. ft. per lineal foot, i.e., 300 lbs. per lineal foot, the latter being treated as a *live* load.

(*c*) Live load in plane of roadway due to wind-pressure $=300$ lbs. per lineal foot.
Fixed load in plane of roadway due to wind-pressure $=150$ lbs. per lineal foot.
Fixed load in plane of other chord due to wind-presure $=150$ lbs. per lineal foot.

Lateral Bracing.—Consider a truss-bridge with parallel chords and panels of length p. Let A be the area of the vertical surface of one truss.

According to (*a*) the lateral bracing in the plane of the roadway is subjected to (1) a panel live load of $360p$ lbs. and (2) a panel fixed load of 30A lbs., while in the plane of the other chord it is subjected to a panel fixed load of

$$50 \times 2A = 100A \text{ lbs.}$$

Thus, if the figure represent the bracing in the plane of the roadway of a ten-panel truss, and if the wind blow upon the side *AB*,

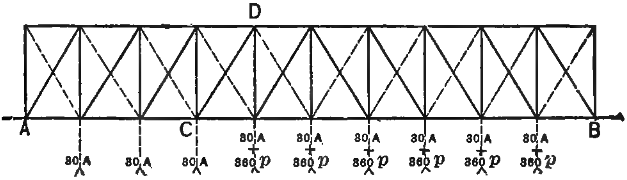


FIG. 684.

the *maximum* horizontal force for which any diagonal, e.g., *CD*, is to be designed is

$=45A$ lbs. due to the horizontal force of 30A lbs. at each panel-point
$+756p$ lbs. due to the horizontal force of 360p lbs. at each panel-point between *C* and *B*.

The dotted lines show the bracing required when the wind blows on the opposite side.

It is sometimes maintained that the wind-forces in the plane of the upper chords of a through-bridge or the lower chords of a deck-bridge are transmitted to the floor-bracing through the posts. This can hardly be correct in the case of long posts, as they do not possess sufficient stiffness. It has, however, been pointed out that, in through-bridges, the cumulative effect of the wind-pressure at the ends of the bridge might produce a serious bending action in the end posts This action would have to be resisted by additional plating on the end posts below the portals, or by an increase of their sectional area.

Under wind-pressure the floor-beams act as posts; hence, if the wind-bracing is attached to the top or compression flange of a floor-beam, the flange's sectional area must be proportionately increased. If the bracing is attached to the lower or tension flange, the stresses in the latter will be diminished.

14. Chords.—The wind-pressure transmitted through the floor-bracing increases the stresses in the several members, or panel lengths, of the leeward chord, the greatest increments being due to a horizontal force of $(360p+30A)$ lbs. at each of the panel-points in AB. The corresponding chord stresses in the ten-panel truss-bridge referred to above are:

$$C_1=0;$$
$$C_2=4\tfrac{1}{2}(360p+30A)\tan\theta \text{ lbs.};$$
$$C_3=C_2+3\tfrac{1}{2}(360p+30A)\tan\theta=8(360p+30A)\tan\theta \text{ lbs.};$$
$$C_4=C_3+2\tfrac{1}{2}(360p+30A)\tan\theta=10\tfrac{1}{2}(360p+30A)\tan\theta \text{ lbs.};$$
$$C_5=C_4+1\tfrac{1}{2}(360p+30A)\tan\theta=12(360p+30A)\tan\theta \text{ lbs.},$$

$90°-\theta$ being the angle between a diagonal and a chord.

Again, the wind-pressure tends to capsize a train and throws an additional pressure of $P\frac{y}{G}$ lbs. per lineal foot upon the leeward rail, P being the pressure in pounds per lineal foot on the train surface, y the vertical distance between the line of action of P and the top of the rails, and G the gauge of the rails.

Thus, the total pressure on leeward rail

$$=\left(\frac{w}{2}+P\frac{y}{G}\right) \text{ lbs. per lineal foot,}$$

and the total pressure on windward rail

$$=\left(\frac{w}{2}-P\frac{y}{G}\right) \text{ lbs. per lineal foot,}$$

w being the weight of the train in pounds per lineal foot.

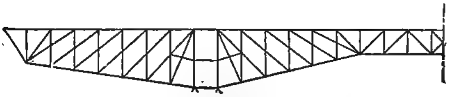
Hence the total vertical pressure at a panel-point of the leeward truss

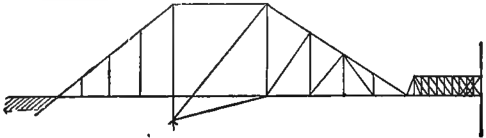
$$=\left(\frac{w}{2}+P\frac{y}{G}\right)p\frac{S+G}{2S}+\left(\frac{w}{2}-P\frac{y}{G}\right)p\frac{S-G}{2S}$$

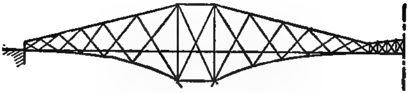
$$=\frac{w}{2}p+pP\frac{y}{G}\frac{G}{S}=\left(\frac{w}{2}p+pP\frac{y}{S}\right) \text{ lbs.,}$$

S being the distance between the trusses.

15. Cantilever Trusses.—A cantilever is a structure supported at one end only, and a bridge of which such a structure forms part may be called a cantilever bridge. Two cantilevers may project from the supports so as to meet, or a gap may be left between them which may be bridged by an independent girder resting upon or hinged to the ends of the cantilevers. The form of the cantilever is subject to considerable variation.


BRIDGE OVER ST. LAWRENCE AT NIAGARA.
FIG. 685.


SUKKUR BRIDGE
FIG. 686.


FORTH BRIDGE.
FIG. 687.

Figs. 688 to 693 represent the simplest forms of a cantilever frame. If the member *AB* has to support a uniformly distributed load as well as a concentrated load at *B*, intermediate stays may be introduced as shown by the full or by the dotted lines in Figs. 690 and 691. Should a live load travel over *AB*, each stay must be

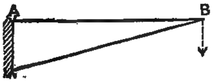


FIG. 688.

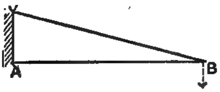


FIG. 689.

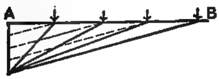


FIG. 690.

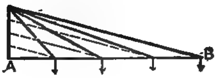


FIG. 691.

designed to bear with safety the maximum stress to which it may be subjected.

Figs. 692 and 693 show cantilever trusses with parallel chords. If the truss is of the double-intersection type, Fig. 693, the stresses in the members terminating in B become indeterminate. They may be made determinate by introducing a short link BD, Fig. 694. Thus, if, in DB produced, BG be taken to represent the resultant stress along the link, and if the parallelogram HK be completed, BK will represent the stress along BE, and BH that along BF.

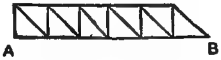


FIG. 692.

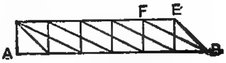


FIG. 693.

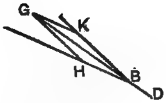


FIG. 694.

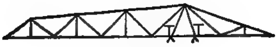


FIG. 695.

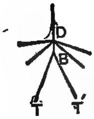


FIG. 696.

This link device has been employed to equalize the pressure on the turntable TT of a swing-bridge (Fig. 695). An "equalizer" or a "rocker-link" BD, Fig. 696, conveys the stresses transmitted through the members of the truss terminating in D to the centre posts BT.

Theoretically, therefore, the pressure over TT will be evenly distributed, whatever the loading may be, if the direction of BD bisects the angle TBT and if friction is neglected.

The joint between the central span and the cantilever requires the most careful consideration and should fulfil the following conditions:

(*a*) The two cantilevers should be free to expand and contract under changes of temperature.

(*b*) The central span should have a longitudinal support which will enable it to withstand the effect of the braking of a train or the pressure of a wind blowing longitudinally.

(*c*) The wind-pressure on the central span should bear equally on the two cantilevers.

(*d*) The connections at both ends should have sufficient lateral rigidity to check undue lateral vibration. Conditions (*a*) and (*c*) would be fulfilled by supporting the central span like an ordinary bridge-truss upon a rocker bolted down at one end and upon a rocker resting on expansion rollers at the other. This, however, would not satisfy condition (*b*). It is preferable to support the span by means of rollers or links at both ends, and to secure it to one cantilever only on the central line of the bridge with a large vertical pin, adapted to transmit all the lateral shearing force. A similar pin at the other end, free to move in an elongated hole, or some equivalent arrangement, as, e.g., a sleeve-joint bearing laterally and with rollers in the seat, is a satisfactory method of transmitting the shearing force at that end also. (If there is an end post, it may be made to act like a hinge so as to allow for expansion, etc.) The points of contrary flexure of the whole bridge under wind-pressure are thus fixed, and all uncertainty as to wind-stresses removed.

Where other spans have to be built adjacent to a large cantilever span, it should not be hastily assumed that it is necessarily best to counterbalance the cantilever by a contiguous cantilever in the opposite direction. If it is possible to obtain good foundations and if piers are not expensive, it might be cheaper to build a number of short independent side spans and to secure the cantilever to an independent anchorage. If this is done, care must be taken to give the abutment sufficient stability to take up the unbalanced thrust along the lower boom of the cantilever.

Suppose that the cantilever is anchored back by means of a single back-stay.

Let W = weight necessary to resist the pull of the back-stay;
h = depth of end post of cantilever;
z = horizontal distance between foot of post and anchorage;
M = bending moment at abutment $= Wz$.

If it is now assumed that the sectional areas of the post and back-stay are proportioned to the stresses they have to bear (which is never the case in practice), the quantity of material in these members must be proportional to

$$W\frac{z^2+h^2}{h}+Wh=W\frac{z^2+2h^2}{h}=M\frac{z^2+2h^2}{hz},$$

which is a minimum when $z=\sqrt{2}h$.

If a horizontal member is introduced between the feet of the back-stay and the post, the quantity of material becomes proportional to

$$W\frac{z^2+h^2}{h}+Wh+W\frac{z^2}{h}=2M\frac{z^2+h^2}{zh},$$

which is a minimum when $z=h$, i.e., when the back-stay slopes at an angle of 45°. By making the angle between the back-stay and the horizontal a little less than 45°, a certain amount of material may be saved in the joints of the back-stays and also in the anchors, which more than compensates for the increased weight of the anchors themselves.

(*Note.*—In these calculations it is assumed that the top chord is horizontal, and that the feet of the post and back-stay are in the same horizontal plane. This is rarely the case in practice.)

According to the above the weight of material necessary for the back-stay is *directly* proportional to the bending moment at the abutment and *inversely* proportional to the depth of the cantilever, other things being equal. A double cantilever has, in general, no anchorage of any great importance.

If the span is very great, a cantilever bridge usually requires less material than any other rigid structure of equal strength, even though anchorage may have to be provided. If two large spans are to be built, a double cantilever, requiring no anchorage, may effect a very considerable saving in material, although a double pier, of sufficient width for stability under all conditions of loading, will be necessary.

Again, where false-works are costly or impossible, the property of the cantilever that it can be made to support itself during erection gives it an immense advantage. If the design of the cantilever is such that it can be built out rapidly and cheaply, it will often be the most economical frame in the end, even if the total quantity of material is not so small as that required for some other type of bridge. In all engineering work *quantity of material* is only *one* of the elements of cost, and this should be carefully borne in mind when designing a cantilever bridge because a want of regard to the method of erection may easily add to its cost an amount much greater than can be saved by economizing material.

In ordinary bridge-trusses the amount of the web metal is greatest at the ends and least at the centre, while the amount of the chord metal is least at the ends and greatest at the centre. Thus the assumption of a uniformly distributed dead load for such bridges is, generally speaking, sufficiently accurate for practical purposes. In the case of cantilever bridges, however, the circumstances are entirely different. In these the amount of the metal both in the web and in the chords is greatest at the support and least at the end. For example, the weight of the cantilevers (exclusive of the weight of platform, viz., ½ ton per lineal foot) for the Indus Bridge, per lineal foot, varies from 6½ tons at the supports to 1 ton at the outer ends. Hence the hypothesis of a uniformly distributed dead load for such structures cannot hold good.

The weight of a cantilever for a given span may be approximately calculated in the following manner:

Determine the stresses in the several members, panel by panel—

(A) For a load consisting of
    (1) a given unit weight, say 100 tons, at the outer end;
    (2) the corresponding dead weight.

(B) For a load consisting of
    (1) the specified live load;
    (2) the corresponding panel dead weight.

Thus the whole weight of a panel will be the sum of the weights deduced in (A) and (B), and the total weight of the cantilever will be the sum of the several panel weights.

This process evidently gives at the same time the weights of cantilevers of one, two, three, etc., panel lengths, the loads remaining the same.

The panel dead weights referred to in (A) and (B) must, in the first place, be assumed. This can be done with a large degree of accuracy, as the dead weight must necessarily *gradually* increase towards the support, and any error in a particular panel may be easily rectified by subsequent calculations.

Again, the preceding remarks indicate a method of finding the most economical cantilever length in any given case.

Take, e.g., an opening spanned by two equal cantilevers and an intermediate girder. Having selected the type of bridge to be employed for the intermediate span, estimate, either from existing bridges or otherwise, the weights of independent bridges of the same type and of different spans. Sketch a skeleton diagram of the cantilever, extending over one half of the whole span, and apply to it the processes referred to in (A) and (B).

If L is the length of the cantilever and P that of a panel, the following table, in which the intermediate span increases by two panel lengths at a time, may be prepared:

Length of Intermediate Span.	Half Weight of Intermediate Span.	Weight on End of Cantilever from Intermediate Span.	Length of Cantilever.	Weight of Cantilever for each 100 Tons at its End.	Weight of Cantilever Due to Load at its End from Intermediate Span.	Weight of Cantilever Due to Specified Live Load and its Own Weight.	Total Weight of One-half Span.
0			L				
$2P$			$L-2P$				
$4P$			$L-4P$				
$6P$			$L-6P$				
$8P$			$L-8P$				
etc.			etc.				
1	2	3	4	5	6	7	8

Weight in col. 3 = *one half* of the weight of the intermediate girder + *one half* of the live load it carries if uniformly distributed. (The proportion will be greater than one half for arbitrarily distributed loads, and may be easily determined in the usual manner.)

Col. 5 gives the weights obtained as in A.

$$\text{Col. } 6 = \text{col. } 5 \times \frac{\text{weight on end of cantilever}}{100}.$$

Col. 7 gives the weights obtained as in B.

Col. 8 = col. 2 + col. 6 + col. 7.

It is important to bear in mind that an increase in the weight of the central span necessitates a corresponding increase in the weights of the cantilevers. Hence, in order that the weight of the structure may be a minimum, the best material with the highest practicable working unit stress should be employed for the centre span.

The table must of course be modified to meet the requirements of different sites. Thus, if anchorage is needed, a column may be added for the weights of the back-stays, etc.

Curve of Cantilever Boom.—Consider a cantilever with one horizontal boom OA, and let x, y be the coordinates of any point P in the other boom, O being the origin of coordinates and A the abutment end of the cantilever.

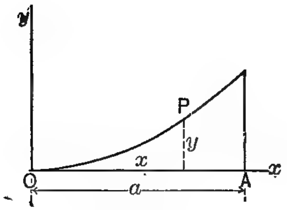


Fig. 697.

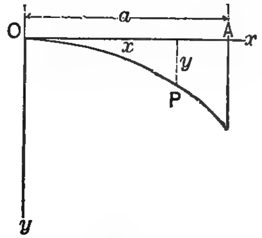


Fig. 698.

Let W be the portion of the weight of an independent span supported at O.

Let w be the *intensity* of the load at the vertical section through P.

Assume (1) that there are no diagonal strains, and, hence, that the web consists of vertical members only;

(2) that the stress H in the horizontal boom is constant, and therefore the bending moment at $P = Hy$;

(3) that the *whole* load is transmitted through the vertical members of the web.

Let k be such a factor that kTl is the weight of a member of length l, subjected to a stress T.

(*Note.*—If l is in feet and T in tons, then k for steel is about .0003, allowance being made for loss of section or increase of weight at connections.)

w consists of two parts; viz., a *constant part* p, due to the weight of the platform, wind-bracing, etc., which is assumed to be uniformly distributed; and a *variable part*, due to the weight of the cantilever, which may be obtained as follows:

Weight of element dx of horizontal boom $=kHdx$.
" " web corresponding to dx $=kwydx$.
" " element of curved boom corresponding to dx

$$=kH\left(\frac{ds}{dx}\right)^2dx.$$

Hence the variable intensity of weight

$$=kH+kwy+kH\left(\frac{ds}{dx}\right)^2,$$

and

$$w=p+kH+kwy+kH\left(\frac{ds}{dx}\right)^2.$$

Again, if M is the bending moment and S the shearing force at the vertical section through P, then

$$\frac{d^2M}{dx^2}=\frac{dS}{dx}=w=H\frac{d^2y}{dx^2}.$$

Therefore $$H\frac{d^2y}{dx^2}=p+kH+kwy+kH\left(\frac{ds}{dx}\right)^2$$

$$=p+kH+kHy\frac{d^2y}{dx^2}+kH\left\{1+\left(\frac{dy}{dx}\right)^2\right\}$$

$$=p+2kH+kH\left\{y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right\}.$$

Integrating twice,

$$Hy=A+Bx+(p+2kH)\frac{x^2}{2}+kH\frac{y^2}{2},$$

A and B being constants of integration.

When $x=0$, $y=0$ and $H\frac{dy}{dx}=W$.

Thus $A=0$ and $B=W$.

Hence $$Hy = Wx + (p + 2kH)\frac{x^2}{2} + kH\frac{y^2}{2}$$

is the equation to the curve of the boom, and represents an ellipse with its major axis vertical, and with the lengths of the two axes in a ratio equal to $\left(\frac{p+2kH}{kH}\right)^{\frac{1}{2}}$.

The depth of the longest cantilever is determined by the vertical tangent at the end of the minor axis, and corresponds to the value of y given by making $\frac{dx}{dy}=0$ in the preceding equation, which gives $y=\frac{1}{k}$.

For a given value of H the curve of the boom is independent of the span. Again, for a given length of cantilever with a boom of this elliptic form, a value of H may be found which will make the total weight a minimum, and which will therefore give the most economical depth. Such an investigation, however, can only be of interest to mathematicians, as the hypotheses are far from being even approximately true in practice, and the resulting depth would be obviously too great.

Assumption (1) no longer holds when a live load has to be considered. Diagonal bracings must then be introduced, which become heavier as the depth increases, in consequence of their increased length. The diagonal bracings are also largely affected by the length of the panels. If the panels are short, and if a great depth of cantilever, diminishing rapidly away from the abutment, is used, the angles of the diagonal bracing, near the abutment, will be unfavorable to economy. This difficulty may be avoided by adopting a double system of triangulation over the deeper part of the cantilever only, or even a treble system for some distance in a large span. The objections justly urged against multiple systems of triangulation in trusses lose most of their force in large cantilevers. In the first place, the method of erection by building out insures that each diagonal shall take its proper share of the dead load; and in the second place, it should be remembered that only in large spans could a double system have anything to recommend it, and then only near the abutment where the stresses are greatest: in such cases the moving load only produces a small portion of the entire stress in the web. In practice a compromise has to be made between different requirements, and the depth must be kept within such limits as will admit of reasonable proportions in other respects, while the diagonal ties or struts may be allowed to vary in inclination, to some extent, from one panel to another.

Again, in fixing the panel length, care must be taken that there is no undue excess of platform weight, as this will produce a corresponding increase in the weight of the cantilever.

An excessive depth of cantilever generally causes an increase in the cost of erection.

Both theory and practice, however, indicate that it will be more advantageous to choose a greater depth for a cantilever than for an ordinary girder bridge.

An ordinary proportion for a large girder bridge would be one ninth to one seventh of the span, and if for the girder were substituted two cantilevers meeting in the middle of the span, the depth might with advantage be considerably increased beyond this proportion at the abutment, if it be reduced to *nil* where the cantilevers meet. When a central span is introduced, resting upon the ends of the two cantilevers, the concentrated load on the end gives an additional reason for still further increasing the depth at the abutment *proportionally to the length of the cantilever.* The greatest economical depth has probably been reached in the Indus bridge, in which the depth at the abutment $=.54\times$length of cantilever. Probably the proportion of one third of the length of the cantilever would be ample, except where the anchorage causes a considerable part of the whole weight, but each case must be considered on its own merits. The reduction of deflection obtained by increasing the depth is also an appreciable consideration.

If a depth be chosen not widely different from that which makes the quantity of material a minimum, the weight will be only slightly increased, while it is possible that great structural advantages may be gained in other directions. In recommending a great depth for a cantilever at its abutment, it is assumed that the depth will be continuously reduced from the abutment outwards. If the load were continuously distributed, it is by no means certain that a cantilever of uniform depth would require more material than one of varying depth, but it has already been pointed out to what extent the weight of the structure itself necessarily varies, and if the concentrated load at the end were separately considered, the economical truss would be a simple triangular frame of very great depth. From economic considerations, it would be well to reduce the depth of the cantilever at the outer end to *nil*, but in many cases it is thought advisable to maintain a depth at this point equal to that at the end of the central span, so that the latter may be built out without false-works, under the same system of erection as is pursued in the case of the cantilever. The post at the ends of the central span and cantilever is sometimes hinged to allow for expansion.

Deflection.—A serious objection urged against cantilever bridges is the excessive and irregular deflection to which they are sometimes subject. They usually deflect more than ordinary truss-bridges, and the deflection is proportionately increased under suddenly applied loads. In the endeavor to recover its normal position, the cantilever springs back with increased force and, owing to the small resistance offered by the weight and stiffness at the outer end, there may result, especially in light bridges, a kicking movement. It must, however, be borne in mind that the deflection, of which the importance in connection with iron bridges has always been recognized, is not in itself necessarily an evil, except in so far as it is an *indication* or a *cause* of over-strain.

16. The Statical Deflection, due to a quiescent load, must be distinguished from what might be called the dynamical deflection, i.e., the additional deflection due to a load in motion. The former should not exceed the deflection corresponding to the statical stresses for which the bridge is designed. The amount of the *dynamical* deflection depends both upon the nature of the loads and upon the manner in which they are applied, nor are there sufficient data to determine its value even approximately. It certainly largely increases the statical stresses and produces other ill effects of which little is known.

Hitherto the question as to the deflection of framed structures has received but meagre attention, and formulæ deduced for solid girders have been employed with misleading results. It would seem to be more scientific and correct to treat each member separately and to consider its individual deformation.

17. Rollers.—One end of a bridge usually rests upon nests of turned wrought-iron or steel friction rollers running between planed surfaces. The diameter of a roller should not be less than 2 ins., and the pressure upon it in pounds per lineal inch should not exceed $500\sqrt{d}$ if made of wrought iron, or $600\sqrt{d}$ if made of steel, d being the diameter in inches.

18. Eye-bars.—In England it has been the practice to roll bars having enlarged ends, and to forge the eyes under hydraulic pressure with suitably shaped dies. In America both hammer-forged and hydraulic-forged eye-bars are made, the latter being called *weldless eye-bars.* Careful mathematical and experimental investigations have been carried out to determine the proper dimensions of the link-head and pin, but owing to the very complex character of the stresses developed in the metal around the eye, an accurate mathematical solution is impossible.

Let d be the width and t the thickness of the *shank* of the eye-bar represented in Fig. 699. Let S be the width of the metal at the sides of the eye, and H the width at the end. Let D be the diameter of the pin.

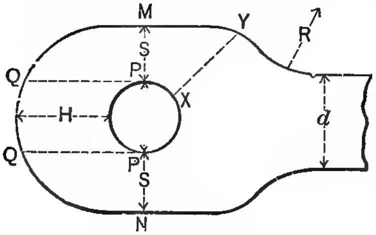


Fig. 699.

The proportions of the *head* are governed by the general condition that each and every part should be at least as strong as the shank.

When the bar is subjected to a tensile stress the pin is tightly embraced, and failure may arise from any one of the following causes:

(*a*) *The pin may be shorn through.*

Hence, if the pin is in double shear, its sectional area should be at least *one half* that of the shank.

It may happen that the pin is bent, but that fracture is prevented by the closing up of the pieces between the pin-head and nut; the efficiency, however, of the connection is destroyed, as the bars are no longer free to turn on the pin.

In practice D for flat bars varies from $\frac{2}{3}d$ to $\frac{4}{5}d$, but usually lies between $\frac{3}{4}d$ and $\frac{4}{5}d$.

The diameter of the pin for the end of a round bar is generally made equal to $1\frac{1}{4}$ times the diameter of the bar.

The pin should be turned so as to fit the eye accurately, but the best practice allows a difference of from $\frac{1}{64}$ to $\frac{1}{100}$ of an inch in the diameters of the pin and eye.

(*b*) *The link may tear across MN.*

On account of the perforation of the head, the direct pull on the shank is bent out of the straight and distributed over the sections S. There is no reason for the assumption that the distribution is uniform, and it is obviously probable that the intensity of stress is greatest in the metal next the hole. Hence the sectional area of the metal across MN must be at least equal to that of the shank, and in practice is always greater.

$$S \text{ usually varies from } .55d \text{ to } .625d.$$

The sectional area through the sides of the eye in the head of a round bar varies from $1\frac{1}{2}$ times to twice that of the bar.

(*c*) *The pin may be torn through the head.*

Theoretically the sectional area of the metal across PQ should be *one half* that of the shank. The metal in front of the pin, however, may be likened to a uniformly loaded girder with both ends fixed, and is subjected to a bending as well as to a shearing action. Hence the *minimum* value of H has been fixed at $\frac{5}{8}d$, and if H is made equal to d, both kinds of action will be amply provided for.

(*d*) *The bearing surface may be insufficient.*

If such be the case, the intensity of the pressure upon the bearing surface is excessive, the eye becomes oval, the metal is upset, and a fracture takes place. Or again, as the hole elongates, the metal in the sections S next the hole will be drawn out, and a crack will commence, extending outwards until fracture is produced.

In practice adequate bearing surface may be obtained by thickening the head so as to confine the maximum intensity of the pressure within a given limit.

(*e*) *The head may be torn through the shoulder at XY.*

Hence XY is made equal to d.

The radius of curvature R of the shoulder varies from $1\frac{1}{2}d$ to $7.6d$.

Note.—The *thickness* of the shank should be $\frac{d}{4}$, or $\frac{2}{7}d$ at least.

The following table gives the eye-bar proportions common in American practice:

Value of d.	Value of D.	Value of S.	
		Weldless Bars.	Hammered Bars.
1.00	.67	1.5	1.33
1.00	.75	1.5	1.33
1.00	1.00	1.5	1.50
1.00	1.25	1.6	1.50
1.00	1.33	1.7	
1.00	1.50	1.85	1.67
1.00	1.75	2.00	1.67
1.00	2.00	2.25	1.75

Also, in weldless bars, $H = S$; in hammered bars $H = d$.

Steel Eye-bars.—Hydraulic-forged steel eye-bars are now being largely made. The steel has an ultimate tenacity of from 60,000 to 68,000 lbs. per square inch, an elastic limit of not less than 50 per cent, and an elongation of from 17 to 20 per cent in a length equal to *ten* times the least transverse dimension.

The Phœnix Bridge Company and the Edge Moor Iron Company give the tables on page 765 of steel eye-bar proportions.

19. Rivets.—A *rivet* is an iron or steel *shank,* slightly tapered at one end (the *tail*), and surmounted at the other by a *cup* or *pan-shaped head* (Fig. 700). It is used to join steel or iron plates, bars, etc. For this purpose the rivet is generally heated to a cherry-red, the shank or *spindle* is passed through the hole prepared for it, and the tail is made into a *button,* or *point.* The hollow cup-tool gives to the point a nearly hemispherical shape, and forms what is called a *snap-rivet* (Fig. 701). Snap-rivets, partly for the sake of appearance, are commonly used in girder-work, but they are not so tight as *conical*-pointed rivets (*staff*-rivets), which are hammered into shape until almost cold (Fig. 702).

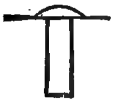


FIG. 700.

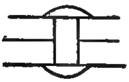


FIG. 701.

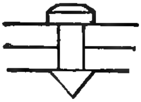


FIG. 702.

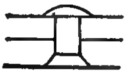


FIG. 703.

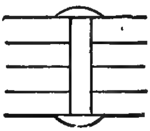


FIG. 704.

Phœnix Bridge Co.			Edge Moor Iron Co.				
Width of Bar *d*.	Diameter of Pin-hole.	Diameter of Head.	Width of Bar *d*.	Diameter of Pin-hole.	Diameter of Head.	Minimum Thickness of Bar.	Excess of Sectional Area of Head along *PP* over Section of Bar.
3	$2\frac{7}{16}$, $2\frac{15}{16}$	7	2	$1\frac{7}{8}$	$4\frac{1}{2}$	1	33%
3	$3\frac{7}{16}$, $3\frac{15}{16}$	8	2	$2\frac{7}{8}$	$5\frac{1}{2}$	1	33
4	$3\frac{7}{16}$	9	$2\frac{1}{2}$	$2\frac{1}{8}$	$5\frac{1}{2}$	1	33
4	$3\frac{15}{16}$, $4\frac{5}{16}$, $4\frac{11}{16}$	10	$2\frac{1}{2}$	$3\frac{1}{8}$	$6\frac{1}{2}$	1	33
5	$3\frac{15}{16}$, $4\frac{5}{16}$	11	3	$2\frac{1}{2}$	$6\frac{1}{2}$	$\frac{3}{4}$	33
5	$4\frac{11}{16}$, $5\frac{5}{16}$	12	3	4	8	$\frac{3}{4}$	33
5	$5\frac{13}{16}$, $6\frac{5}{16}$	13	4	$4\frac{1}{8}$	$9\frac{1}{2}$	$\frac{3}{4}$	33
6	$4\frac{11}{16}$, $4\frac{15}{16}$	13	4	$5\frac{1}{8}$	$10\frac{1}{2}$	$\frac{3}{4}$	33
6	$5\frac{5}{16}$, $5\frac{9}{16}$, $5\frac{13}{16}$	14	5	$4\frac{5}{8}$	$11\frac{1}{2}$	$\frac{3}{4}$	37
6	$6\frac{5}{16}$, $6\frac{11}{16}$, $6\frac{15}{16}$	15	5	$5\frac{5}{8}$	$12\frac{1}{2}$	$\frac{3}{4}$	37
7	$5\frac{9}{16}$	15	6	$5\frac{1}{4}$	$13\frac{1}{2}$	$\frac{7}{8}$	37
7	$5\frac{13}{16}$, $6\frac{5}{16}$, $6\frac{11}{16}$	16	6	$6\frac{1}{4}$	$14\frac{1}{2}$	$\frac{7}{8}$	37
7	$6\frac{15}{16}$, $7\frac{5}{16}$, $7\frac{11}{16}$	17	7	$5\frac{5}{8}$	$15\frac{1}{2}$	$\frac{15}{16}$	40
8	$6\frac{5}{16}$	17	7	$7\frac{1}{8}$	17	$\frac{15}{16}$	40
8	$6\frac{11}{16}$, $6\frac{15}{16}$, $7\frac{5}{16}$	18	8	$5\frac{3}{4}$	17	1	40
8	$7\frac{11}{16}$, $8\frac{3}{8}$	19	8	$6\frac{3}{4}$	18	1	40
8	$8\frac{7}{8}$, $9\frac{3}{8}$	20					
9	$7\frac{5}{16}$, $7\frac{11}{16}$	20					
9	$8\frac{3}{8}$, $8\frac{7}{8}$	21					
10	$8\frac{3}{8}$	22					
10	$8\frac{7}{8}$, $9\frac{3}{8}$	23					
10	10, $10\frac{1}{2}$	24					

In both the Phœnix and Edge Moor bars the thickness of the head is the same as that of the body of the bar, or does not exceed it by more than $\frac{1}{16}$ inch.

When a smooth surface is required, the rivets are *countersunk* (Fig. 703). The countersinking is drilled and may extend *through* the plate, or a shoulder may be left at the inner edge.

Cold-riveting is adopted for the small rivets in boiler-work and also wherever heating is impracticable, but tightly driven turned bolts are sometimes substituted for the rivets. In all such cases the material of the rivets or bolts should be of superior quality.

Loose rivets are easily discovered by tapping, and, if very loose, should be at once replaced. It must be borne in mind, however, that expansions and contractions of a complicated character invariably accompany *hot-riveting*, and it cannot be supposed that the rivets will be perfectly tight. Indeed, it is doubtful whether a rivet has any hold in a straight drilled hole, except at the ends.

Riveting is accomplished either by hand or machine, the latter being far the more effective. A machine will squeeze a rivet, at almost any temperature, into a most irregular hole, but the exigencies of practical conditions often prevent its use, except for ordinary work, and its advantages can rarely be obtained where they would be most appreciated, as, e.g., in the riveting up of connections.

Dimensions of Rivets.—The *diameter* (d) of a rivet in ordinary girder work varies from $\frac{3}{4}$ in. to 1 in., and rarely exceeds $1\frac{1}{8}$ ins. The *thickness* of a plate in ordinary girder-work should never be less than $\frac{1}{4}$ in., and a thickness of $\frac{3}{8}$ in. is preferable.

According to Fairbairn, d should be about $2t$ if $t<\frac{1}{2}$ in., and should be about $1\frac{1}{2}t$ if $t>\frac{1}{2}$ in.

According to Unwin, d should lie between $(\frac{3}{4}t+\frac{5}{16})$ in. and $(\frac{7}{8}t+\frac{3}{8})$ in. when t varies from $\frac{1}{4}$ in. to 1 in.

When the rivets join several plates, $d=\left(\frac{T}{8}+\frac{5}{8}\right)$ in., T being the total plate thickness.

According to French practice, the diameter of a rivet-head $=1\frac{2}{3}d$, and the length of the rivet from the head $=T+1\frac{2}{3}d$.

According to Rankine, the size of the head $=\frac{2}{3}d$, and the length of the rivet from the head $=T+2\frac{1}{2}d$.

The diameter of the rivet-hole exceeds that of the shank by from $\frac{1}{32}$ to $\frac{1}{8}$ in., so as to allow for the expansion of the latter when hot.

There seems to be no objection to the use of long rivets provided that they are properly heated and secured.

Strength of Punched and Riveted Plates.—Experiment shows that the tenacity of iron and steel plates is considerably diminished by punching. This deterioration in tenacity seems to be due to a molecular change in a narrow annulus of the metal around the hole. The removal of the annulus largely neutralizes the effect of the punching, and therefore the holes are sometimes punched $\frac{1}{8}$ in. less in diameter than the rivets and are subsequently rimered or drilled out to the full size. The original strength may also be almost entirely restored by annealing, and, generally, in steel-work, either this process is adopted or the annulus referred to above is removed.

Punching does not sensibly affect the strength of Landore-Siemens unannealed plates, and only slightly diminishes the strength of thin steel plates, but causes a considerable loss of tenacity in thick steel plates; the loss, however, is less than for iron plates.

The harder the material the greater is the loss of tenacity.

Iron seems to suffer more from punching when the holes are near the edge than when removed to some distance from it, while mild steel suffers less when the hole is one diameter from the edge than when it is so far that there is no bulging at the edge.

The injury caused by punching may be avoided by drilling the holes. In important girder-work and whenever great accuracy of workmanship is required, a uniform pitch may be insured and the full strength of the metal retained by the use of multiple drills. Drilling is a necessity for first-class work when the diameter of the holes is less than the thickness of the plate, and also when several plates are *piled*. It is impossible to punch plates, bars, angles, etc., in spite of all expedients, in such a manner that the holes in any two exactly correspond, and the irregularity becomes intensified in a pile, the passage of the rivet often being completely blocked. A *drift*, or *rimer*, is then driven through the hole by main force, cracking and bending the plates in its passage, and separating them one from another.

The holes may be punched for ordinary work, and in plates of which the thickness is less than the diameter of the rivets. Whenever the metal is of an inferior quality the holes should be drilled.

20. Riveted Joints.—In *lap*-joints (Figs. 705 and 708) the plates overlap and are riveted together by one or more rows of rivets which are said to be in *single shear*, as each rivet has to be sheared through one section only.

In *fish-* (or *butt-*) joints (Figs. 706 and 707) the rivets are in *double shear*, i.e., must be each sheared through two sections. Thus they are not subjected to the one-sided pull to which rivets in single shear are liable.

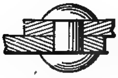


FIG. 705.

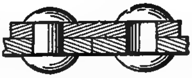


FIG. 706.

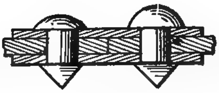


FIG. 707.

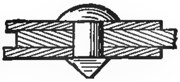
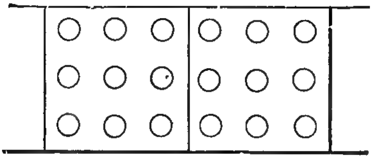


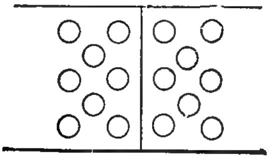
FIG. 708.

In *fish*-joints the ends of the plates meet, and the plates are riveted to a single cover (Fig. 706), or to two covers (Fig. 707), by means of one or more rows of rivets on each side of the joint.

A fish-joint is properly termed a *butt*-joint when the plates are in compression. The plates should butt evenly against one another, although they seldom do so in practice. Indeed, the mere process of riveting draws the plates slightly apart, leaving a gap which is often concealed by calking. A much better method is to fill up the space with some such hard substance as cast zinc, but the best method, if the work will allow of the increased cost, is to form a *jump-joint*, i.e., to plane the ends of the plates carefully, and then bring them into close contact, when a short cover with one or two rows of rivets will suffice to hold them in position.

The riveting is said to be *single, double, triple,* etc., according as the joint is secured by *one, two, three,* or more rows of rivets.


CHAIN
FIG. 709.


ZIGZAG
FIG. 710.

Double, triple, etc., riveting may be *chain* (Fig. 709) or *zigzag* (Fig. 710). In the former case the rivets form straight lines longitudinally and transversely, while in the latter the rivets in each row divide the space between the rivets in adjacent rows. Experiments indicate that chain is somewhat stronger than zigzag riveting.

Figs. 711 to 713 show forms of joint usually adopted for bridge-work. In boiler-work the rivets are necessarily very close together,

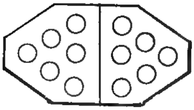


FIG. 711.

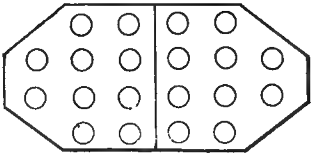


FIG. 712.

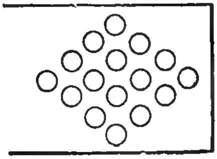


FIG. 713.

and if the strength of the solid plate be assumed to be 100, the strength of a single-riveted joint hardly exceeds 50, while double-riveting will only increase it to 60 or 70. Fairbairn proposed to make the joint and unpunched plate equally strong by increasing the *thickness* of the punched portion of the plate, but this is somewhat difficult in practice.

The stresses developed in a riveted joint are of a most complex character and can hardly be subjected to exact mathematical analysis. For example, the distribution of stress will be necessarily irregular (*a*) if the pull upon the joint is one-sided; (*b*) when local action exists, or the plates stretch, or internal strains are in the metal before punching; (*c*) if there is a lack of symmetry in the arrangement of the rivets, so that one rivet is more severely strained than another; (*d*) when the workmanship is defective.

The joint may fail in any one of the following ways:

(1) The rivets may shear.

(2) The rivets may be forced into and crush the plate.

(3) The rivets may be torn out of the plate.

(4) The plate may tear in a direction transverse to that of the stress.

The resistance to rupture should be the same in each of the four cases, and always as great as possible.

The shearing and tensile strengths of plate-iron are very nearly equal. Thus iron with a tenacity of 20 tons per square inch has a shearing strength of 18 to 20 tons per square inch. Rivet-iron is usually somewhat stronger than plate-iron.

Again, the shearing strength of steel per square inch varies from about 24 tons for steel, with a tenacity of about 30 tons, to about 33 tons for steel, with a tenacity of about 50 tons; an average value for rivet-steel with a tenacity of 30 tons being 24 tons.

Hence, if 4 be a factor of safety, the working coefficients become

For wrought iron { 5 tons per square inch in shear, and
5 " " " " " tension.

For steel. . . . . { 6 tons per square inch in shear, and
7½ " " " " " tension.

Allowance, however, must be made for irregularity in the distribution of stress and for defective workmanship, and in riveting wrought-iron plates together it is a common practice to make the aggregate section of the rivets at least equal to and sometimes 20 per cent greater than the net section of the plate through the rivet-holes.

Hence, the working coefficients are reduced to

4 or $4\frac{1}{2}$ tons per square inch for wrought iron,

and

5 or $5\frac{1}{2}$ " " " " " steel,

according to the character of the joint.

There is very little reliable information respecting the indentation of plates by rivets and bolts, and it is most uncertain to what extent the tenacity of the plates is affected by such indentation. Further experiments are required to show the effect of the crushing pressure upon the bearing area (i.e., *the diameter of the rivet multiplied by the thickness of the plate*), although a few indicate that the shearing strength of the rivet diminishes after the intensity of the bearing pressure exceeds a certain maximum limit.

21. Theoretical Deductions.

Let S be the total stress at a riveted joint;

f_1, f_2, f_3, f_4 be the safe tensile, shearing, compressive, and bearing unit stresses, respectively;

t be the thickness of a plate, and w its width;

N be the total number of rivets on one side of a joint;

n be the total number of rivets in one row;

p be the pitch of the rivets, i.e., the distance centre to centre;

d be the diameter of the rivets;

x be the distance between the centre line of the nearest row of rivets and the edge of the plate.

Value of x.—It has been found that the minimum safe value of x is d, and this in most cases gives a sufficient *overlap* ($=2x$), while $x=\frac{3}{2}d$ is a maximum limit which amply provides for the bending and shearing to which the joint may be subjected. Thus the overlap will vary from $2d$ to $3d$.

x may be supposed to consist of a length x_1 to resist the shearing action, and a length x_2 to resist the bending action. It is impossible to determine theoretically the exact value of x_2, as the straining at the joint is very complex, but the metal in front of each rivet (the rivets at the ends of the joint excepted) may be likened to a uniformly loaded beam of length d, depth $x_2-\frac{d}{2}$, and breadth t, with both ends *fixed*. Its moment of resistance is therefore $\frac{f}{6}t\left(x_2-\frac{d}{2}\right)^2$, f being the maximum unit stress due to the bending. Also, if P is the load upon the rivet, the *mean* of the bending moments at the end and centre is $\frac{P}{8}d$.

Hence, *approximately*,

$$\frac{P}{8}d = \frac{f}{6}t\left(x_2 - \frac{d}{2}\right)^2, \quad \text{or} \quad P = \frac{4}{3}\frac{ft}{d}\left(x_2 - \frac{d}{2}\right)^2.$$

It will be assumed that the shearing strength of the rivet is equal to the strength of a beam to resist cross-breaking.

Single-riveted lap and single-cover joints (Figs. 705 and 706).

$$\frac{\pi d^2}{4}f_2 = (p-d)tf_1 = dtf_4; \quad 2x_1tf_2 = \frac{\pi d^2}{4}f_2; \quad \text{therefore } x_1 = \frac{\pi}{8}\frac{d^2}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \tag{1}$$

$$\frac{4}{3}\frac{ft}{d}\left(x_2 - \frac{d}{2}\right)^2 = \frac{\pi d^2}{4}f_2; \quad \text{therefore } x_2 = \frac{d}{2} + \frac{1}{4}\sqrt{3\pi\frac{d^3}{t}\frac{f_2}{f}}. \quad . \quad . \quad . \quad . \quad . \quad . \tag{2}$$

As already pointed out, these joints are weakened by the bending action developed, and possibly also by the concentration of the stress towards the inner faces of the plates.

Single-riveted double-cover joints (Fig. 707).

$$2\frac{\pi d^2}{4}f_1 = (p-d)tf_1 = dtf_4; \quad \text{therefore} \quad 2x_1tf_2 = 2\frac{\pi d^2}{4}f_2. \quad x = \frac{\pi}{4}\frac{d^2}{t}. \quad . \quad . \quad . \quad . \tag{3}$$

$$\frac{4}{3}\frac{ft}{d}\left(x_2 - \frac{d}{2}\right)^2 = \frac{1}{2}\frac{\pi d^2}{4}f_2; \quad \text{therefore} \quad x_2 = \frac{d}{2} + \frac{1}{4}\sqrt{\frac{3}{2}\pi\frac{d^3}{t}\frac{f_2}{f}}. \quad . \quad . \quad . \quad . \quad . \quad . \tag{4}$$

These joints are much stronger than joints with *single* covers. Also, equation (3) shows that the bearing unit stress in a double-cover joint is twice as great (*theoretically*) as in a single-cover joint (eq. 1), so that rivets of a larger diameter may be employed in the latter than is possible in the former for corresponding values of $\frac{d}{t}$.

Chain-riveted joints (Fig. 709).

$$f_1(w-nd) = S = f_4Ndt; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{5}$$

$$S = N\frac{\pi d^2}{4}f_2 \text{ when there is } \textit{one} \text{ cover only}; \quad . \quad . \quad . \quad . \quad . \quad . \tag{6}$$

$$S = N\frac{\pi d^2}{2}f_2 \text{ when there are } \textit{two} \text{ covers}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{7}$$

This class of joint is employed for the flanges of bridge girders, the plates being piled as in Figs. 714, 715, 716, and n being usually 3, 4, or 5.

In Fig. 715 the plates are grouped so as to *break joint*, and opinions differ as to whether this arrangement is superior to the *full butt* shown in Fig. 716.

The advantages of the latter are that the plates may be cut in uniform lengths, and the flanges built up with a degree of accuracy which cannot be otherwise attained, while the short and awkward pieces accompanying broken joints are dispensed with.

A good practical rule, and one saving much labor and expense, is to make the lengths of the plates, bars, etc., multiples of the pitch, and to design the covers, connections, etc., so as to interfere with the pitch as little as possible.

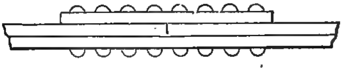


FIG. 714.

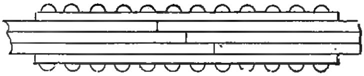


FIG. 715.

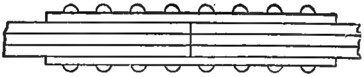


FIG. 716.

The distance between two consecutive joints of a group (Fig. 713) is generally made equal to *twice* the pitch.

An excellent plan for lap and single-cover joints is to arrange the rivet as shown in Figs. 709 to 713.

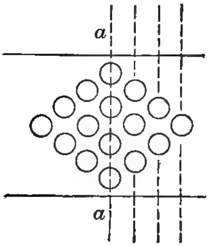


FIG. 717.

The strength of the plate at the joint is only weakened by *one* rivet-hole, for the plate cannot tear at its weakest section, i.e., along the central row of rivets (*aa*), until the rivets between it and the edge are shorn in two.

Let there be m rows of rivets, 1 1, 2 2, 3 3, . . . (Fig. 717).

The total number of rivets is evidently m^2.

Let f_1, q_2, q_3, q_1, . . . be the unit tensile stresses in the plate along the lines 1 1, 2 2, 3 3, . . . , respectively. Then

$$S=(w-\ d)tf_1=\frac{\pi d^2}{4}m^2f_2, \qquad \text{for the line 1 1;}$$

$$=(w-2d)tq_2=\frac{\pi d^2}{4}(m^2-1)f_2, \text{`` `` `` 2 2;}$$

$$=(w-3d)tq_3=\frac{\pi d^2}{4}(m^2-3)f_2, \text{`` `` `` 3 3;}$$

$$=(w-4d)tq_4=\frac{\pi d^2}{4}(m^2-6)f_2, \text{`` `` `` 4 4;}$$

. . . .

Therefore $$S=(w-d)tf_1=(w-2d)\frac{m^2}{m^2-1}tq_2=\ldots$$

Assume that $f_1 = q_2$. Then $w=(m^2+1)d$.

Hence, by substituting this value of w in the first of the above relations, $\frac{d}{t}=\frac{14}{11}\frac{f_1}{f_2}$. Since q_3, q_4, . . . are each less than f_1, the assumption is justifiable.

Covers.—In *tension* joints the strength of the covers must not be less than that of the plates to be united. Hence a *single* cover should be at least as thick as a single plate; and if there are two covers, each should be at least half as thick.

When two covers are used in a tension pile it often happens that a joint occurs in the top or bottom plate, so that the greater portion of the stress in that plate may have to be borne by the nearest cover. It is, therefore, considered advisable to make its thickness five eighths that of the plate.

The number of the joints should be reduced to a minimum, as the introduction of covers adds a large percentage to the dead weight of the pile.

Covers might be wholly dispensed with in *perfect-jump* joints, and a great economy of material effected, if the difficulty of forming such joints and the increased cost did not render them impracticable. Hence it may be said that covers are required for all *compression* joints, and that they must be as strong as the plates; for, unless the plates butt closely, the whole of the thrust will be transmitted through the covers. In some of the best examples of bridge construction the tension and compression joints are identical.

22. Efficiency of Riveted Joints.—The efficiency of a riveted joint is the ratio of the maximum stress which can be transmitted to the plates through the joint to the strength of the solid plates.

Denote this maximum efficiency by η.

Let p be the pitch of the rivets;
d ‘‘ diameter of the rivets;
t ‘‘ thickness of the plates;
f_t ‘‘ tenacity of the solid plate;
mf_t ‘‘ ‘‘ ‘‘ ‘‘ riveted plate;
f_s ‘‘ shearing strength of the rivets;
N ‘‘ number of rivets in a pitch length;
e ‘‘ ratio of the strength of a rivet in double shear to its strength in single shear.

Then η_1 = efficiency as regards the plates $=\frac{(p-d)tmf_t}{ptf_t}$

$$=\frac{m(p-d)}{p}. \qquad . \quad . \quad . \quad . \quad . \qquad (1)$$

$$\eta_2 = \text{efficiency as regards the rivets} = \frac{eN\frac{\pi}{4}d^2f_s}{ptf_t}. \qquad . \quad . \quad . \quad . \quad . \qquad (2)$$

The efficiency of the joint is, of course, the smaller of these two values; and the joint is one of maximum efficiency when $\eta_1=\eta_2=\eta$; that is, when

$$m\frac{p-d}{p}=\frac{eN\frac{\pi}{4}d^2f_s}{ptf_t},$$

or

$$(p-d)tmf_t=eN\frac{\pi}{4}d^2f_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

In this expression the quantities m, f_t, N, and e are constants for any given joint, being of necessity known, or having been fixed beforehand; and the equation thus expresses one condition governing the relations of the three variables, p, d, and t to each other. It is obvious, however, that, in order to determine the values of any two of these variables in terms of the third, another relation between them must be postulated. In short, in designing a joint, the value of one of the three ratios $\frac{p}{d}$, $\frac{p}{t}$, and $\frac{d}{t}$ must be fixed.

Case I.—Suppose that the ratio $\frac{p}{d}$ has a certain value. This is very frequently the quantity predetermined; but it is most usually done by fixing the value of η, η very obviously involving $\frac{p}{d}$; in fact $\eta=m\left(1-\frac{d}{p}\right)$.

Equation (3) may be written

$$p=eN\frac{\pi}{4}\frac{d^2}{t}\frac{f_s}{mf_t}+d,$$

or

$$p=d\left(eN\frac{\pi}{4}\frac{d}{t}\frac{f_s}{mf_t}+1\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

If the ratio $\frac{d}{t}$ be denoted by k, then

$$\frac{p}{d}=eN\frac{\pi}{4}k\frac{f_s}{mf_t}+1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

But since $\eta=\frac{m(p-d)}{p}$,

$$\frac{p}{d}=\frac{m}{m-\eta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Therefore, substituting in (5),

$$\frac{\eta}{m-\eta}=eN\frac{\pi}{4}k\frac{f_s}{mf_t}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

and, ultimately,

$$k=\frac{4}{eN\pi}\frac{mf_t}{f_s}\frac{\eta}{m-\eta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

The process of designing a joint of maximum efficiency for a boiler of given diameter and pressure of steam, when η $\left(\text{or the ratio } \frac{p}{d}\right)$ is fixed, is then as follows: Settle the number of rivets per pitch (i.e., N); the value to be allowed for e (depending on the nature of the shearing stress on the rivets); and the values of m, f_t, and f_s. Then k is known from equation (8).

But t may be found from the relation

$$\text{pressure} \times \text{diameter} = \eta \times 2tf_t,$$

or

$$t = \frac{\text{pressure} \times \text{diameter}}{2\eta f_t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Hence, since $k = \frac{d}{t}$ is known, d may be found; and since $\frac{p}{d} = \frac{m}{m - \eta}$ is known, p is also fixed.

CASE II.—When $\frac{p}{t}$, the ratio of rivet pitch to plate thickness, is given, equation (5) must be otherwise manipulated.

Multiplying it by $\frac{d}{p}$, and substituting for d its value kt, we have

$$1 = \frac{eN\pi}{4} \frac{t}{p} k^2 \frac{f_s}{mf_t} + \frac{t}{p} k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Putting this in the form of a quadratic equation in k,

$$k^2 + \frac{4}{eN\pi} \frac{mf_t}{f} k - \frac{4}{eN\pi} \frac{mf_t}{f_s} \frac{p}{t} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

For brevity, substituting A for $\frac{4}{eN\pi}$, T for $\frac{mf_t}{f_s}$, and R for $\frac{p}{t}$, and solving the quadratic,

$$k = -\frac{AT}{2} \pm \frac{1}{2}\sqrt{A^2T^2 + 4ATR}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The method of designing the joint is, then, as follows:

A, T and R being known, k may be found by substituting their values in equation (12), the positive sign of the second term being taken.

Now, $$\eta = m\left(1 - \frac{d}{p}\right) = m\left(1 - \frac{kt}{p}\right) = m\left(1 - \frac{k}{R}\right);$$

and since both k and R are now known, the thickness of plate (t) may be found, as in Case I, by equation (9). The values of the diameter and pitch of rivets follow at once from the known values of k and R.

This method of designing a joint appears to be the most rational of the three. For the greatest pitch for which a joint will remain steam-tight depends mainly on the relation of pitch of rivets to thickness of plates; although it is also affected by the relative size of rivets and of rivet-heads.

CASE III.—If $\frac{d}{t}$, or k, be predetermined, the value of ρ must first be obtained in order that the plate thickness may be found by means of equation (9). Now, $\eta = m\frac{p-d}{p}$ may be put into the form

$$p = \frac{md}{m-\eta};$$

and if this value is substituted for p in equation (4),

$$\frac{md}{m-\eta} = \left(\frac{eN\pi}{4}k\frac{f_s}{m f_t}+1\right)d.$$

From this is finally deduced

$$\eta = m\frac{eN\pi kf}{eN\pi kf_s + 4mf_t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The plate thickness may now be found by equation (9); the diameter of rivet from $d = kt$, and the pitch from $p = \frac{md}{m-\eta}$. In the above investigations no account has been taken of the effect of the bearing pressure on the rivets or plate.

If f_c be the allowable bearing pressure per projected square inch of rivet surface, the following relation must obtain:

$$(p-d)tmf_t = Ndtf_c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This may be written

$$f_c = \frac{(p-d)mf_t}{Nd}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Then if f_c be estimated by this equation, and if it should be greater than 43 tons per square inch in a lap-joint, or 45 to 50 tons in a butt-joint, such joint will fail by the rivets shearing before the full strength of the plate is exerted, as Kennedy's experiments show that with these values of f_c the rivets do not attain their natural ultimate shearing strength (viz., f_s), but fail at shearing stresses much below this.

Again, the maximum allowable ratio $\frac{d}{t}$ (i.e., k) as the preliminary datum for the design of a joint, may be fixed by using the expression

$$\frac{d}{t} = \frac{f_c}{e\frac{\pi}{4}f_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

deduced from the obvious relation—similar to (14)—

$$eN\frac{\pi}{4}d^2 f_s = Ndtf_c.$$

(Unwin suggests the relation $d = \frac{6}{5}\sqrt{t}$.)

In designing the joint by any of the methods given above, any value obtained for k greater than that supplied by (16) should be rejected.

Note on Friction of Riveted Joints.—Elaborate experiments on the small displacements produced by loads on riveted joints of all kinds have recently been made by Considère (Annales des Ponts et Chaussées, 1886), Bach (Zeit. d. Ver., 1892, 1894, 1895), Dupuy (Annales des Ponts et Chaussées, 1895), and by Van der Kolk (Zeit d. Ver., June 1897).

These show that the frictional resistance produced by the great pressure of the riveting is, in a well-made joint, sufficient to transmit the required amount of force across the joint. In fact the stanchness and durability of the joint depend upon the plate friction, and not upon the shearing strength of the rivets or the tearing strength of the plates. When the rivet cools it contracts lengthwise, and the longitudinal tension thereby produced induces a cross-contraction which, added to the diametral contraction due to cooling, makes the rivet in the finished joint a loose fit in its hole. The shearing strength of the rivet does not, therefore, come into play until the plates have moved sufficiently to cause the rivets again to bear against the sides of the hole. Even when this does happen, it is evident that, at first, only a few of the rivets in a given joint will bear, and these must be deformed, or must give way more or less, before the rest of the rivets can come into action. Hence it is the frictional grip of the plates upon each other which prevents this slipping, and which is the true criterion of the strength of the joint. When once the plates slip, a slackness of the whole joint will be produced by a reversal of load. and in the case of a joint which is required to retain a fluid under pressure leakage will take place.

The latest experiments by Van der Kolk were made on the joints of bridge struts and ties with double-butt straps. The breaking load was not determined in these experiments, as it was not considered of sufficient importance. As a rule the stresses in riveted joints of bridges are much less than the breaking loads of plates of even the lowest tenacity. The question is not which kind of joint has the greatest statical strength, but which joint is least likely to become slack under the action of reversed loading.

The displacements observed in the joints under load were of two kinds—elastic (or disappearing) and permanent. The former were considered the more crucial in defining the best form of joint, as the permanent set, once taken, is hardly increased by repeated loading.

The elastic extensions were smallest (1) in the case of hand-riveting, and with holes somewhat too large for the rivets, and (2) in the case of machine-riveting when the pressure on the dies was much greater than is usual in practice.

It is very remarkable that hand-riveted joints, with rivets a good fit in their holes, allowed large elastic displacements under comparatively small loads. Riveting with the machine causes the rivets to fill the holes, and, unless a very great pressure is applied and maintained, produces the same bad results as to elastic movement.

Ex. 17.—*Design of 127-ft. through riveted span.*

Divide the span into five panels of 25.4 ft. each. Take depth centre to centre of chords = 28 ft. and width centre to centre of trusses = 17 ft. The live load will consist of two consolidation engines followed by a uniform train load of 3400 lbs. per lineal foot of track as shown in Fig. 718. In this figure, to facilitate calculation, there is written under each wheel the B.M. of all preceding wheels up to that point.

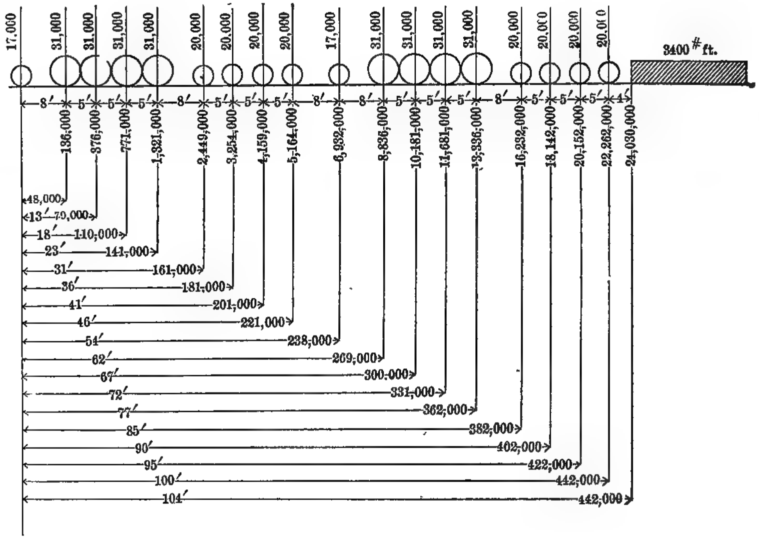


Fig 718.

Also, on the horizontal lines is shown the distance from the front of the engine up to each wheel, and the sum of the loads up to and including the wheel in question is given.

Cooper's specifications will be used and all bending moments and shears will be computed from the actual wheel concentrations.

B.M. at Panel-point 1, Fig. 719.—Consider the load as coming upon the span from the right. According to the *criterion* deduced in Art. 8, Chapter II, the B.M. at panel-point 1 will be a maximum when the average load on the left is equal to the average load on the whole span. Place the 3d driver of

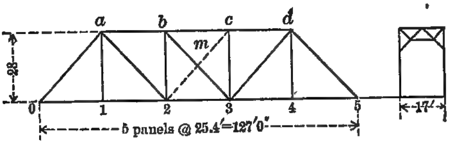


Fig. 719.

the first engine at 1. Then x, the length of span covered by the uniform load, $=15.6$ ft., so that total load on span $=442{,}000+3400+15.6=495{,}040$ lbs.

The average load per panel $=495{,}040\div 5=99{,}008$ lbs.

With the 3d driver on the right of 1, load in panel 01 $=79{,}000$ lbs. $<99{,}008$. When 3d driver is on the left of 1, load in panel 01 $=110{,}000$ lbs. $>99{,}008$ lbs. so that the criterion is satisfied with the 3d driver at point 1. Hence

$$R=\frac{1}{127}\left(24{,}030{,}000+442{,}000\times 15.6+\frac{3400\times 15.6^2}{2}\right)=\frac{31338912}{127}\text{ lbs.,}$$

and
$$\text{B.M.}=25.4R-771{,}000=5{,}496{,}782\text{ ft.-lbs.}$$

Panel-point (2).—Criterion is satisfied with 3d tender wheel of 1st engine at 2. Then $x=13.2$ ft. Therefore

$$R=\frac{1}{127}\left(24{,}010{,}000+442{,}000\times 13.2+\frac{3400\times 13.2^2}{2}\right)=\frac{30160608}{127}\text{ lbs.,}$$

and
$$\text{B.M.}=50.8R-4{,}159{,}000=7{,}905{,}242\text{ ft.-lbs.}$$

Panel-point (3).—Try 2d driver of 2d engine at 3. Then $x=13.8$ ft.

and
$$R=\frac{1}{127}\left(24{,}030{,}000+442{,}000\times 13.8+\frac{3400\times 13.8^2}{2}\right)=\frac{30453348}{127}\text{ lbs.,}$$

and
$$\text{B.M.}=76.2R-10{,}181{,}000=8{,}091{,}010\text{ ft.-lbs.}$$

The B.M. at 3 is larger than at 2. We should, however, obtain the same moment at 2 by letting the train advance from the left, and hence we must use the larger moment, 8,091,010, in computing the stresses for the members whose centres of moments are at panel-point 2.

Maximum Shears for Truss Members.—By Art. 8, Chapter II, the maximum shear in any panel will occur when the load in that panel is equal to the average load per panel on the whole span. For panel-point 1 this occurs when the 3d driver is at the panel-point. Then $x=15.6$ ft. and, as above.

$$R=\frac{31338912}{127}=246{,}763\text{ lbs.}$$

Therefore
$$S=R-\frac{17000\times 18+31000\times(10+5)}{25.4}=216{,}409\text{ lbs.}$$

Panel-point 2.—3d driver at panel-point 2.

$$R=\frac{1}{127}(18{,}142{,}000+402{,}000\times 4.2)=156{,}145\text{ lbs.,}$$

and therefor
$$S=125{,}791\text{ lbs.}$$

Panel-point 3.—2d-driver at 3.

$$R=\frac{1}{127}(8{,}836{,}000+269{,}000\times 1.8)=73{,}387 \text{ lbs.},$$

and therefore $S=58{,}584$ lbs.

B.M. for stringers, the length being 25.4 *ft.*

The greatest B.M. will occur when four drivers are on the stringer, as shown in Fig. 720. (Art. 8, Chapter II.) Then

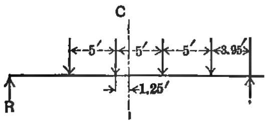


FIG. 720.

$$R=\frac{1}{25.4}(31{,}000\times 45.8)=55{,}897 \text{ lbs.},$$

and

$$\begin{aligned}\text{B.M.}&=55{,}897\times 11.45-31{,}000\times 5\\&=485{,}020 \text{ ft.-lbs.}\end{aligned}$$

The greatest floor-beam concentration will occur with the loads so placed as to give the max. B.M. at the centre of a span of two panel lengths. Place 3d driver at panel-point. Then load on floor-beam

$$=\frac{1}{25.4}(31{,}000\times 81.6+17{,}000\times 7.4+20{,}000\times 22.2)=122{,}023 \text{ lbs.}$$

For greatest end shear on stringer place 1st driver at panel-point. Then

$$R=S=\frac{1}{25.4}\{20{,}000\times 2.4+31{,}000(10.4+15.4+20.4+25.4)\}=89{,}276 \text{ lbs.}$$

TABLE OF MOMENTS AND SHEARS FOR LIVE LOADS.

Panel-points.	Moments for Span.	Shears for Span.	Moments for Truss.	Shears for Truss.
1	5,496,782 ft.-lbs	216,409 lbs.	2,748,391 ft-.lbs.	108,205 ft.-lbs.
2	8,091,010 "	125,791 "	4,045,505 "	62,895 "
3	8,091,010 "	58,584 "	4,045,505 "	29,292 "

Max. B.M. for stringer $=485{,}020\div 2=242{,}510$ ft.-lbs.
" concentration for floor-beam $=122{,}023$ lbs., or 61,012 lbs. at each end.
" end shear for stringer $=89{,}276\div 2=44{,}638$ lbs.

Allowable stresses:

Timber (extreme fibre)........................ 1,000 lbs. per sq. in.
Medium steel, tensile stresses:
Lateral sway-bracing for wind strains............. 18,000 " " " "
Bottom flanges, riveted floor-beams and stringers... 10,000 " " " "

	Live Load.	Dead Load.
Bottom chords, main diagonals, and long verticals..	10,000	20,000

Compressive stresses:

	Live Load.	Dead Load.
Chord segments	$10{,}000-45\frac{l}{r}$	$20{,}000-90\frac{l}{r}$
Posts of through-bridges	$8{,}500-45\frac{l}{r}$	$17{,}000-90\frac{l}{r}$
Lateral struts and rigid bracing (wind strains)		$13{,}000-60\frac{l}{r}$.

Rivets 9000 lbs./sq. in. in shear, 15,000 lbs. in bearing; 80 per cent of above for floor system; reduce *one-third* for field rivets.

Ties and Guard-rails.—Spacing the stringers 7 ft. centre to centre, we may use the same ties and guard-rails as for the plate-girder span Example 44, Chapter VII.

Stringers.—Assume dead load, including weight of stringer, as 700 lbs. per foot of span or 350 lbs. per foot of stringer.

Then B.M. (dead load) $=\frac{1}{8}\times350\times25.4^2=$ 28,225 ft.-lbs.
" (live ") = 242,510 "

Total " 270,735 "

Take a $38''\times\frac{3}{8}''$ web. The effective depth $=36''$ or 3 ft.

Then flange stress $=270{,}735\div3=90{,}245$ lbs.

and " area $=90{,}245\div10{,}000=9.02$ sq. in. net.

Use two $6''\times4''\times\frac{9}{16}''$ angles $=10.62$ sq. in. gross area $=9.5$ sq. in. net area for each flange, the flanges being assumed, under the specifications, to carry the entire bending action.

Use $3\frac{1}{2}''\times3\frac{1}{2}''\times\frac{3}{8}''$ angles for intermediate stiffeners, spaced 3 ft. centres, and $3\frac{1}{2}''\times3\frac{1}{2}''\times\frac{1}{2}''$ angles for end stiffeners. For stringer bracing use $3\frac{1}{2}''\times3\frac{1}{2}''\times\frac{3}{8}''$ angles, as shown in Fig. 721.

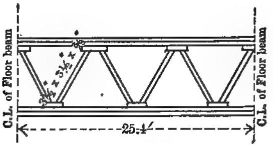


FIG. 721.

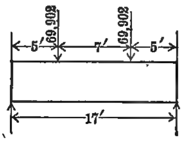


FIG. 722

Floor-beams.—Dead-load concentr. per stringer $=350\times25.4=$ 8,890 lbs.
Live " " " " $=61{,}012$ lbs.

Total " " " $=69{,}902$ lbs.

Assume weight of floor-beam $=2600$ lbs. Then, Fig. 722,
B.M. due to weight of floor-beam $=\frac{1}{8}\times2600\times17$ = 5,525 ft.-lbs.
" " concentration $=69{,}902\times5$ $=349{,}510$ "

Total B.M. 355,035 "

Assume a 48″×$\frac{3}{8}$″ web. The effective depth = 3.84 ft., so that

$$\text{the flange stress} = 355{,}035 \div 3.84 = 92{,}457 \text{ lbs.}$$

and " " area = 92,457 ÷ 10,000 = 9.25 sq. in.

Use two 6″×4″×$\frac{9}{16}$″ angles = 10.62 sq. in. gross area = 9.5 sq. in. net area. End stiffeners 3½″×3½″×½″; intermediate stiffeners 3½″×3½″×$\frac{3}{8}$″, spaced about 4 ft. centre to centre.

Stringer Details.—The end shear in the stringer must be transmitted from the web of the stringer through the end stiffening angles to the floor-beam and thence in a similar way to the post. The rivets connecting the end stiffeners to the stringer web are in double shear, hence their bearing value on the $\frac{3}{8}$-inch web plate will govern.

The bearing value of a $\frac{7}{8}$-in. rivet on a $\frac{3}{8}$-in. plate is 15,000×$\frac{3}{8}$×.8 for floor system (see specification) = 3938 lbs. for shop riveting, and = 2619 lbs. for field riveting.

End shear on stringer = 44,638 lbs.

Hence the required number of rivets = 44,638 ÷ 3938 = 11 +, say 12.

These may be placed partly in the angles *A* and partly in the fillers *B*, Fig. 723.

Value of $\frac{7}{8}$-in. rivet in single shear = .6013×9000×0.8 = 4329 lbs. for shop riveting or 2886 lbs. for field riveting.

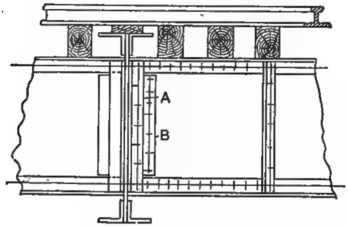


FIG. 723.

In the connection of the stringers to the floor-beams, *bearing* will govern; and since the riveting is done in the field the bearing value per rivet is only 2619 lbs. The concentration from two stringers = 69,902 lbs. Hence the number of rivets required = 69,902 ÷ 2619 = 26 +, say 28.

The rivet spacing in the flanges may be determined as in the case of the plate-girder Ex. 44, Chapter VII.

Connection of End Stiffener to Web.—The load = 69,902 + 1300 = 71,202 lbs. and the required number of rivets = 71,202 ÷ 3938 = 18 +, say 19. Put 10 in row *C*, 9 in row *D*, Fig. 724.

End Stiffeners to Post.—Single shear will govern. The value of each rivet = 2886 lbs. and the required number of rivets = 71,202 ÷ 2886 = 24 +, say 26.

Dead-load Stresses Fig. 719.—Assume a dead load of 900 lbs. per lineal foot of truss. Let D = panel dead load. Then $D = 900 \times 25.4 = 22{,}860$ lbs.

Also, $\tan\theta = .907 \qquad \sec\theta = 1.35$
$D \tan\theta = 20{,}734 \qquad D\sec\theta = 30{,}860.$

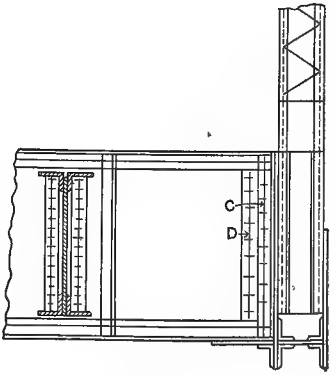


FIG. 724.

Assume $\frac{1}{3}$ of the load concentrated at the top chord.

Then, stress in 01–12 $=2D \tan\theta=41{,}468$ lbs. T.
" " 2–3 $=3D \tan\theta=62{,}202$ " T.
" " ab, $bc=3D \tan\theta=62{,}202$ " C.
" " $0a=2D \sec\theta=61{,}720$ " C.
" " $a2=D \sec\theta=30{,}860$ " T.
" " $a1=\frac{2}{3}D=15{,}240$ " T.
" " $b2=\frac{1}{3}D=7{,}620$ " C.

Live-load Stresses.

	Equivalent Live-Load Method (for comparison).
Stress in 01–12 $=2{,}748{,}391\div28=98{,}157$ T.	96,760 T.
" " 2, 3 $=4{,}045{,}505\div28=144{,}482$ T.	145,014 T.
" " ab, $bc=4{,}045{,}505\div28=144{,}482$ C.	145,140 C.
" " $0a=108{,}205 \sec\theta=146{,}076$ C.	144,020 C.
" " $a2=62{,}895 \sec\theta=84{,}910$ T.	86,412 T.
" " $b3=29{,}292 \sec\theta=39{,}544$ T. or C.	43,206 T. or C.
" " $a1$ = floor-beam concentration = 61,012 T.	61,295 T.
" " $b2$ = (shear at panel-point 2) = 29,292 C.	29,028 C.

Wind-load Stresses.—Specification.—"To provide for wind strains and vibrations from high-speed trains, the bottom lateral bracing in through bridges will be proportioned to resist a lateral force of 600 lbs. for each foot of the span; 450 lbs. of this being treated as a moving load, and as acting on a train of cars, at a line 6 feet above the base of rail.

The top lateral bracing in through-bridges will be proportioned to resist a lateral force of 150 lbs. per lineal foot for spans up to 300 ft.

The stresses in truss members from assumed wind forces need not be considered except (*a*) when the wind stresses exceed 30 per cent of the maximum stresses due to the dead and live loads upon the same member and the section is then to be increased until the total stress per square inch does not exceed by more than 30 per cent the maximum fixed for dead and live loads only, and (*b*) when the wind stress can neutralize or reverse the stress in any member."

Lower Lateral System.—We shall use a double system of rigid lower lateral bracing (Fig. 725) capable of carrying either compression or tension.

Dead wind load = 150 lbs. per lineal foot, so that the panel load $=150\times25.4=3810$ lbs. We shall assume *one half* of this load or 1905 lbs. ($=w$) lbs. to be carried by each of the systems Figs. 726 and 727, and superpose the results to obtain the total stresses shown in Fig. 725. We have

$$w\tan\theta=2846 \qquad w\sec\theta=3423$$
$$\tan\theta=1.494 \qquad \sec\theta=1.797$$

Then, Fig. 726 Stress in $01=ab=2w\tan\theta=5692$ lbs.
" " $12=23=bc=3w\tan\theta=8538$ "
" " $0a=2w\sec\theta=6846$ "
" " $1b=w\sec\theta=3423$ "

Similarly the stresses in Fig. 727 may be found and hence combined as in Fig. 725. To find the chord stresses from the moving wind load of 450 lbs. per foot run we have only to multiply the dead load wind stresses by 3. Thus

the panel load $w' = 450 \times 25.4 = 11{,}430$ lbs.

FIG. 725.

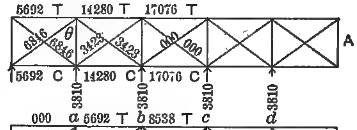


FIG. 726.

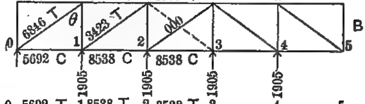
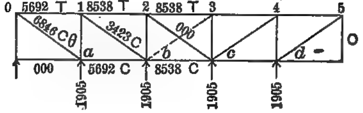


FIG. 727.



Suppose this moving load to come upon the span from the right, Fig. 728. The shear and hence the diagonal stress in each panel will be greatest when the load covers the span to the right of that panel. We shall assume that the shear is divided equally between the two diagonals in each panel.

Hence in panel 1 diagonal stress $= \frac{1}{2} \times \frac{10w'}{5} \sec\theta = 20{,}540$ lbs.

" " " 2 " " $= \frac{1}{2} \times \frac{6w'}{5} \sec\theta = 12{,}324$ "

" " " 3 " " $= \frac{1}{2} \times \frac{3w'}{5} \sec\theta = \ 6{,}162$ "

The combined wind loads in diagonals of the lower lateral system are shown in Fig. 729.

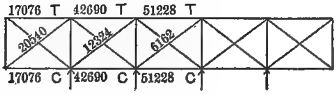


FIG. 728.

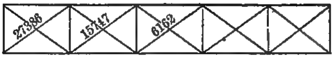


FIG. 729.

Upper Lateral System.—The stresses in the upper lateral system, Fig. 730, due to a load of 150 lbs. per lineal foot may be found similarly. The chord stresses need not be computed as they will be much less than 30 per cent of the combined live and dead load stresses and will be further reduced by the effect of the overturning moment of the wind.

Overturning Moment Due to Wind.—*(a) On Upper Chord.*—The wind load carried by the upper lateral system to each hip $= 2 \times 3810 = 7620$ lbs. This will increase the reaction at the ends of the leeward truss by an amount $R = 7{,}620 \times \frac{28}{17} = 12{,}550$ lbs. (Fig. 731.)

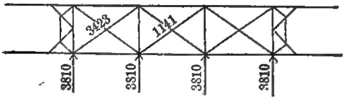


FIG. 730.

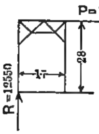


FIG. 731.

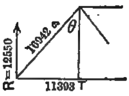


FIG. 732.

The corresponding compressive stress in the inclined end post = 12,550 sec θ = 16,942 lbs. (Fig. 732.) The corresponding tensile stress in the bottom chord = 12,550 tan θ = 11,393 lbs., the latter being constant throughout the length of the bottom chord. We shall have a compressive stress of the same magnitude throughout the leeward top chord, but as the direct effect of the wind is to cause tension in that chord, the two will tend to neutralize each other and need not be considered.

(*b*) *On Train.*—The wind load on the train (450 lbs. per ft.) is assumed to act 6 ft. above the base of rail or, say, 11 ft. above the plane of the bottom laterals, Fig. 733.

$$\text{Panel load } P = 25.4 \times 450 = 11{,}430.$$

Hence R, the reaction at leeward end of floor-beam $= \dfrac{11430 \times 11}{17} = 7396$ lbs.

Thus the overturning moment of the wind on the train produces a load of 7396 lbs. per panel on leeward truss.

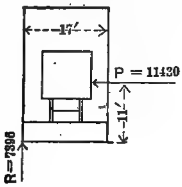


Fig. 733.

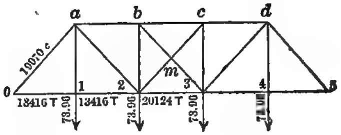


Fig. 734.

Fig. 734 shows the corresponding stresses in the end post and bottom chord. For reasons already mentioned we need not consider the effect on the top chord, while the resulting stresses in the web members are sufficiently small to be disregarded.

Table of Stresses.

Member.	Dead Load. 1.	Live Load. 2.	Fixed Wind Load. 3.	Moving Wind Load. 4.	Over-turning Moment Due to W. L. on Top Chord. 5.	Over-turning Moment Due to W. L. on Train. 6.	Maximum Combined Stresses.
0*a*.	61720C	146076C	. . . .	. . . .	16942C	19970C	244708C
01 leeward.	41468T	98157T	5692T	17076T	11393T	13416T	187202T
12 "	41468T	98157T	14230T	42690T	11393T	13416T	221354T
23 "	62202T	144482T	17076T	51228T	11393T	20124T	306505T
							Minimum.
01 windward. . .	41468T	98157T	5692C	(17076C)	11393C	(13416C)	24383T
12 " . . .	41468T	98157T	14230C	(42690C)	11393C	(13416C)	15845T
ab–bc.	62202C	144482C					
*a*2.	30860T	84910T					
*b*3	0	39544T					
*a*1.	15240T	61012T					
*b*2.	7620C	29292C					

N. B.—Minimum loads in 01, 12, are evidently obtained by combining columns 1, 3, and 5. As there is no approach to reversal of stress, no other members need be examined for reversal.

Main Truss Sections.—Bottom chord 01, 12. We shall use the same section in these two panels.

Section for dead load stress 41,468 ÷20,000 = 2.07 sq. ins.
" " live " " 98,157 ÷10,000 = 9.82 " "

Total load 139,625; total section = 11.89 " "

Average unit stress for dead and live load 139,625 ÷ 11.89 = 11,743 lbs. Section required for combined load = 221,354 ÷ (11,743 × 1.3) = 14.5 sq. in. net. Hence combined load governs. Use four 6″×4″×½′ angles = 19 sq. ins. gross area. Deducting 8 rivet-holes, each ½ sq. in., we have 19 − 4 = 15 sq. ins. net area, Fig. 735.

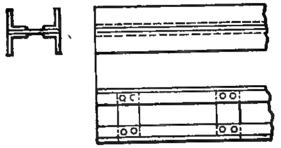


FIG. 735.

The angles will be connected by stay-plates spaced about 3 ft. 0 in. centre to centre.

Bottom Chord 23.

Section for dead load 62,202 ÷ 20,000 = 3.11 sq. ins.
" " live " 144,482 ÷ 10,000 = 14.45 " "

Total load = 206,684; total section = 17.56 " "

Average unit stress for dead and live loads 206,684 ÷ 17.56 = 11,770 lbs. Section required for combined loads = 306,505 ÷ (11,770 × 1.3) = 20.0 sq. ins. Use four 6″×4″×$\frac{11}{16}$″ angles = 25.64 sq. ins. gross area = 20.14 sq. ins. net area.

Diagonals. *a*2. Section for dead load 30,860 ÷ 20,000 = 1.54 sq. ins.
" " live " 84,910 ÷ 10,000 = 8.49 " "
Total section = 10.03 " "

Use two 12″—25-lb. channels = 15 sq. ins gross area = 12.75 sq. ins. net area.

bm–2*m*. Dead load = 0.
Live " 39,544 ÷ 10,000 = 3.95 sq. ins.

Use four 3½″×2½″×⅜″ angles = 8.44 sq. ins. gross area = 6.94 sq. ins. net.

This section is much larger than is actually required, but cannot well be reduced without using sections below the specified limits.

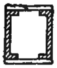


FIG. 736.

*a*1. Section for dead load 15,240 ÷ 20,000 = 0.76 sq. ins.
" " live " 61,012 ÷ 10,000 = 6.10 " "

Total section = 6.86 " "

Use four 5″×3″×⅜″ angles = 11.44 sq. ins. gross area = 8.44 sq. ins. net area

Top Chord.—The same section is required throughout.

Try two 15″—40-lb. channels, Fig. 737, laced top and bottom = 23.52 sq. ins. r about axis AB = 5.43 ins. (Carnegie). l = 25 ft. = 300 ins., and $\frac{l}{r} = 56$. Hence

$$20{,}000-90\frac{l}{r}=14{,}960 \text{ lbs.}$$

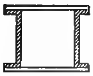


FIG. 737.

and

$$10{,}000-45\frac{l}{r}=\ \ 7{,}480 \text{ ''}$$

The section for dead load = 62,202 ÷ 14,960 = 4.15 sq. ins.
" " " live " = 14,482 ÷ 7,480 = 19.32 " "
Total section = 23.47 " "

Hence the assumed section is satisfactory.

Vertical Post b2.—The lightest convenient section we can use under the specifications will be two 12″—25-lb. channels, which will be found to exceed the requirements for mere strength.

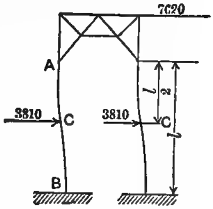


FIG. 738.

End Post oa. Fig. 738.—In addition to the compressive stresses in the end post, there is a bending moment due to the wind pressure on the top chord. The wind load carried to each hip, as already shown, is 7620 lbs., of which we may assume that *one-half*, or, 3810 lbs., will be carried to the pier by each post. It may also be assumed that the post is fixed at A, Fig. 738, by the portal bracing, and at B by the anchorage, so that there is a point of inflection at C and the B.M. on the post is $3810\times\frac{l}{2}=3810\times180=685{,}800$ in.-lbs., as l is about 30 ft. or 360 ins. The extreme fibre stress caused by this B.M., added to that caused by the compressive stress, must not exceed the specified limits.

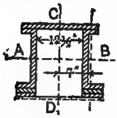


FIG 739.

Try the following sections, Fig. 739:

One cover plate 20″ × ½″ = 10.00 sq. ins.
Two 15″—50-lb. channels = 29.42 " "
Four flats 4″ × ⅝″ = 10.00 " "
Total section = 49.42 " "

The cover plate is used to make the section capable of resisting bending in a transverse direction. The flats are riveted to the bottom of the channels to balance the cover-plate and keep the centre of gravity of the section at the centre of the channels, so that the centre-of-gravity line of the end posts will intersect that of the other members at connecting points.

Moment of Inertia about AB.

Cover-plate $= \frac{1}{12}\times 20\times(\frac{1}{2})^3 + 10\times 7.75^2 \quad = 601.9$
Channels $= 2\times 402.7 \quad = 805.4$
Flats $= 2\times\frac{1}{12}\times 4\times 1.25^3 + 10\times 8.125^2 = 661.5$

Total moment of inertia $= 2068.2$

Moment of Inertia about CD.

Cover-plate $= \frac{1}{12}\times\frac{1}{2}\times 20^3 \quad = 333.3$
Channels $= 2\times 11.22 + 29.42\times(6.87)^2 \quad = 1412.4$
Flats $= \frac{1}{12}\times\frac{5}{4}\times 2\times 4^3 + 10\times 8.25^2 \quad = 693.9$

Total moment of inertia $= 2439.6$

or about 2200, allowing for rivet-holes.

About AB, $r=\sqrt{\frac{2068.2}{49.42}}=6.47$ ins.; l (unsupported length in vertical plane) $=453$ ins. Hence $\frac{l}{r}=70$ so that

$$17{,}000-90\frac{l}{r}=10{,}700 \text{ lbs.}$$

and

$$8{,}500-45\frac{l}{r}=5{,}350 \text{ ''}$$

Section for dead load $= 61{,}720\div 10{,}700 \quad = 5.77$ sq. ins.
" " live " $= 146{,}076\div 5{,}350 \quad = 27.30$ " "

Total load $= 207{,}796$ Total section $= 33.87$ " "

Allowable stress for dead and live loads
$= 207{,}796 \div 33.05 = 6283$ lbs.
Allowable stress for combined loads $= 6283\times 1.30 = 8168$ lbs.
The fibre stress due to bending $= \frac{685800\times 10.25}{2200} \quad = 3195$ "
The stress due to combined compression loads $\quad = 4952$ "

Total fibre stress $\quad 8147$ "

And as this is slightly less than the allowable intensity (8168 lbs.) the section is satisfactory.

Chord Splices.—In the top chord, for convenience in erection, only one splice will be used, placed in the middle panel. The abutting ends should be planed so as to insure perfect contact, and in that case full reliance may be placed on their bearing against each other, the function of the splice-plates being to hold the two sections in place. The bottom chord will be spliced in the second panel from each end. The rivets may be arranged in double shear, so that their bearing will limit their value. Bearing value of a $\frac{7}{8}$-in. rivet on $\frac{1}{2}$-in. angle $=15{,}000\times\frac{7}{8}\times\frac{1}{2}=6562$ for shop rivets or 4375 for field rivets. Thirty per cent extra where wind loads are included gives the value of a field rivet = 5688 lbs. Hence 221,354 ÷ 5688 = 40, which is the number of rivets required in the splice.

Lower Lateral Bracing.—The lower lateral bracing (Fig. 740) will be riveted to the bottom flanges of the stringers at A, B, C, and D, the points of intersection; so that the greatest unsupported length in a vertical plane will be AC or BD, that is, about 151 ins. The greatest stress in a diagonal is 27,386 lbs. compression or tension. The ratio $\frac{l}{r}$ must not exceed 120, and we shall therefore use two $4'' \times 3'' \times \frac{3}{8}''$ angles (= 4.9 sq. ins.) with the 4-in. legs riveted back to back at intervals of 1 ft. (Fig. 741.) $r = 126$, therefore $151 \div 1.26 = 120$. Allowable stress = $13,000 - 60\frac{l}{r} = 5800$ lbs., so that the section required is $27,386 \div 5800 = 4.7$ sq. ins., or slightly less than the section used. This section will of course be larger than is necessary for the panels nearer the centre of the span; but any reduction would make $\frac{l}{r}$ too great, so that the same section will be used for all bottom laterals. Transverse angles AD and BC should also be used, to provide for the longitudinal thrusts arising from braked trains. At the intersection O one of the diagonals must be spliced as shown in Fig. 742, and a sufficient number of rivets should be used here as well as in the end connections not only to carry the maximum stress but to develope the full strength of the diagonal.

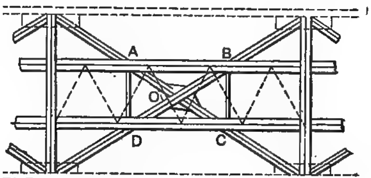


FIG. 740.

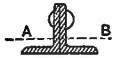


FIG. 741.

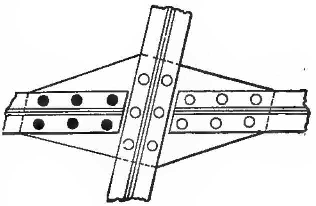


FIG. 742.

Upper Lateral Bracing.—The stresses in the upper laterals are very light, and the ratio $\frac{l}{r}$ will be the governing consideration.

The distance between the outer rivets in the connecting plates may be taken as 153 ins. If two angles $4'' \times 3'' \times \frac{3}{8}''$ are used, arranged as in Fig. 743, $r = 1.2$ and $\frac{l}{r} = 121$.

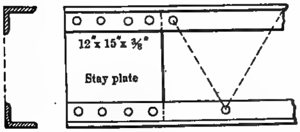


FIG. 743.

The distance back to back of the angles is 15 ins., the same as the depth of the top chord. For the transverse struts use four angles $3'' \times 2\frac{1}{2}'' \times \frac{3}{8}''$, with a single line of lacing.

Portals.—The total wind load at each hip is 7620 lbs. Assuming that the end posts are fixed at the lower end, there will be a point of contraflexure at A, Fig. 744. Each post may be assumed to transfer to the pier one half of the wind load at the hip, and therefore H, the horizontal reaction at A, = 3810 lbs.

Also, taking moments about A, $V=\dfrac{7620\times 31.15}{17}=13{,}962$ lbs. Stress in $OY=V\sec 45°=19{,}742$ lbs. Or the stress in OY may be obtained by taking moments about U_2, whence

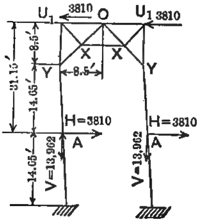


FIG. 744.

$$\text{stress in } OY=\frac{3810\times 31.15}{8.5}\sec 45°=19{,}742 \text{ lbs. C.}$$

Taking moments about O of the forces to the left,

$$\text{stress in } XX=\frac{3810\times 31.15-13{,}962\times 8.5}{4.25}=0.$$

Hence XX and, consequently, U_1X have no stress. Taking moments about Y,

$$\text{stress in } V_1O=\frac{3810(14.65+8.5)}{8.5}=13{,}962 \text{ lbs. T.}$$

The same section may be used for all the portal members as for the upper laterals.

(*Problem.*—Check the strength of the portal members and design the connections.)

The stresses in the sway-bracing at the centre panel-points are indeterminate. These braces make the structure more rigid and diminish vibration. Beyond securing a satisfactory value for the ratio $\dfrac{l}{r}$, their design is a matter of judgment. At the point of their connection with the post a diaphragm must be inserted between the two channels of the post, to avoid undue bending stresses.

Shoes.—The total weight of the span, including the track, will be about 225,000 lbs., of which 112,500 will be carried at each end. The greatest reaction due to the live load at either end will be 282,000 lbs. Hence the total load to be carried by each shoe will be

$$\tfrac{1}{2}(112{,}500+282{,}000) \text{ or } 197{,}000 \text{ lbs., nearly.}$$

The allowable pressure on the masonry is 250 lbs. per square inch. Hence bearing area required $=197{,}000\div 250=788$ sq. ins.

The base-plate will be made $27''\times 30''=810$ sq. ins.

The shoe at one end must rest upon rollers, so as to provide for expansion. The roller shoe is shown in Figs. 745, 746, the angles and vertical plates being of ¾-in. metal, and the horizontal plates ⅞-in. thick. A 5¼-in. pin may be used, the method of designing this pin will be shown in a subsequent example. The allowable bearing pressure for this pin per inch of length $=5\frac{1}{4}\times 15{,}000$ $=78{,}750$ lbs., so that the length of bearing required on the vertical plates is $197{,}000\div 78{,}750=2.5$ ins., while the four ¾-in. plates provide 3 ins. The strength of the vertical plates should also be checked as columns. These plates must transfer the wind pressure to the pier. The wind pressure on the span is $150+600=750$ lbs. per lineal foot, so that the horizontal pressure at each pedestal $=\frac{1}{4}(127\times 750)=23{,}810$ lbs., half of which, or say 12,000 lbs., must be carried by each web. The weakest section will be at *B*, and the B.M. at $B=12{,}000\times 7\frac{1}{2}=90{,}000$ in.-lbs.

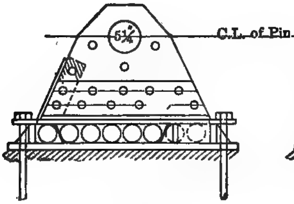


FIG. 745.

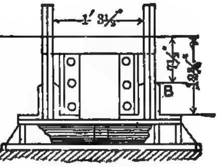


FIG. 746.

$$\text{Hence fibre stress}=\frac{90000\times 6}{24\times(1.5)^2}=10{,}000 \text{ lbs. per square inch.}$$

As this combined with the direct compression would be somewhat large, a diaphragm should be provided. The smallest diameter permissible for the friction rollers for a 127-ft. span is 3⅛ ins. and the allowable pressure per lineal inch is $3\frac{1}{8}\times 300=937.5$ lbs. Hence the total length of roller required for one shoe is $197{,}000\div 937.5=210$ ins. We shall use eight rollers 26¼ ins. long.

Provision should be made for a change of length of about 1¾ ins. in either direction, or 3½ ins. in all.

The best detail for a roller bearing is a cast-steel box in which the rollers may be placed. The box is then filled with oil to protect the rollers against rust and accumulations of dirt. For the fixed shoe a cast-steel gridiron may be used, the spaces between the ribs being filled with cement. Each shoe should be fastened to the masonry by means of anchor-bolts 1¼ ins. in diameter.

Camber.—Sufficient camber may be provided by lengthening the top chord ⅛ in. in every 10 ft. In one panel length this will amount to $\frac{5}{16}$ in., so that the length of the panel at the top chord will be $25'\ 4\frac{13}{16}''+\frac{5}{16}''=25'\ 5\frac{1}{8}''$. The diagonals must of course be lengthened in proportion. The make up of the members and a general elevation of the span are shown in Fig. 790, page 812.

Ex. 18. *Design of a 520-ft. swing span.*

The span will be designed to carry a double track railway between the trusses, while on cantilevers outside of the trusses provision will be made for combined highway and motorway traffic, and for pedestrians. The following live loads will be used:

For railway stringers, Waddell's Class " R";

For railway floor-beams, hangers, and subdiagonals, Waddell's Class " S" combined with the effects of the loads from the cantilevers;

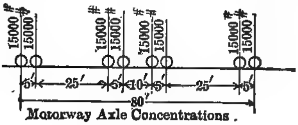


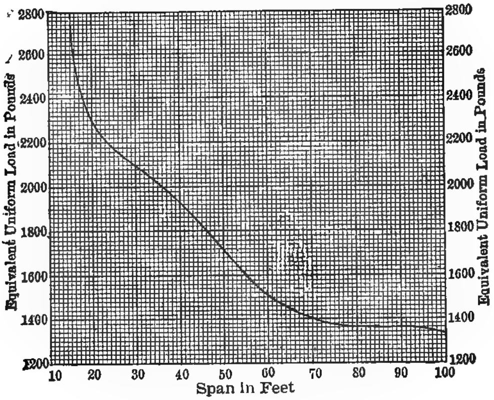
FIG. 747.

For main truss members, Waddell's Class " V " combined with the effects of cantilever loads;

For the roadway portion of the cantilever brackets, Class " B";

For each motorway track, the live load shown in Fig. 747.

Figs. 748 and 749 give the curves of Equivalent Uniform load for this last. The equivalent uniform loads for other classes may be found in Waddell's " De Pontibus";



E. U. MOTORWAY LOADS FOR PLATE GIRDERS

FIG. 748.

For the sidewalks, Class " C."

The span, Fig. 791, p. 813, will consist of one tower panel of 30 ft. 8 ins. and fourteen panels of 34 ft. $11\frac{7}{16}$ ins., or altogether 520 ft. $\frac{1}{8}$ in. In computing the loads the panels will be assumed 35 ft. long, the distance from centre to centre of trusses being 31 ft. 8 ins.

The details of the floor and handrails are in part governed by the specifications. The ties may be computed as in previous examples. The general arrangement of the floor is as shown in Fig. 750.

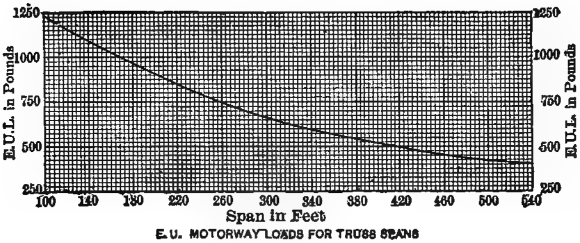


FIG. 749.

Stringers.—(a) Sidewalk Stringer.—Referring to Fig. 750 it will be seen that the sidewalk stringer will carry one half of the sidewalk. The dead

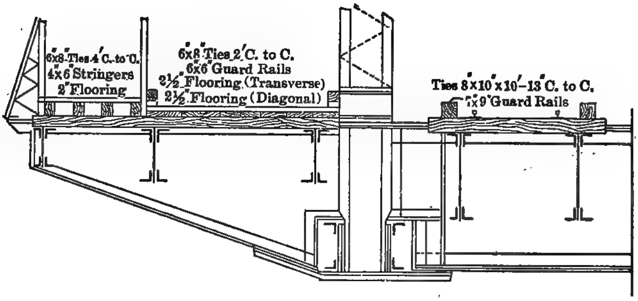


FIG. 750.

load will consist of the handrail, one half the floor and the weight of the girde itself, say 139 lbs. per lineal foot of girder.

Live load, Class " C," for 35-ft. span $=76.5$ lbs. per square foot. Henc we have

Dead load. . . . . $=139$ lbs. per linear foot of stringer
Live " $=76.5\times2.5$. . . . . . . $=191$ " " " " " "
Impact (35-ft. span) $=54\%$. . $=103$ " " " " " "

Total load. . . . . $=433$ " " " " " "

The max. B.M. $=\frac{1}{8}\times433\times35^2=66{,}400$ ft.-lbs.

The stringer may be designed as in previous examples. We shall us a $30''\times\frac{5}{16}''$ web; four angles $3''\times2\frac{1}{2}''\times\frac{5}{16}''$ for flanges, and the usual en stiffeners and fillers, the total weight being 2130 lbs., or about 60 lbs. per lineal foot of stringer.

(*b*) *Motorway Stringers.*—Outer stringer. The dead load will be as follows:

Handrail.	60	lbs. per foot of stringer
Floor-timber.	204	“ “ “ “ “
Rail and splices.	30	“ “ “ “ “
Weight of stringer (say).	125	“ “ “ “ “
Total dead load.	=419	“ “ “ “ “
Live load from sidewalk.	191	“ “ “ “ “
Impact load from sidewalk (54%).	103	“ “ “ “ “
Live load from motorway.	1020	“ “ “ “ “
Impact load from motorway (74.8%).	761	“ “ “ “ “
Total load.	=2494	“ “ “ “ “

The max. B.M. $=\frac{1}{8}\times 2494\times 35^2=382{,}000$ ft.-lbs.

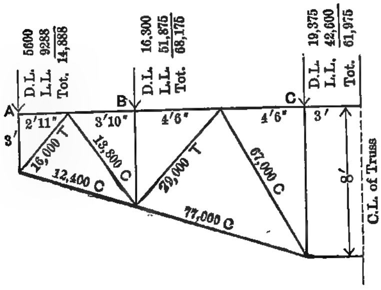


FIG. 751.

We shall use a 36″×$\frac{5}{16}$″ web and 6″×3½″×½″ angles for flanges, the total weight of the stringer with details being 4157 lbs., or 119 lbs. per lineal foot of span. The inner stringer will carry the same load, less that due to the sidewalk. The flange angles may be reduced to 5″×3½″×½″, the total weight being 3915 lbs.

Intermediate Cantilever Brackets.—Concentration from sidewalk stringer at *A*, Fig. 751.

Dead load = 139×35.	= 4865	lbs.
Cantilever and laterals (say).	= 735	“
Total dead load.	= 5600	“
Live load (70-ft. span) 73×2.5×35.	= 6388	“
Impact (70-ft. span) 45.4%.	= 2900	“
Total load.	=14,888	“

Similarly assuming that the portions of the weights of the cantilevers and laterals concentrated at *B* and *C* are 1600 lbs. and 4500 lbs. respectively, we have the total concentration at *B* = 68,175 lbs. and at *C* = 61,975 lbs.

The stresses may now be determined graphically and the sections and details designed in the usual way. The weight of each intermediate cantilever will be about 510 lbs. The general dimensions for the cantilevers at each end of the span will be the same, but as they have to carry only the load upon half of one panel, instead of half of two panels, they may be made lighter.

The weight of each will be about 4350 lbs. Hence the total weight of met in the cantilevers will be $5100\times28+4350\times4=160,200$ lbs., or 310 lbs. p lineal foot of span.

(*c*) *Railway stringers:*

Equivalent live load, Class "R" (35-ft. span)...	=	8450 lbs.
Impact 74.77%..........................	=	6320 "
Dead load (including floor)..................	=	1100 "
Total load..........................	=	15,870 "

per lineal foot of track, or 7935 lbs. per lineal foot of stringer.

$\text{B.M.}=\frac{1}{8}\times7935\times35^2=1,215,000$ ft.-lbs.

Assume a web of $62''\times\frac{3}{8}''$; the effective depth $=4.87$ ft.

Flange stress $=1,215,000\div4.87=250,000$ lbs. (about).

" area $=250,000\div14,000-(\frac{1}{8}\text{ of web})=17.8-2.9$ $=14.9$ sq. ins.

Use two $6''\times6''\times\frac{3}{4}''$ angles $=16.88$ sq. ins. gross area $=15.38$ sq. ins. n area.

The weight of one stringer will be 8637 lbs. or (say) 250 lbs. per line foot of stringer. The tower span is only 30 ft. 8 ins. in length, so that th stringers will weigh about 6250 lbs., each.

The stringer bracing will consist of $4''\times4''\times\frac{3}{8}''$ angles, attached to th top flanges and arranged as shown in Fig. 752, with a cross-frame at the centr of each panel, Fig. 753. Including the necessary connecting plates and rivets it will weigh 80 lbs. per lineal foot of span. Hence the total weight of stringe will be $8750\times56+6250\times4=515,000$ lbs. $=990$ lbs. per lineal foot of spa Or, total weight for stringers and bracing $=990+80=1070$ lbs. per line foot of span.

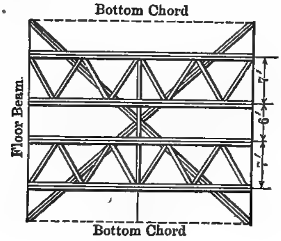


FIG. 752.

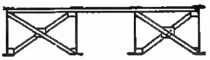


FIG. 753.

Intermediate Floor-beams.—Concentration from railway stringers:

Equivalent live load (70-ft. span), Class "S"...	=	6320	lbs.	per	lineal	foo
Impact 70.18%..........................	=	4450	"	"	"	"
Dead load (not including flooring)............	=	900	"	"	"	"
Total load..........................	=	11,670	"	"	"	"

Hence the dead-load concentration for each stringer is $\frac{900}{2} \times 35 = 15{,}750$ lbs., and the corresponding live-load concentration $= \frac{10770}{2} \times 35 = 188{,}500$ lbs.

The floor-beam and its cantilevers will then be loaded as shown in Fig. 754, to which must be added the weight of the floor-beam itself, which will be assumed as 18,500 lbs. uniformly distributed.

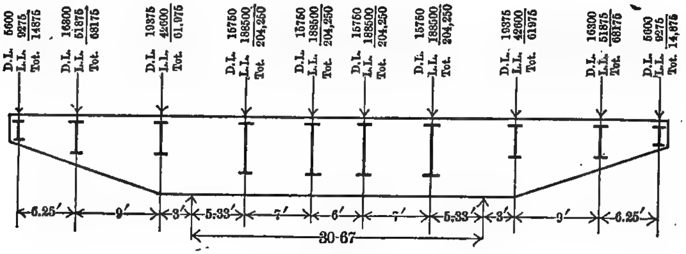

D.L. 5600 L.L. 9275 Tot. 14875
D.L. 16300 L.L. 51875 Tot. 68175
D.L. 19375 L.L. 42600 Tot. 61,975
D.L. 15750 L.L. 188500 Tot. 204,250
D.L. 15750 L.L. 188500 Tot. 204,250
D.L. 15750 L.L. 188500 Tot. 204,250
D.L. 15750 L.L. 188500 Tot. 204,250
D.L. 19375 L.L. 42600 Tot. 61975
D.L. 16300 L.L. 51875 Tot. 68175
D.L. 5600 L.L. 9275 Tot. 14,875
6.25′
9′
3′
5.33′
7′
6′
7′
5.33′
3′
9′
6.25′
30·67

FIG. 754.

The maximum positive B.M. will occur in the floor-beam when the latter carries its full load and the cantilevers carry only the dead load. The minimum positive, or maximum negative B.M. (should reversal take place) will occur when the cantilevers are fully loaded, and the floor-beam carries only the dead load.

The B.M. at centre of floor-beam due to full load on same
$= 408{,}500 \times 12.33 - 204{,}250 \times 7 + \frac{1}{8} \times 18{,}500 \times 30.67 = +3{,}678{,}000$ ft.-lbs.
Neg. B.M. due to dead load on cantilevers
$= 19{,}375 \times 3 + 16{,}300 \times 12 + 5{,}600 \times 18.25 = -\ \ 356{,}000$ "
Maximum positive B.M. . . . . $= +3{,}322{,}000$ "
Pos. B.M. due to dead load on floor-beam
$= 31{,}500 \times 12.33 - 15{,}750 \times 7 + \frac{1}{8} \times 18{,}500 \times 30.67$ . . $= +\ \ 349{,}000$ "
Neg. B.M. due to full load on cantilevers
$= 1{,}975 \times 3 + 68{,}175 \times 12 + 14{,}875 \times 18.25 = -1{,}275{,}500$ "
Maximum negative B.M. . . . . $= -\ \ 926{,}500$ "

The flange stress will reverse, and hence (Waddell's Specifications, p. 17) we must add to the sectional area required for the greater stress three fourths of that required for the less. Or we may add three fourths of the smaller B.M. to the larger, and design the section for the sum, viz.,

$$3{,}322{,}000 + \tfrac{3}{4} \times 926{,}500 = 4{,}017{,}000 \text{ ft.-lbs.}$$

Assume a web of $96'' \times \frac{1}{2}''$; the effective depth $= 8$ ft.; then flange stress $= 4{,}017{,}000 \div 8 = 502{,}125$ lbs., and flange area $= 502{,}125 \div 14{,}000 - (\frac{1}{8}$ of web$) = 35.87 - 6.0 = 29.87$ sq. ins.

Use two angles, $8'' \times 8'' \times \frac{9}{16}''$. . . . . $=17.52$ sq. ins.
" one cover-plate, $16'' \times \frac{9}{16}''$. . . . . $= 9.00$ " "
" one plate, $16'' \times \frac{1}{2}''$. . . . . $= 8.00$ " "

Total area. . . . . $=34.52$ " " gross section

$=34.52-4.37=30.15$ sq. ins. net section, which is sufficient.

It will be found that the outer cover-plate must extend to about the outer stringer as shown, Fig. 755.

The floor-beam must be cut away at the lower corner to make room for the bottom chord and pin. The web will be reinforced by two plates $55'' \times \frac{9}{16}'' \times 5'5''$ as shown.

The floor-beam will weigh 17,500 lbs. (assumed weight 18,500). Total weight of floor-beams $=17,500 \times 14$ $=245,000$, or 470 lbs. per linear foot of span.

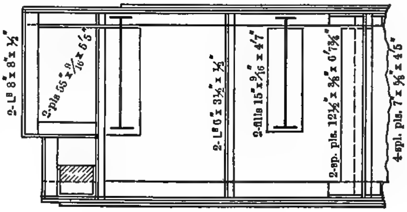


FIG. 755.

(N.B. The weights of the two floor-beams at the tower will not be considered now, since they will be carried to the pier without inducing stress in the trusses.)

Lower Laterals.—Three cases must be considered for the wind stresses in the lower lateral system.

(*a*) A dead load of 30 lbs. per square foot of exposed surface when the span is open. This, from a rough computation, will amount to 670 lbs. per lineal foot of span.

Hence panel load $=w=670 \times 35=23,450$ lbs.; $\sec\theta=1.516$;

$$\tan\theta=1.139;\ \frac{w\sec\theta}{2}=17,800;\ w\tan\theta=26,700.$$

The stresses will be as shown in Fig. 756, the stresses in the diagonals beginning at the left being successively $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$ . . $6\frac{1}{2}$ times $\frac{w\sec\theta}{2}$ and the chord stresses in the same way being $\frac{1}{4}$, $1\frac{1}{4}$, $3\frac{1}{4}$, $6\frac{1}{4}$, $10\frac{1}{4}$, $15\frac{1}{4}$, $21\frac{1}{4}$, $28\frac{1}{4}$ times $w\tan\theta$.

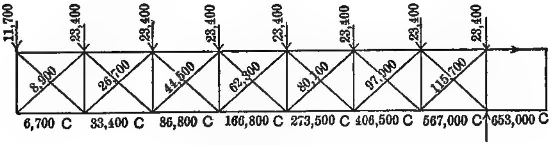


FIG. 756.

(*b*) The span closed, and a moving load of 1100 lbs. per lineal foot on one arm only; $w=$ panel load $=1100 \times 35=38,500$. The stresses, Fig. 757, may be found as in the riveted truss, Ex. 17.

(*c*) The span closed, and a moving load of 1100 lbs. covering both arms. In this case we have a girder continuous over four supports.

Panel load $=1100\times 35\doteq 38{,}500$ lbs

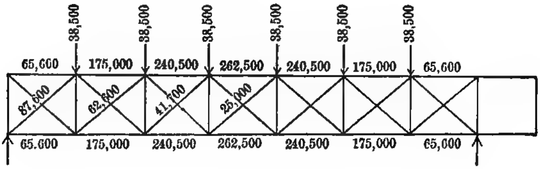


FIG. 757.

The following table gives the reactions in pounds at the end and centre for loads at each panel-point of one arm, and at the corresponding panel-point of the other arm, Fig. 760.

Points Loaded.	Reaction at End in Lbs.	Reaction at Centre in Lbs.
1–1	30,800	7,700
2–2	23,500	15,000
3–3	16,200	22,300
4–4	10,400	28,100
5–5	5,400	33,100
6–6	1,900	36,600

The stresses for each panel load may readily be obtained either analytically or graphically. For example, consider a load at panel point 3; $R=16{,}200$ lbs. Hence the shear in the first three panels is 16,200 lbs. and the corresponding stress in each diagonal

$$\frac{16200\sec\theta}{2}=12{,}290\text{ lbs.}$$

The shear in the remaining panels is $16{,}200-38{,}500=-22{,}300$ lbs.

Hence diagonal stress $=-\dfrac{22300}{2}\sec\theta=-17{,}570$ lbs.

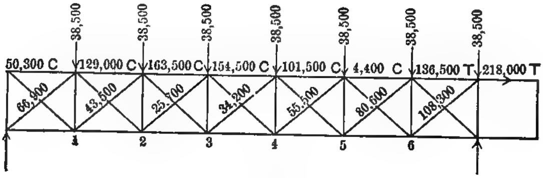


FIG. 758.

Fig. 758 shows the stresses found by combining the effects of the several loads.

The greatest stress in the lower laterals is thus 115,700 lbs. T or C, in the panel next the tower.

We shall use four angles connected by angle lacing, Fig. 759.

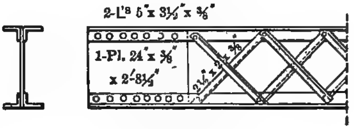


FIG. 759.

The top angles will be riveted to the stringers at their intersections, but the lower angles will be unsupported in a horizontal plane for nearly half their length, or say 240 ins. Try four angles each 5″×3½″×⅜″, arranged as shown, Fig. 761; area = 12.2 sq. ins.; $r = 2.4$ ins.; $\frac{l}{r} = \frac{240}{2.4} = 100$. Hence

$$\text{allowable stress per sq. in.} = 16{,}000 - 60\frac{l}{r} = 10{,}000 \text{ lbs.},$$

and the section required = 115,700 ÷ 10,000 = 11.57 sq. ins., so that the above section is ample. For the sake of rigidity, and to avoid the use of a large number of different sections, we shall use the above for all lower laterals.

The total weight of the lower lateral system will be about 84,000 lbs. or, say, 160 lbs. per lineal foot of span.

The loads on the upper laterals will be much lighter, but the sections will be determined by the ratio $\frac{l}{r}$, which for wind bracing must not exceed 140. The section may be similar to that for the lower laterals, using, however, 4″×3″×⅜″ angles, and 2¼″×½″ bars instead of angles for lacing, in the diagonal members U_1U_2, etc., and 5″×3½″ angles for the transverse struts U_2U_2, etc., Fig. 762.

Instead of making the upper lateral bracing continuous throughout the entire length of the span, it is omitted in the panels U_5U_6 and U_6U_7. Bracing shown by the dotted lines, Fig. 760, is inserted in the plane of U_5M_6 (Fig. 761),

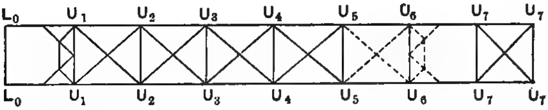


FIG. 760.

and a portal similar to that in the plane of U_1L_0 is introduced in the plane M_6L_7. The object of this is to transfer the wind load on the top chord to the pier in the most direct manner possible, which tends to secure both economy and rigidity. The bracing in the tower panel U_7U_7 consists of adjustable rods, simply intended to keep the tower posts in position. The portals in panels U_1L_0 and M_6L_7 may be designed as in the riveted span Ex 17. All the portal struts consist of four angles, 5″×3½″×⅜″, arranged as in the laterals.

The total weight of metal in the portals and upper laterals will be about 104,000 lbs., or 200 lbs. per lineal foot of span.

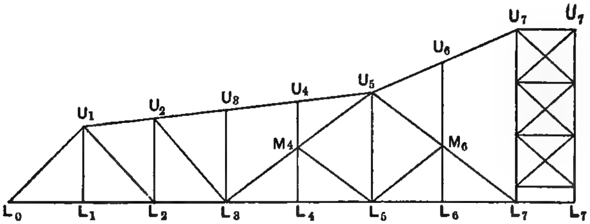


FIG. 761.

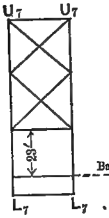


FIG. 762.

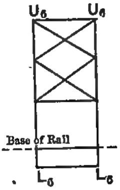


FIG. 763.

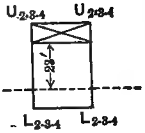


FIG. 764.

The stresses in the vertical sway bracing between the trusses are indeterminate, and the ratio $\frac{l}{r}$ will be the governing consideration in their design. We shall use struts of the same section as the diagonals of the upper lateral system, the general arrangement being as shown in Figs. 762–766. The horizontal transverse strut is brought to within 23 ft. (the specified clearance) of the base of rail.

The total weight of metal in sway-bracing will be about 75,400 lbs., or 145 lbs. per lineal foot of span. Hence the total weight of upper and lower laterals, portals, and sway-bracing will be $160+200+145=505$ lbs. per lineal foot of span, of which about 460 lbs. will be carried by the trusses.

The dead load carried by the trusses may now be in part computed and in part assumed.

Dead Loads.

Tracks and flooring	1910	lbs. per lineal foot
Sidewalk stringers, 2 at 60 lbs.	120	" " " "
Motorway " 2 at 119 lbs.	238	" " " "
" " 2 at 112 lbs.	224	" " " "
Cantilevers	100	" " " "
" lateral bracing for	100	" " " "
Railway stringers and bracing	1070	" " " "
Floor-beams	470	" " " "
Handrails	300	" " " "
Bolts for floor	48	" " " "
Lateral system	460	" " " "
Trusses and operating machinery (assumed)	3700	" " " "
Total dead load	=8950	" " " "

or 4475 lbs. per lineal foot of each truss.

The average panel load would be 4475×35=156,600 lbs., but the panel load will evidently increase from the ends toward the tower, so that we shall assume the loading given in Fig. 765.

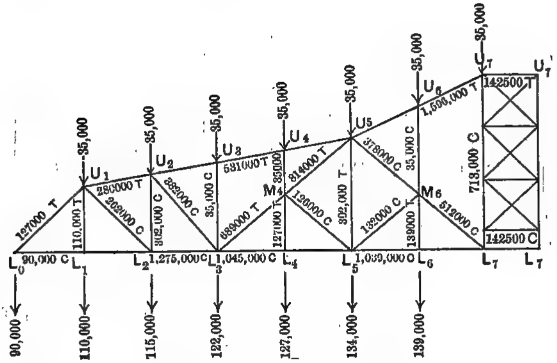


FIG. 765.

When the span is open the entire weight rests on the centre pier, and we have two cantilever arms balancing each other. When the span is closed the same is true, except in so far as provision is made for lifting the ends and giving them a firm bearing by means of wedges or equivalent devices. The end-lifting gear will be designed to exert an uplift of 120,000 lbs. at each end of each truss.

The stresses will be considered under four conditions of loading.

(1) Span open, dead load only acting. The resulting stresses are readily obtained graphically, Fig. 766, and are shown in Fig. 765.

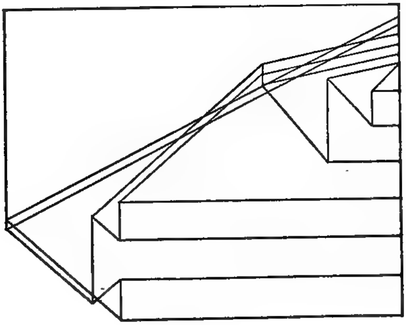


FIG. 766.

(2) An uplift of 120,000 lbs. at the end of the truss. The resulting stresses, also found graphically, are indicated in Fig. 767. As this uplift may not always act, the resulting stresses are considered only when they *increase* the stresses due to other conditions of loading.

(3) Live load on one arm only, which is then assumed to act as a single span.

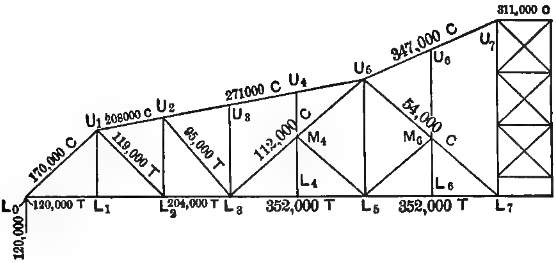


FIG. 767.

Equivalent live load, Class "V," 245-ft. span = 4525 lbs. lin. ft. of truss
" " " motorway " " = 775 " " " " "
" " " sidewalks " " = 290 " " " " "

Total live load .... = 5590 " " " " "

Referring to Fig. 754 it will be seen that the most unfavorable loading on a truss will occur when the motorways and sidewalks outside of that truss and the railway tracks between the trusses all carry their maximum load, while the motorway and sidewalk on the other side are empty. This condition of loading may readily prevail over one or two panels, but is quite unlikely to do so over the whole span. Its effect will therefore be considered in computing the stresses in the floor-beam hangers and sub-diagonals, but will be ignored in proportioning the main truss members.

Panel live load = 5590 × 35 = 195,650 lbs.

The chord stresses will be greatest when the live load covers the whole span. The reaction at each end is then 195,650 × 3 = 586,950 lbs., and a single stress diagram (Fig. 768) gives all the chord stresses.

To find the stresses in U_1L_2, U_2L_2, U_2L_3, assume a reaction of 100,000

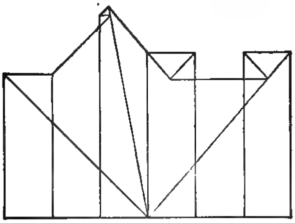


FIG. 768.

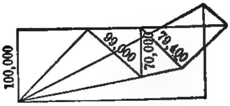


FIG. 769.

lbs. at the *left* end of the truss. Fig. 769 gives the resulting stresses, viz., 99,000, 70,000, and 79,400 lbs., respectively. Now the stress in U_1L_2 is a maximum when L_2 and all the panel-points to the *right* of it carry the full live panel load. The corresponding reaction at L_0 is $\frac{1}{7}\times 195{,}650\times(1+2+3+4+5)=419{,}250$ lbs. Hence the stress in

$$U_1L_2=\frac{419250\times 99000}{100000}=415{,}000 \text{ lbs. (nearly).}$$

U_2L_2 and U_2L_3 will have a maximum stress when the live load extends from the *right* to L_3.

Corresponding reaction at $L_0=\frac{1}{7}\times 195{,}650\times(1+2+3+4)=279{,}500$ lbs.,

so that $$\text{stress in } U_2L_2=\frac{279500\times 70000}{100000}=195{,}000 \text{ lbs.,}$$

and $$\text{stress in } U_2L_3=\frac{279500\times 79400}{100000}=222{,}000 \text{ lbs.}$$

Similarly the tensile stress in L_3M_4 will be a maximum when the live load extends from the *left* end of the span up to L_3. Reaction at $L_7=167{,}700$. From Fig. 770 the stress in L_3M_4 due to a reaction of 100,000 lbs. at L_7 is 235,000 lbs. Hence the maximum tensile stress in $L_3M_4=235{,}000\times\frac{167700}{100000}$ $=394{,}000$ lbs.

When the live load advances to L_4, it will be best, on account of the secondary members, to make a special diagram, Fig. 771. Reaction at $L_7=279{,}500$ lbs., whence stress in $M_4U_5=465{,}000$ lbs.

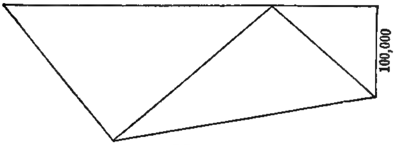


FIG. 770.

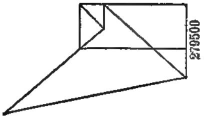


FIG. 771.

The stresses in the hangers U_1L_1, M_4L_4, and M_6L_6 will be greatest when the live load covers two panels.

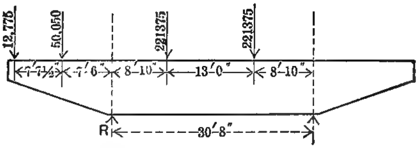


FIG. 772.

Equivalent live load for sidewalk (70-ft. span, Class " C ") $=73\times 5=365$ lbs. per lineal foot of sidewalk.

Concentration on floor-beam $=365\times 35=12{,}775$ lbs.

Equivalent live load for motorway $=1430$ lbs. per foot of track.

Concentration from one track on floor-beam $=1430\times 35=50{,}050$ lbs.

Equivalent live load for railway (Class " S ") $=6235$ lbs. per foot of track.

Concentration from one track on floor-beam $=6325\times 35=221{,}375$ lbs.

Assuming one sidewalk and motorway unloaded, we have the floor-beam and cantilevers loaded as in Fig. 774, where R is the reaction on the hanger, the concentration from each track being placed at the centre line of the track.

Hence $R\times30.67=221{,}375\times30.67+50{,}050\times38.17+12{,}775\times45.79$, or $R=$ 303,000 lbs. (nearly) = stress in U_1L_1, M_4L_4, and M_6L_6. The sub-diagonals M_4L_5, M_6L_5 will each transfer one half of the load in the hanger. Hence stress in each is $\frac{1}{2}\times303{,}000 \sec\theta=233{,}000$ lbs. C.

The stress in the hanger U_5L_5 will be greatest when the live load extends from L_3 to L_7. Taking the equivalent uniform loads for a span of 140 ft. in the same way as above, we find the stress to be 535,500 lbs. T. Fig. 773 shows the stresses in all members for a live load covering one arm only.

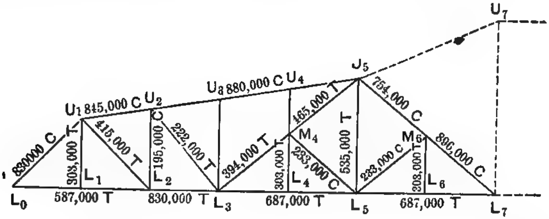


FIG. 773.

(4) Live load covering both arms, which then form a girder continuous over four supports.

Equivalent live loads 520-ft. span:

Sidewalks, 43.5×5........ = 218 lbs. per lineal foot of truss
Motorway................. = 425 " " " " " "
Railway (Class "V")....... = 4190 " " " " " "

Total load.......... = 4833 " " " " " "

Panel load $=4833\times35=169{,}155$ lbs. $=w$. It will be assumed that a panel load is placed successively at panel-points 1-1, 2-2, etc., Fig. 774.

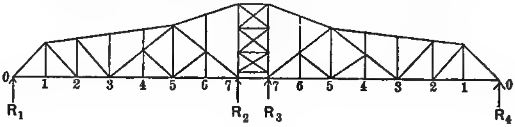


FIG. 774.

The reactions in pounds for each position of the live load may be computed by the Theorem of Three Moments, as follows:

Live Load at Panel-point.	$R_1=R_4$.	$R_2=R_3$.	Stress Diagram.
1-1	135,300	33,800	Fig. 768
2-2	102,800	66,300	" 769
3-3	71,500	97,600	" 770
4-4	45,100	124,000	" 771
5-5	24,000	145,100	" 772
6-6	9,000	160,100	" 773

A separate stress diagram is required to give the stresses for each position of the load, Figs. 778–83. The stresses in lbs. are as follows:

Member.	Panel Load at						Total Tension.	Total Compression.
	Panel-point 1–1.	Panel-point 2–2.	Panel-point 3–3.	Panel-point 4-4.	Panel-point 5–5.	Panel-point 6–6.		
U_1L_3	190000C	144000C	100000C	64000C	34000C	13000C	....	545000C
L_0L_2	135000T	103000T	715 0T	45000T	24000T	9000T	387500T	
L_2L_3	84000T	162000T	119000T	77000T	410C0T	16000T	499000T	
L_3L_5	....	4000T	11000T	132000T	71000T	27000T	245000T	
L_5L_7	....	4000T	11000T	35000T	71000T	124000T	245000T	
U_1U_2	85000C	175000C	121000C	77000C	42000C	15000C	....	515000C
U_2U_5	48000C	102000C	157000C	102000C	54000C	21000C	....	484000C
U_5U_7	29000T	54000T	75000T	78000T	70000T	45000T	351000T	
U_1L_2	70000C	100000T	70000T	45000T	24000T	9000T	248000T	70000C
U_2L_2	49000T	98000T	49000C	31000C	17000C	6000C	147000T	103000C
U_2L_3	56000C	112000C	55000T	35000T	20000T	7000T	117000T	168000C
L_3M_4	65000T	130000T	194000T	41000T	22000C	9000C	389000T	72000C
M_4U_5	65000T	130000T	194000T	87000T	22000C	9000C	476000T	31000C
U_5M_6	32000C	65000C	97000C	136000C	174000C	86000C	....	590000C
M_6L_7	32000C	65000C	97000C	136000C	174000C	215000C	....	719000C
U_7U_7	25000T	46000T	65000T	70500T	63000T	39000T	309000T	
U_7L_7	13000C	24000C	34500C	35000C	32000C	20000C	....	158000C

To allow for impact a percentage I is to be added to the live loads. For railway and motorway loads $I=\frac{400}{l+500}$, and for sidewalk loads $I=\frac{100}{l+150}$, l being the length of span which must be covered to produce the maximum stress

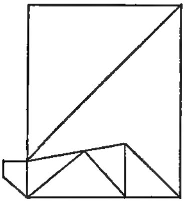


FIG. 775.

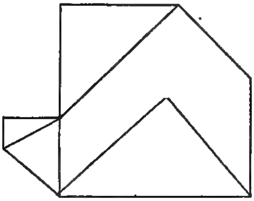


FIG. 776.

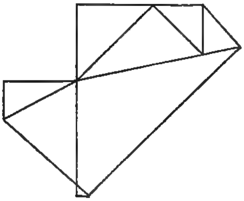


FIG. 777.

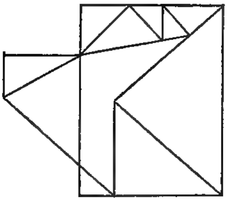


FIG. 778.

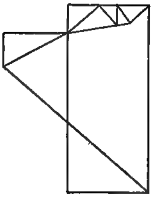


FIG. 779.

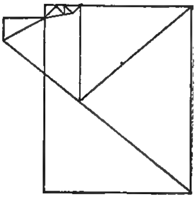


FIG. 780.

in the member under consideration. In the present case the following average values may be taken:

No. of Panels Loaded.	I.
1.	73.0 per cent.
2.	68.1 " "
3.	64.1 " "
4.	60.25 " "
5.	57.6 " "
6.	54.8 " "
7.	51.2 " "
Both arms	36.8 " "

The various stresses may now be combined as in the following table:

COMBINED STRESSES.

Member.	Case 1, Dead Load, in Lbs.	Case 2, Uplift Load, in Lbs.	Case 3, Live Load, One Arm, in Lbs.	Case 4, Live Load, Both Arms, in Lbs.	No. of Panels Loaded for Max.	Impact Factor.	Impact, in Lbs.	Total.	
								Tension, in Lbs.	Compression, in Lbs.
L_0U_1	127,000T	170,000C	830,000C	(545,000C)	7	.512	425,000C	127,000	1,298,000
L_0L_2	90,000C	120,000T	587,000T	(387,500T)	7	.512	301,000T	918,000	90,000
L_2L_3	275,000C	204,000T	830,000T	(499,000T)	7	.512	425,000T	1,184,000	275,000
L_3L_5	1,045,000C	352,000T	687,000T	(245,000T)	7	.512	352,000T	346,000	1,045,000
U_2U_2	280,000T	208,000C	845,000C	(515,000C)	7	.512	433,000C	280,000	1,206,000
U_2U_5	531,000T	271,000C	880,000C	(484,000C)	7	.512	450,000C	531,000	1,070,000
U_1L_1	110,000T	....	303,000T	....	2	.681	206,000T	619,000	
U_1L_2	262,000C	119,000T	415,000T	(248,000T)	15	.576	239,000T	511,000	
				70,000C	15	.368	26,000C	....	358,000
U_2L_2	302,000T	84,000C	195,000C	(103,000C)	4	.6025	117,500C	....	945,000
				147,000T	15	.368	54,000T	503,000	
U_2L_3	382,000C	95,000T	222,000T	(117,000T)	4	.6025	133,500T	68,500	
				168,000C	15	.368	62,000C	....	612,000
U_3L_3	35,000C	....	....	....	....	....	....	....	35,000
L_3M_4	689,000T	112,000C	394,000T	(389,000T)	3	.641	253,000T	1,336,000	
M_4U_5	814,000T	112,000C	465,000T	(476,000T)	4	.6025	280,000T	1,559,000	
U_4M_4	35,000C	....	....	....	....	....	....	....	35,000
M_4L_4	127,000T	....	303,000T	....	2	.681	206,000T	636,000	
M_4L_5	126,000C	....	233,000C	....	2	.681	158,000C	....	517,000
U_5L_5	302,000T	....	535,000T	....	4	.6025	323,000T	1,160,500	
U_5M_6	378,000C	54,000C	754,000C	(590,000C)	7	.512	386,000C	....	1,572,000
U_6M_6	35,000C	....	....	....	....	....	....	....	35,000
M_6L_6	139,000T	....	303,000T	....	2	.681	206,000T	648,000	
M_6L_7	512,000C	54,000C	896,000C	(719,000C)	7	.512	459,000C	....	1,921,000
U_7U_7	1,425,000T	....	....	311,000T	15	.368	114,500T	1,850,500	
U_7L_7	713,000C	....	....	156,000C	15	.368	57,500C	....	926,500
U_5U_7	1,596,000T	347,000C	....	347,000T	15	.368	128,000T	2,071,000	

N.B.—Figures in parentheses ( ) are not included in totals.

The wind stresses in the chords will not affect the section required, and are therefore not considered in this table.

Where a member carries stresses of opposite kinds the area of section required must be computed for tension and compression separately, and three fourths of the smaller area added to the greater. Thus, for the top chord, try the following section:

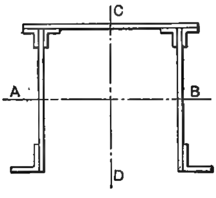


FIG. 781.

One cover-plate $36'' \times \frac{9}{16}''$	20.25	sq ins.
Four top angles $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{3}{8}''$.	9.92	" "
Two webs $30'' \times \frac{13}{16}''$	48.75	" "
Two bottom angles $6'' \times 6'' \times \frac{3}{4}''$.	16.88	" "
Total section	95.80	" "

Unsupported length, $l = 35' - 6\frac{15}{32}''$, say 427 ins.

Moment of inertia about axis $CD = 18{,}117$, and about axis $AB = 13,290$. Therefore $r = \sqrt{\frac{13290}{95.80}} = 11.8$ and $\frac{l}{r} = 36.2$.

Hence the allowable stress per sq. in. $= 18,000 - 70\frac{l}{r} = 15,470$ lbs.

Thus the section required for compression U_1U_2

$$= 1,206,000 \div 15,470 = 78.0 \text{ sq. ins.};$$

the section required for tension U_1U_2

$$= 280,000 \div 16,000 = 17.5 \text{ sq. ins.},$$

and the total section required

$$= 78.0 + \tfrac{3}{4} \times 17.5 = 91.1 \text{ sq. ins.}$$

The section required for compression U_2U_5

$$= 1,070,000 \div 15,470 = 69.3 \text{ sq. ins.};$$

the section required for tension U_2U_5

$$= 531\,000 \div 16,000 = 33.2 \text{ sq. ins.},$$

and the total section

$$= 69.3 + \tfrac{3}{4} \times 33.2 = 94.2 \text{ sq. ins.}$$

The above section may be used for the entire top chord.

Top chord section U_5U_6 carries tension only, and eye-bars may be used. Section required $= 2,071,000 \div 18,000 = 115.1$ sq. ins. Use four bars 12″ × $1\frac{5}{8}$″ and two bars 12″ × $1\frac{9}{16}$″ = 115.5 sq. ins.

All other members may be designed in a similar manner. Taking, for example, the members which meet at U_5, we may use for M_4U_5 four bars 12″ × $1\frac{13}{16}$″ = 87.0 sq. ins.; for U_5L_5 four bars 10″ × $1\frac{13}{16}$″ = 72.5 sq. ins.; and for U_5M_6 the following section, Fig. 782:

One cover-plate 36″ × $\frac{9}{16}$″	20.25 sq. ins.
Four top angles $3\frac{1}{2}$″ × $3\frac{1}{2}$″ × $\frac{7}{16}$″	11.48 " "
Four webs (in pairs) 30″ × $\frac{1}{2}$″	60.00 " "
Two bottom angles 6″ × 6″ × $\frac{3}{4}$″	16.88 " "
Total section	= 108.61 " "

The stresses from all these members are transmitted through the pin at U_5. The allowable bearing on the pin is 22,000 lbs. per square inch of the surface obtained by multiplying together the diameter of the pin and the length over which the member bears upon it. Assume 10 ins. as the diameter of pin U_5. Then allowable bearing per inch of length $= 22,000 \times 10 = 220,000$ lbs. The largest eye-bars are 12 ins. wide, so that they can safely carry $12 \times 18,000 = 216,000$ lbs. per inch of thickness. The 220,000 lbs. in bearing afforded by a 10-in. pin is accordingly ample for the eye-bars.

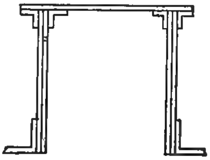


FIG. 782.

The length of bearing on the pin required for $U_5M_6=1,572,000 \div 220,000 = 7.15$ ins., of which the webs provide 2 ins., so that 5.15 ins. in the whole member or 2.57 (say $2\frac{5}{8}$) in. on each side must be made up by means of *pin-plates.* We shall reinforce each web with six plates $\frac{7}{16}''$ thick arranged as in Figs. 783–4. These plates should be of sufficient length and width to distribute the pressure from the pin uniformly over the entire section of the member. The two inside plates in this member are extended so as to form a hinge. Plates of the following dimensions may be used:

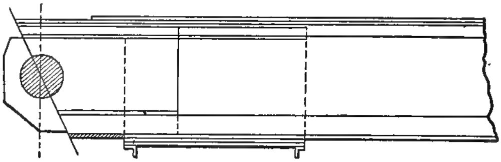


FIG. 783.

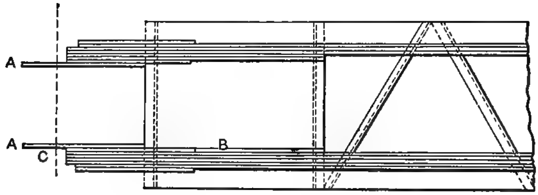


FIG. 784.

Two plates $26'' \times \frac{7}{16}'' \times 6'\ 0''$ inside
" " $29'' \times \frac{7}{16}'' \times 6'\ 1\frac{1}{2}''$ "
" " $25\frac{3}{4}'' \times \frac{7}{16}'' \times 3'\ 6''$ "
" " $20\frac{3}{4}'' \times \frac{7}{16}'' \times 2'\ 8\frac{1}{2}''$ hinge-plates
" " $20\frac{1}{2}'' \times \frac{7}{16}'' \times 3'\ 6''$ outside
" " $23\frac{1}{2}'' \times \frac{7}{16}'' \times 3'\ 7\frac{1}{2}''$ "

The pressure on each plate will be $\frac{7}{16} \times 220,000 = 96,000$ lbs. Hence the number of rivets required to transfer the stress in plate A across plane CB

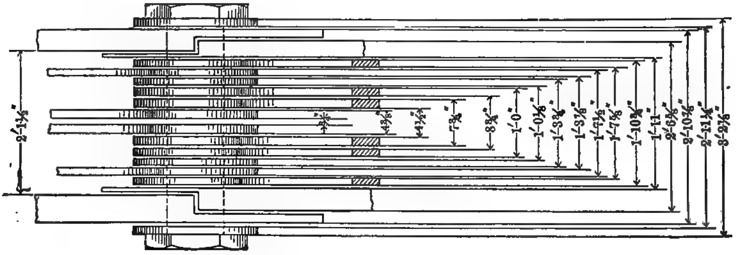


FIG. 785.

$=96,000 \div 6013$, say 16, and at least sixteen additional rivets will be required at each successive plate until the web is reached.

Figs. 785–6 show the joint at U_5 in plan and elevation.

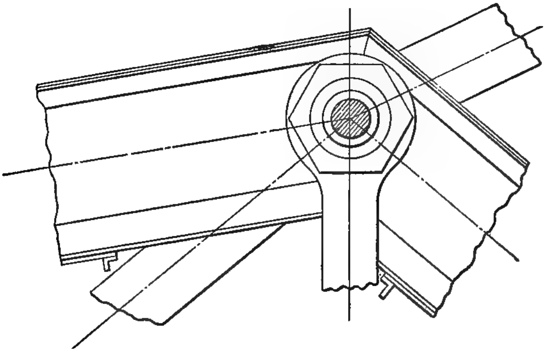


FIG. 786.

The greatest stress in the pin U_5 will evidently occur when both arms of the span are fully loaded. The stresses in the various members will then be as shown in Fig. 787.

Following are the horizontal and vertical components H and V:

	H	V
U_4U_5	+ 128,900	+ 23,100
U_5U_6	+1,851,500	+925,700
U_5M_4	−1,080,000	−926,400
U_5M_6	− 899,400	+771,400
U_5L_5		−764,000

Positive stresses are supposed to act to the right and upward; negative to the left and downward. Distributing the stress in each member among its components in proportion to their sectional areas, Figs. 788 and 789 show

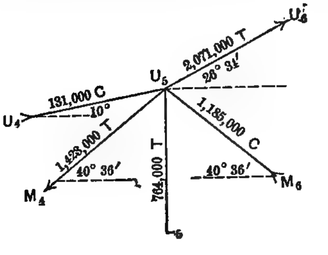


FIG. 787.

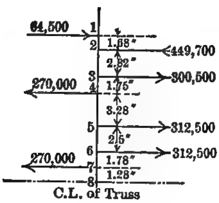


FIG. 788.

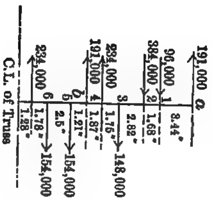


FIG. 789.

the loading of the pin in horizontal and vertical planes respectively. Only one half of the pin is shown, as the arrangement of the members is symmetrical about the centre line of the truss. The load from each piece is also assumed to be concentrated at the centre of its bearing upon the pin. The assumed dead load of 35,000 lbs. at U_5 may be distributed over the pin, reducing each upward component and increasing each downward one by about 2000 lbs. The bending moments in a horizontal and vertical plane may now be computed for each point in the pin.

BENDING MOMENTS IN HORIZONTAL PLANE (M_H).

Point.	Load.	Shear.	Lever-arm.	Increment of B.M.	Total B.M.
		0			
1	+ 64,500	+ 64,500	1.68 ins.	+ 108,360	+ 108,360 in.-lbs
2	−449,700	−385,200	2.82 "	−1,086,264	− 977,904 "
3	+300,500	− 84,700	1.75 "	− 148,225	−1,126,129 "
4	−270,000	−354,700	3.28 "	−1,163,416	−2,289,545 "
5	+312,500	− 42,200	2.5 "	− 105,500	−2,395,045 "
6	+312,500	+270,300	1.78 "	+ 481,134	−1,913,911 "
7	−270,000				

BENDING MOMENTS IN VERTICAL PLANE (M_V).

a	−191,000	−191,000	3.44 ins.	− 657,040	− 657,040 in.-lbs.
1	+ 9,600	−181,400	1.68 "	− 304,752	− 961,792 "
2	+384,000	+202,600	2.82 "	+ 571,332	− 390,460 "
3	+148,000	+350,600	1.75 "	+ 613,550	+ 223,040 "
4	−234,000	+116,600	1.87 "	+ 218,042	+ 441,132 "
b	−191,000	− 74,400	1.21 "	− 90,024	+ 351,108 "
5	+154,000	+ 79,600	2.5 "	+ 199,000	+ 550,108 "
6	+154,000	+233,600	1.78 "	+ 415,808	+ 965,916 "
7	−234,000				

The greatest resultant B.M. will evidently be at 6, where B.M. $=\sqrt{M_H{}^2+M_V{}^2}$ $=\sqrt{(2,395,045^2+550,108^2)}=2,457,000$ in.-lbs.

Allowable fibre stress in pin is 27,000 lbs. per square inch.

Hence $2,457,000=\frac{27000}{r}I=\frac{27000\pi r^3}{4}$, where r is the radius of the pin, and therefore $r=4.87$ ins. and diameter of pin $=9.74$ ins., so that the assumed diameter, 10 ins., is on the safe side.

Other joints and members may be designed in the same way.

It will be necessary to determine the deflection of the ends of the span due to its own weight and other causes, to ascertain how much the eye-bars in panels U_5U_6, U_6U_7 must be shortened in order to bring the centre of the pins at the ends of the span to the same elevation as those near the centre.

The deflection Δ of a given point due to the distortion of any member is given by the formula

$$\Delta=\Sigma\frac{pul}{E},$$

where p is the stress per square inch in the member;
l " length of the member;
u " stress in the member due to a unit load placed at the point whose deflection is sought;
E = the coefficient of elasticity (Young's modulus).

Downward deflections and tensile stresses will be considered *positive;* upward deflections and compressive stresses, *negative.*

Deflection at End of Span Due to Dead Load.—p is obtained by dividing the dead-load stress in each member by the sectional area of the member; u can be obtained graphically by placing a unit load at the end of the span; $E=29{,}000{,}000$ lbs. The following Table can now be prepared:

Member.	Stress.	Area.	p	l	$\frac{pl}{E}$	u	$\frac{pul}{E}$
L_0L_2	− 90,000	71.095	− 1,226	838.4	.0367	−1.000	+ .0367
L_2L_3	− 275,000	104.095	− 2,642	419.4	.0382	−1.695	+ .0649
L_3L_5	−1,045,000	99.97	−10,453	838.4	.3020	−2.912	+ .8800
L_5L_7	−1,039,500	99.97	−10,398	838.4	.3000	−2.912	+ .8736
L_7L_7	−1,425,000	470.40	− 3,030	184.0	.0193	−2.579	+ .0496
U_1U_2	+ 280,000	95.8	+ 2,921	426.06	.0430	+1.723	+ .0742
U_2U_5	+ 531,000	95.8	+ 5,539	1278.18	.2442	+2.238	+ .5465
U_5U_7	+1,596,000	115.5	+13,818	938.13	.4480	+2.879	+1.2900
U_7U_7	+1,425,000	103.5	+13,780	184.00	.0875	+2.5735	+ .2252
L_0U_1	+ 127,000	99.55	+ 1,276	593.53	.0262	+1.414	+ .0369
U_5M_6	− 378,000	108.61	− 3,483	552.75	.0664	+ .445	− .0300
M_6L_7	− 512,000	142.38	− 3,596	552.75	.0686	+ .445	− .0305
U_1L_2	− 262,000	68.43	− 3,835	593.53	.0784	− .984	+ .0784
U_2L_3	− 382,000	65.18	− 5,830	648.81	.1315	− .790	+ .1040
L_3M_4	+ 689,000	75.00	+ 9,187	552.75	.1749	+ .928	+ .1630
M_4U_5	+ 814,000	87.00	+ 9,356	552.75	.1784	+ .928	+ .1660
U_2L_2	+ 302,000	46.98	+ 6,440	495.0	.110	+ .695	+ .0764
U_7L_7	− 713,000	70.44	−10,150	1140.0	.399	−1.228	+ .5150

Hence total deflection due to dead load $=5.1199$ ins. $\left(=\sum\frac{pul}{E}\right)$.

Similarly the deflection due to the uplift at the ends may be obtained. The amount is -1.6662 ins.

The pin-holes will be about $\frac{1}{32}$ in. larger than the pins, and hence there will be a play at each end of a pin-connected member $=\frac{1}{64}$ or .0156 in.

Considering this play as a distortion of the member, corresponding to $\frac{pl}{E}$ in the above table, it may be shown that the deflection at the end due to this cause is $+0.5419$ in.

The temperature of the top chords and web members will often exceed that of the bottom chords. Assuming a mean difference of 15° F. for the top chords and $7\frac{1}{2}$° F. for the posts and diagonals, the elongation of the former per unit of length will be $15\times0.00000667=0.0001$ in. and of the latter 0.00005 in. This will cause a deflection of the ends $=+0.6831$ in.

Similarly the lengthening of the top chords and diagonals on account of camber will cause a deflection at the ends $=3.1964$ ins. Hence the total deflection at the ends will be

$$5.1199-1.6662+0.5419+0.6831+3.1964=7.8751 \text{ ins.}$$

This will be taken up by shortening the eye-bars in U_5U_6 and U_6U_7. The value of u for $U_5U_7=2.879$; hence the amount each panel must be shortened $=\frac{1}{2}(7.8751\div2.879)=1.368=1\frac{11}{32}$ ins.

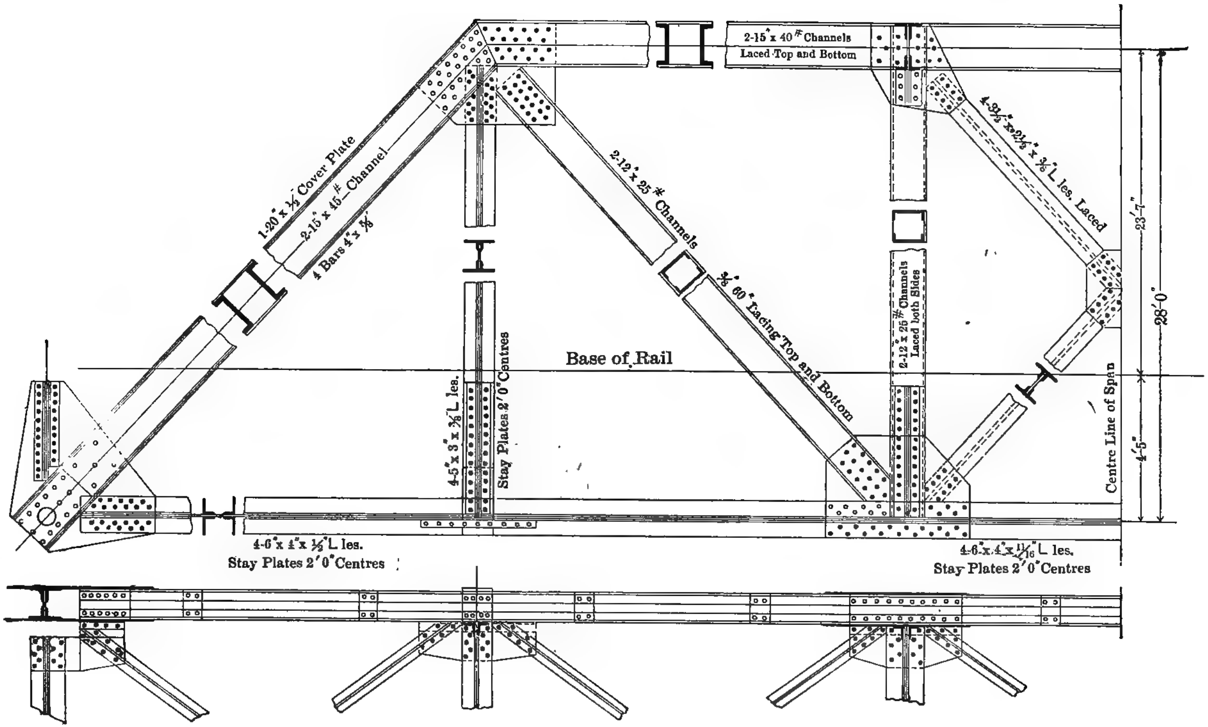


FIG. 790.—DESIGN FOR 127-FT. SPAN.

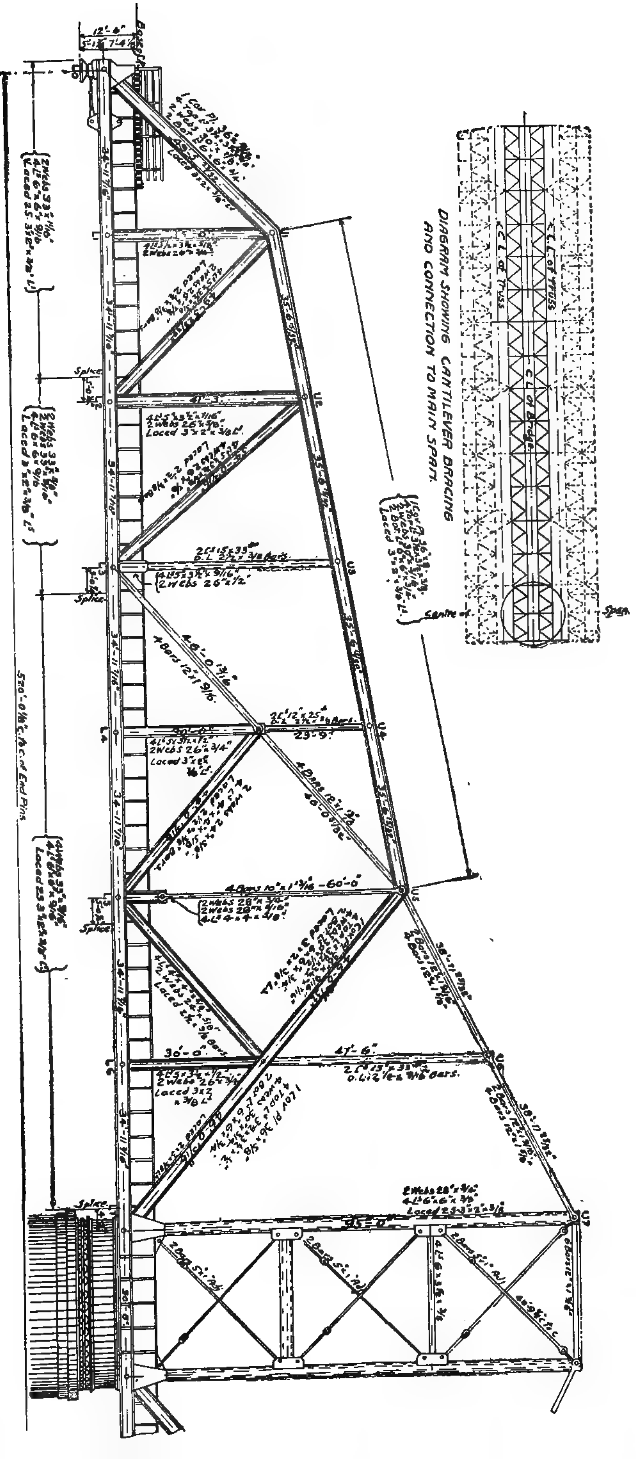


FIG. 791.—DESIGN FOR 520-FT. SWING-SPAN.

TABLE OF ACTUAL WEIGHTS OF MODERN PLATE-GIRDER BRIDGES.

(From data kindly supplied by Messrs. Macdonald, Wilson, Peterson, Yeatman, and others.)

No. of Spans.	Length.	Depth.	No. of Tracks.	Deck, *D*, Half Through, *H*, or Through, *T*.	Weight in Lbs.	Weight per Lineal Foot of Each Track in Tons of 2000 Lbs.	Engineer, Builder, or Owner.	Remarks.
1	26′		1	*D*	8212		P. McK. & Y. R.R.	Add to *dead load* 400 lbs. per foot for ties and rails. *Live load.* Two engines of 161,000 lbs. in 108 ft., followed by 2240 lbs. per lineal foot.
1	30′		1	*D*	10160		P. McK. & Y. R.R.	
1	30′		1	*D*	10000		Phœnix I. Co.	
1	33′		1	*D*	12500		Dominion B. Co.	
1	34′ 8″		1	*D*	15609		Edge Moor I. Co.	
1	34′ 8″	3′	1	*D*	16550		D. B. Co.	
1	37′ 7½″	3′ 6″	2	*D*		.375	Wilson	
1	38′		1	*D*	15153		P. McK. & Y. R.R.	*Dead load* and *live load.* As for 26′ girder.
1	44′ 8″		1	*D*	23383		Edge Moor I. Co.	
1	44′ 8″	3′ 6″	1	*D*	25750		D. B. Co.	
1	48′		1	*D*	23900		P. McK. & Y. R.R.	*Dead load* and *live load.* As for 26′ girder.
1	54′ 2″		2		103720		Phœnix I. Co.	
1	56′ 6″	5′	2	*T*	106500		D. B. Co.	
1	58′		1	*D*	35000		P. McK. & Y. R.R.	*Dead load* and *live load.* As for 26′ girder.
1	59′	5′	1	*D*	44400		D. B. Co.	
1	59′		2		122000		Phœnix I. Co.	
1	60′		1	*D*	39418		P. McK. & Y. R.R.	*Dead load* and *live load.* As for 26′ girder.
1	60′	5′	1	*D*	37000		Phœnix I. Co.	
1	63′	5′	2	*H*		.6	Wilson	
1	64′	5′	1	*D*	43800		D. B. Co.	
1	64′ 6″	6′	1	*D*	52530		"	
1	66′	5′ 4″	2	*D*		.54	Wilson	Girders = 799 lbs.; floor system = 293 lbs.
1	67′	5′ 4″	2	*D*		.52		Girders = 755 lbs.; floor system = 293 lbs.
1	68′		1	*D*	46340		P. McK. & Y. R.R.	*Dead load* and *live load.* As for 26′ girder.
1	68′	5′	2	*D*		.55	Wilson	
1	64′—69′ 4½″	5′ 4″	1	*D*		.53	"	Girders = 1125 lbs.; floor system = 275 lbs.
4	68′ 8½″ to 71′ 4½″	6′	2	*D*		.525	"	Girders = 935 lbs.; floor system = 275 lbs.
5	69′ 10¾″ to 71′ 0¾″	5′ 9″	2	*D*		.55	"	
1	75′	5′ 9″	2	*D*		.55	"	
1	82′	5′ 6″	2	*D*		.4	"	Girders = 1131 lbs.; floor system = 275 lbs.
1	82′	6′	2	*H*		.7	"	
1	85′	6′ 6″	1	*D*	82340		D. B. Co.	
1	85′	7′	1	*T*	119000		"	*Live load.* Two 98-ton engines.
1	86′	6′	1	*T*	118600		"	
1	87′ 6″	7′	1	*D*	84100		"	*Live load.* Two 98-ton engines.
1	88′	7′	1	*D*	87300		"	*Live load.* Two 98-ton engines.
1	90′	7′	2	*D*		.65	Wilson	

TABLE OF ACTUAL WEIGHTS OF MODERN OPEN-TRUSS BRIDGES.

(From data kindly supplied by Messrs. Cooper, Wilson, Macdonald, Peterson, Yeatman, and others.)

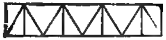
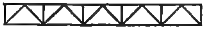
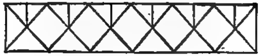
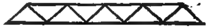
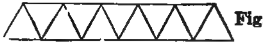
No. of pans.	Length C. to C. of Pins	Depth.	No. of Tracks.	Deck, *D*, Half Through, *H*, or Through, *T*.	Weight in Lbs.	Weight per Lineal Foot of Each Track, in Tons of 2000 Lbs.	Engineer, Builder, or Owner.	Remarks.
1	28′		1	*D*	12000		Alden	
1	30′		1	*D*	11000		Edge Moor I. Co.	
1	32′		1	*D*	11500		Passaic	
1	40′ 6″		1	*D*	31900		Edge Moor I. Co.	
1	43′		1	*T*	33100		Phœnix I. Co.	
1	50′		1	*D*	19800		Edge Moor I. Co.	
1	53′ 4½″	10′ 3″	2	*D*		.585	Wilson	Single intersection.
1	54′		1	*D*	36000		Alden	
1	61′ 4″	10′ 3″	2	*D*		.59	Wilson	Single intersection.
1	71′ 9″		2	*T*	146000		Alden	
3	73′	9′ 11½″	1	*D*		.5		 Fig. 792.
1	77′ 6″	24′	1	*D*	78448		C. B. R. R.	 Fig. 793.
1	82′ 6″		1	*D*	52800		Edge Moor I. Co.	
3	82′ 6″	8′ 8″	2	*D*		.54	Wilson	
1	89′ 6″—91′ 6″	10′ 3″	2	*D*		.72	"	Single intersection.
1	90′ 1″	17′ 7″	2	*D*		.61	"	Single intersection; 8 panels.
1	94′	17′	2	*D*		.506	"	 B Fig. 794.
1	94′ 0¼″ to 96′ 8½″	15′	2	*D*		.69	"	Single intersection.
1	97′	10′	2	*H*		.76	"	 Flooring 1 track = 480 lbs.
1	98′	11′			130100		G. T. R. R.	Iron 1 " =1525 "
1	100′		1	*T*	91000		Phœnix I. Co.	
1	100′		1	*T*	76000		"	Fig. 795.
1	102′ 6″	24′	1	*D*	112855		D. B. Co.	
1	103′		1	*T*	92000		Passaic	
1	106′		1	*D*	128000		Phœnix I. Co.	
2	115′ 6″	17′	2	*D*		.54	Wilson	Truss similar to B.
1	116′ 10⅛″	20′ 7″	2	*T*		.65	"	 Fig. 796.

TABLE OF ACTUAL WEIGHTS OF MODERN OPEN-TRUSS BRIDGES.

(From data kindly supplied by Messrs. Cooper, Wilson, Macdonald, Peterson, Yeatman, and others.)

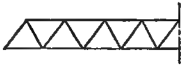
No. of Spans.	Length C. to C. of Pins.	Depth.	No. of Tracks.	Deck, *D*, or Through, *T*.	Weight in Lbs. of Each Span.	Weight per Lineal Foot of Each Track, in Tons of 2000 Lbs.	Engineer, Builder, or Owner.	Remarks.
1	123′ 2½″ to 125′ 10¼″	15′	2	*D*		.75	Wilson	Single intersection.
1	125′	24′	1	*D*	136271		D. B. Co.	
1	126′ 8″	17′	2	*D*		.57	Wilson	Truss similar to B.
1	134′		1	*T*	144368		P. McK. & Y. R.R.	
2	131′ 1″ to 134′ 7″	18′	2	*D*		.76	Wilson	Single intersection.
1	316′		1	*T*	143000		Passaic	
3	136′ 9″	17′ 7″	2	*D*		.76	Wilson	Single intersection; 12 panels.
3	136′	17′	2	*D*		.59	"	Truss similar to B.
1	137′ 1$\frac{5}{16}$″	21′	2	*T*		.8	"	 Half truss. Fig. 797.
4	137′ 4″	17′ 10″	1	*D*		.8	"	Truss similar to B.
4	142′ 8″	17′ 10″	1	*D*		.8	"	Truss similar to B.
5	147′ 6″	17′	2	*D*	376600		Cooper	
1	150′		1	*T*	310000		Keystone B. Co.	
1	150′	24′	2	*T*	327870		P. McK. & Y. R.R.	
1	151′ 4″	17′	2	*D*		.5	Wilson	Drawbridge—truss similar to B.
1	152′		1	*D*	161000		Phœnix I. Co.	
1	153′		1	*T*	180000			
1	154′ 7″	28′	1	*D*	221200		D. B. Co.	
3	154′		1	*D*	267160		Peterson	Single intersection.
3	156′	20″	2	*D*		.86	Wilson	Single intersection.
1	156′		1	*D*	161000		Phœnix I. Co.	
3	156′ 9″	25′ 4″	1	*D*	222400		Cooper	
23	157′ 2½″	18′	2	*D*		.65	Wilson	Truss similar to B.
1	161′ 3″		1	*T*	260700		Edge Moor I. Co.	
1	161′ 10½″	27′	1	*D*	229458		D. B. Co.	
1	162′ 9″		1	*T*	208340		Edge Moor I Co.	
1	163′ 2⅛″	25′	2	*T*		.82	Wilson	Floor system—1 track = 300 lbs.; iron = 1340 lbs. Single intersection; 10 panels.
1	163′ 5⅛″	20′ 4″	1	*D*		.85	"	Floor system—1 track = 380 lbs.; iron = 1320 lbs. Single intersection; 13 panels.

TABLE OF ACTUAL WEIGHTS OF MODERN OPEN-TRUSS BRIDGES.

(From data kindly supplied by Messrs. Cooper, Wilson, Macdonald, Peterson, Yeatman, and others.)

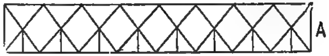
No. of Spans.	Length C. to C. of End Pins.	Depth.	No. of Tracks.	Deck, *D*, or Through, *T*.	Weight of Each Span.	Weight per Lineal Foot of Each Track, in Tons of 2000 Lbs.	Engineer, Builder, or Owner.	Remarks.
3	166′	27′	1	*T*		.89	Wilson	 Fig. 798.
1	167′ 7″	28′	2	*T*	440000		Cooper	
1	175′		1	*T*	250000			
1	190′ 3¾″	30′	2	*T*		.91	Wilson	Track and floor ties = 300 lbs.; iron = 1520 lbs. Single intersection; 9 panels.
1	192′	30′	2		609600		D. B. Co.	*Live load.* Two 98-ton engines followed by 3000 lbs. per lineal foot.
1	195′ 8½″		1	*D*	251000		Phœnix I. Co.	
1	197′		1	*D*	316000		Lassig	
2	206′	33′	2	*T*	612000		Cooper	
3	214′ 6″		1	*T*	454763		Peterson	Double intersection (pin).
1	208′ 4″	33′	2	*T*	655000		Cooper	
1	245′ 7″	40′	2	*T*	806000		"	
1	253′		2	*T*	825000		Edge Moor I. Co.	
1	250′		1	*T*	450000		"	
1	260′	26′	1	*T*		1.24	Wilson	Truss similar to A; upper chord with 9 divisions.
1	270′	27′	1	*T*		1.4	"	Truss similar to A; upper chord with 12 divisions.
1	342′	40′			1906000		Cooper	Aqueduct bridge for Panama Canal.
1	396′ 1″		1	*T*	789000		Phœnix I. Co.	
1	525′		2	*T*	4100000		Edge Moor I. Co.	
1	550′	2 R. R. 2 Rdwys. 2 F'tways		*T*	8815000		Phœnix I. Co.	

TABLE OF ACTUAL WEIGHTS OF MODERN OPEN-TRUSS BRIDGES.

(From data kindly supplied by Messrs. Cooper, Wilson, Macdonald, Peterson, Yeatman, and others.)

Bridge.		Length C. to C. of Pins.	No. of Tracks.	Deck or Through.	Weight in Pounds.	Remarks.
Kentucky River Bridge.	1 Suspended.	260′	1	*D*	435000	35′ deep, 24′ wide, 10 panels.
	2 Arms.	130′	1	*D*	ea. 415000	Depths, 60′ and 35′; 31′ over towers.
	2 Arms.	208′	1	*D*	ea. 513500	Depths, 60′ and 25′; 8 panels.
	22 Trestle girders.	30′	1	*D*	ea. 8500	*Unit stresses.* Floor, soft steel $= 9000\left(1+\frac{\text{min.}}{\text{max.}}\right)$.
	2 Main towers.				ea. 309000	
	Trestle bents.	928 ft. vert.			230000	Trusses, hard steel $=12000\left(1+\frac{\text{min.}}{\text{max.}}\right)$.
	Floor of cantilever system included in above.					
	2 Anchor towers.	92′ 9″ base to cap.			ea. 79250	*Live load.* Two 86-ton engines followed by 3000 lbs.
	2 Anchorages.				ea. 10750	
Kanawha Bridge.	2 Shore arms.	240′	1	*T*	ea. *550345	Depths of shore arms at end = 24′, at centre = 44′; width = 20′.
	2 River arms.	160′	1	*T*	ea. 317066	Depths of river arms at pier = 68′, at end = 24′; width = 20′; 8 panels.
	1 Suspended.	160′			211810	Depth = 24′; width = 20′; 8 panels.
						Live load. Two consolidated engines of 171 tons in 108′.
						Wind-pressure. 450 lbs. per foot of loaded bridge. 250 lbs. per foot of unloaded bridge.
Poughkeepsie Bridge.	Shore arm.	190′ 10½″	2	*D*	749007	
	Cantilever.	160′	2	*D*	636668	
	Suspended.	212′	2	*D*	384970	
	Connecting.	509′	2	*D*	2929052	
	River towers.	100′ ea.			423095	
	Shore towers.	85′ ea.			270799	
	E. Viaduct.	175′	2	*D*	375734	
	E. "	161′	2	*D*	312380	
	W. "	145′	2	*D*	253144	
	E. "	115′	2	*D*	183392	
	E. "	116′ 9″	2	*D*	180163	
Thames River Bridge.	West approach.	164′ 6″	2	*T*	342894	*Live load.* Two consolidated engines of 171 tons in 103 ft., followed by 3000 lbs. per lineal foot.
	River bridge.	148′	2	*D*	250174	*Stresses.* All tension $=10000\left(1+\frac{\text{min.}}{\text{max.}}\right)$.
	River bridge.	306′ 7″	2	*T*	1062228	Alternate tension or compression $=10000\left(1-\frac{\text{min.}}{2\text{ max.}}\right)$.
	River bridge (draw).	497′ 7″	2	*T*	1781598	Posts, pin ends $=\left(10000-60\frac{l}{r}\right)\left(1+\frac{\text{min.}}{\text{max.}}\right)$.
	Turn-table, shafting.				322830	Chords (compression) $=\left(10000-40\frac{l}{r}\right)\left(1+\frac{\text{min.}}{\text{max.}}\right)$.

* Includes backstays connecting tower with span.

TABLE OF ACTUAL WEIGHTS OF MODERN OPEN-TRUSS BRIDGES.

(From data kindly supplied by Messrs. Cooper, Wilson, Macdonald, Peterson, Yeatman, and others.)

Location.	No. of Spans.	Length C. to C. of Pins.	Depth.	No. of Tracks.	Deck, *D*, or Through, *T*.	Weight in Lbs. of Each Span.	Type.	Engineer, Builder, or Owner.
St. Lawrence River, Lachine	3	80′		1	*D*	70527	Plate girder	Peterson and C. Shaler Smith
	8	240′		1	*D*	482162	Pin, double intersection	
	2	269′		1	*D*	706631	" " "	
	2	408′		1	*T*	1343021	" " "	
	1	119′		1	*D*	170314	Pin, single intersection	
Ottawa River, Vaudreuil...	8	100′		1	*D*	108478	Lattice, double intersection	Peterson
	7	71′		1	*D*	64337	Plate girder	
	2	65′		1	*D*	55300	" "	
Ottawa River, St. Anne's..	1	324′		1	*T*	931749	Pin, single intersection	Peterson
	3	104′		1	*T*	176870	Lattice, double intersection	
	2	100′		1	*D*	108478	" " "	
	8	66′		1	*D*	55541	Plate girders	
Sault Ste. Marie..........	1	53′	27′ ends 40′ centre	1	*D*	41138	Plate girder	Peterson
	10	238′ 7½″		1	*T*	465936	Pin, single intersection	
	2	104′		1	*T*	163476	Lattice, double intersection	
Prince of Wales Bridge, Ottawa River..........	2	255′			*D*	478388	Pin, double intersection	Peterson
	1	160′			*D*	199303	" " "	
	10	150′			*D*	174354	" " "	
	1	135′			*D*	145224	" " "	
Blair....................	1	330′		1	*T*	492000 (I.) 288000 (S.)		Morison
	1	110′		1	*D*	780000 120000 (I.)		
Bismarck...............	1	400′		1	*T*	600782 (I.) 348797 (S.) 25777 (C. I.)		Morison
	1	113′		1	*D*	975356 97515		

TABLE OF LOADS FOR HIGHWAY BRIDGES.

Span in Feet.	City and Suburban Bridges Liable to Heavy Traffic	Bridges in Manufacturing Districts. Ballasted Roads.	Bridges in Country Districts. Unballasted Roads.
100 and under	100 lbs. per sq. ft.	90 lbs. per sq. ft.	70 lbs. per sq. ft.
100 to 200	80 " " "	60 " " "	60 " " "
200 to 300	70 " " "	50 " " "	50 " " "
300 to 400	60 " " "	50 " " "	45 " " "
above 400	50 " " "	50 " " "	45 " " "

EXAMPLES.

1. A bridge of N equal spans crosses a span of L feet; the weights in tons per lineal foot of the main girders of the platform, permanent way, etc., and of the live load are w_1, w_2, w_3, respectively. Show that

$$w_1=\frac{LA}{N-LB},$$

where $$A=w_2(pk+r)+w_3(pk+q) \quad \text{and} \quad B=pk+r,$$

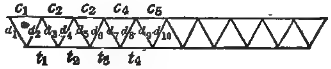
k being the ratio of span to depth, and p, q, r numerical coefficients. Hence also determine the limiting span of a girder.

If X is the cost of a pier and if Y is the cost per ton of the superstructure, find the value of N which will make the total cost per lineal foot a minimum, and prove that this is approximately the case when the spans are so arranged that the cost of one span of the bridge structure is equal to the cost of a pier.

Ans. A span $<\dfrac{1}{pk+r}$. Cost is a minimum when $N-LB=L\sqrt{\dfrac{AY}{X}}$, and the minimum cost of the span $=X\left(1-\dfrac{LB}{N}\right)=X$, approximately.

2. The platform of a single-track bridge is supported upon the top chords of two Warren girders; each girder is 100 ft. long, and its bracing is formed of ten equilateral triangles (base 10 ft.); the dead weight of the bridge is 900 lbs. per lineal foot; the greatest total stress in the seventh sloping member from one end when a train crosses the bridge is 41,394.8 lbs. Determine the weight of the live load per lineal foot. Prepare a table showing the greatest stress in each bar and bay when a single load of 15,000 lbs. crosses the girder.

Ans.



$2771\frac{3}{7}$ lbs. per lin. ft.

FIG. 799.

Stresses in diagonals:

$$d_1=d_2=4\tfrac{1}{2}\sqrt{3};\quad d_3=d_4=4\sqrt{3};$$
$$d_5=d_6=3\tfrac{1}{2}\sqrt{3};\quad d_7=d_8=3\sqrt{3};$$
$$d_9=d_{10}=2\tfrac{1}{2}\sqrt{3}\text{ tons.}$$

Stresses in compression chord: $c_1 = 2\frac{1}{4}\sqrt{3}$; $c_2 = 6\sqrt{3}$; $c_3 = 8\frac{3}{4}\sqrt{3}$;
$c_4 = 10\frac{1}{2}\sqrt{3}$; $c_5 = 11\frac{1}{4}\sqrt{3}$ tons.
Stresses in tension chord: $t_1 = 4\frac{1}{2}\sqrt{3}$; $t_2 = 8\sqrt{3}$; $t_3 = 10\frac{1}{2}\sqrt{3}$;
$t_4 = 12\ \sqrt{3}$ tons.

3. A Warren girder composed of eight equilateral triangles has its upper chord in compression and has every joint loaded with a weight of 2 tons, the loads being transmitted to the joints in the lower chord by means of vertical struts. The span = 80 ft. Find the stresses in all the members.

Ans. Bays in compression chord: 1st = $5\sqrt{3}$; 2d = $13\sqrt{3}$;
3d = $18\frac{1}{3}\sqrt{3}$; 4th = $21\sqrt{3}$ tons.
Bays in tension chord: 1st = $9\frac{1}{3}\sqrt{3}$; 2d = $16\sqrt{3}$;
3d = $20\sqrt{3}$; 4th = $21\frac{1}{3}\sqrt{3}$ tons.
Stresses in verticals: In each vertical = 2 tons.
Stresses in diagonals: 1st = $10\sqrt{3}$; 2d = $8\frac{2}{3}\sqrt{3}$;
3d = $7\frac{1}{3}\sqrt{3}$; 4th = $6\sqrt{3}$;
5th = $4\frac{2}{3}\sqrt{3}$; 6th = $3\frac{1}{3}\sqrt{3}$;
7th = $2\sqrt{3}$; 8th = $\frac{2}{3}\sqrt{3}$ tons.

4. A Warren girder, with a platform on the lower boom, carries a load of 20 tons at the centre. Find the stress in each member, and also find the weight at each joint of lower boom which will give the same stresses in the centre bays. There are six bays in the lower chord.

Ans. Stress in each diagonal = $\frac{20}{3}\sqrt{3}$ tons.
Tens. chord: stress in 1st bay $\frac{10}{3}\sqrt{3}$; 2d = $10\sqrt{3}$; 3d = $\frac{50}{3}\sqrt{3}$ tons.
Comp. chord: stress in 1st bay = $\frac{40}{3}\sqrt{3}$; 2d = $\frac{40}{3}\sqrt{3}$; 3d = $20\sqrt{3}$ tons.
Weight at each joint = $5\frac{15}{17}$ tons.

5. A Warren girder for a single-track railway bridge consists of eight equilateral triangles and has to cross a span of 96 ft.; the platform is on the bottom chord; the loads per lineal foot for which the truss is to be designed are 2250 lbs. due to engine, 1500 lbs. due to train, and 450 lbs. due to bridge. Determine the maximum stresses (both tensile and compressive) in the members met by vertical planes immediately on the right of the second, third, and fourth apices in the compression chord. Also, find how many $\frac{7}{8}$-in. rivets are required to connect the diagonals met by these planes with the chords and to prevent any tendency to longitudinal slip between the support and the first apex, and between the first and second apices in the tension chord. (Shear strength of rivets being 10,000 lbs. per square inch.)

6. A Warren girder with its bracing formed of nine equilateral triangles (base = 10 ft.) is 90 ft. long, and its dead weight is 500 lbs. per lineal foot. Determine the maximum stresses in each member when a live load of 1350 lbs. per lineal foot, preceded by a concentrated load of 18,000 lbs., crosses the girder, assuming that every joint is loaded. The diagonals and verticals are riveted to angle-irons forming part of the flanges.

How many $\frac{3}{4}$-in. rivets are required for the connection of the several members meeting at the third apex in the upper chord? (23, 6, and 13.) How many are required in the first bay of each chord to prevent longitudinal slip? (15 in tension chord and 18 in compression chord.)

Ans.

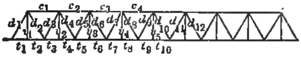


FIG. 800.

$t_1 = t_2 = 51{,}529$ lbs.; $t = t_4 = 123{,}280$ lbs.; $t_5 = t_6 = 178{,}260$ lbs.;
$t_7 = t_8 = 209{,}580$ lbs.; $t_9 = t_{10} = 219{,}540$ lbs.;
$c_1 = 91{,}222$ lbs.; $c_2 = 154{,}590$ lbs.; $c_3 = 196{,}590$ lbs.;
$c_4 = 217{,}230$ lbs.;
$d_1 = 103{,}060$ lbs.; $d_2 = 92{,}088$ lbs.; $d_3 = 81{,}551$ lbs.;
$d_4 = 71{,}447$ lbs.; $d_5 = 61{,}777$ lbs.; $d_6 = 52{,}539$ lbs.;
$d_7 = 43{,}735$ lbs.; $d_8 = 35{,}363$ lbs.; $d_9 = 27{,}424$ lbs.;
$d_{10} = 19{,}919$ lbs.; $d_{11} = 12{,}846$ lbs.; $d_{12} = 6{,}207$ lbs.

The stresses d_{10}, d_{11}, and d_{12} are of an opposite kind to those due to the dead load. The maximum load on each vertical $= 20{,}500$ lbs.

7. If a force of 5000 lbs. strike the bottom chord of the girder in the preceding question at 20 ft. from one end and in a direction inclined at 30° to the horizontal, determine its effect upon the several members.

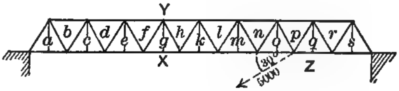


FIG. 801.

Ans.

$$Xa = \frac{Xc}{3} = \frac{Xe}{5} = \ldots = \frac{Xo}{13} = \frac{5000\sqrt{3}}{27} \text{ lbs.};$$

$$Yb = \frac{Yd}{2} = \frac{Yf}{3} = \ldots = \frac{Yp}{7} = \frac{10000\sqrt{3}}{27} \text{ lbs.};$$

$$Ya = ab = bc = \ldots = no = op = \frac{10000\sqrt{3}}{27} \text{ lbs.};$$

$$Ys = sr = rq = qp = \frac{35000\sqrt{3}}{27} \text{ lbs.};$$

$$Zs = \frac{85000\sqrt{3}}{27} \text{ lbs.}; \quad Zq = \frac{40000\sqrt{3}}{27} \text{ lbs.};$$

$$Yr = \frac{35000\sqrt{3}}{27} \text{ lbs.}; \quad Yp = \frac{70000\sqrt{3}}{27} \text{ lbs.}$$

8. A lattice girder 200 ft. long and 20 ft. deep, with two systems of right-angle triangles, carries a dead load of 800 lbs. per lineal foot. Determine the greatest stresses in the diagonals and chords of the fourth bay from one end when a live load of 1200 lbs. per lineal foot passes over the girder.

Ans. If *riveted:* Diagonal stress $= 37{,}200\sqrt{2}$ lbs.;
Chord stress $=450{,}000$ lbs.
If *pin-connected:* Diagonal stress $= 44{,}800\sqrt{2}$ and $29{,}600\sqrt{2}$ lbs.;
Chord stress $=460{,}000$ lbs. in compression and
$=440{,}000$ lbs. in tension.

9. A lattice girder 80 ft. long and 8 ft. deep carries a uniformly distributed load of 144,000 lbs. Find the flange inch-stresses at the centre, the sectional area of the top flange being $56\frac{1}{4}$ sq. ins. gross, and of the bottom flange 45 sq. ins. net.

What should be the camber of the girder, and what extra length should be given to the top flange, so that the bottom flange of the loaded girder may be truly horizontal? ($E=29{,}000{,}000$ lbs.)

Ans. 3185.8 lbs.; 4000 lbs.

$x_1=.29735$ in.; $x_2=.2987$ in.; $s_1-s_2=\frac{1624}{81925}$ ft.

10. A lattice girder 80 ft. long and 10 ft. deep, with four systems of right-angle triangles, carries a dead load of 1000 lbs. per lineal foot. Determine the greatest stresses in the diagonals met by a vertical plane in the *seventh* bay from one end when a live load of 2500 lbs. per lineal foot passes over the girder. Design the flanges, which are to consist of plates riveted together.

The lattice bars are riveted to angle-irons. Find the number of $\frac{7}{8}$-in. rivets required to connect the angle-irons with the flanges in the first bay, 10,000 lbs. per square inch being the safe shearing strength of the rivets.

Ans. If *riveted:* Diagonal stress $=10{,}664\frac{1}{16}\sqrt{2}$ lbs.
If *pin-connected:* " " $=9062\frac{1}{2}\sqrt{2}$; $6250\sqrt{2}$; $15{,}468\frac{3}{4}\sqrt{2}$;
$11{,}875\sqrt{2}$ lbs.

11. A lattice girder of 40 ft. span, 5 ft. depth, and with horizontal chords has a web composed of two systems of right-angle triangles and is designed to support a dead and a live load, each of $\frac{1}{2}$ ton per lineal foot, upon the bottom chord. Determine the maximum stresses in the members of the third bay from one end met by a vertical plane.

Ans. If *riveted:* Diagonal stress $=\frac{135}{32}\sqrt{2}$ tons;
Chord stress $=33\frac{3}{4}$ tons.
If *pin-connected:* Diagonal stress $=\frac{85}{16}\sqrt{2}$ and $\frac{50}{16}\sqrt{2}$ tons;
Chord stress $=32\frac{1}{2}$ tons in tension,
35 tons in compression.

12. A lattice truss of 100 ft. span and 10 ft. depth has a web composed of four systems of right-angle triangles. The maximum stress in the diagonal joining the sixth apex in the upper chord to the fourth apex in the lower is 16 tons. Find the dead load, the live load being 1 ton per lineal foot, assuming the truss to be (*a*) riveted, (*b*) pin-connected.

Ans. (*a*) .554 ton; (*b*) 1.062 tons.

13. A lattice girder of 40 ft. span has a web composed of two systems of triangles (base $=10$ ft.) and is designed to carry a live load of 1600 lbs. per lineal foot and a dead load of 1200 lbs. per lineal foot. Defining the stress-length of a member to be the product of its length into the stress to which

it is subjected, find the depth of the truss so that its *total* stress-length may be a minimum. *Ans.* Riveted, 11.94 ft.; pin-connected, 11.9 ft.

14. Prepare a table giving the stresses in the several members of a single-intersection deck-truss for a double-track bridge of 342 ft. span, 33 ft. depth, and with eighteen panels. The panel engine, live, and dead loads are 121,000, 65,000, and 40,000 lbs., respectively, per truss.

15. Prepare a table giving the stresses in the several members of a double-intersection through-truss of 342 ft. span, 33 ft. depth, with eighteen panels and a double track. The panel engine, train, and dead load are 121,000, 65,000, and 40,000 lbs., respectively, per truss.

16. Prepare a table giving the stresses of the several members of a double-intersection through-truss of 154 ft. span, 20 ft. depth, and with eleven panels. The panel engine, live, and bridge loads are 91,000, 48,000, and 23,000 lbs., respectively, per truss.

17. Prepare a table giving the stresses in the several members of a through-truss for a double-intersection double-track bridge of 342 ft. span, 40 ft. depth, and with nineteen panels. The panel engine, live, and dead loads are 96,000, 53,000, and 43,200 lbs., respectively.

18. A horizontal eye-bar of length l in. and weighing w pounds per lineal inch, carries a force of P pounds. Taking

M_0 = moment of resistance of section $M_1 - M_2 = \frac{f_1 I}{e}$;
M_1 = bending moment at centre of eye-bar due to its own weight $= \frac{1}{8}wl^2$;
M_2 = bending moment from direct stress P into the lever-arm $d = Pd$;
f_1 = unit stress on extreme fibre at centre of span due to M_1 and M_2 acting together;
e = distance from neutral axis to extreme fibre;
d = maximum deflection due to all forces acting together;
I = moment of inertia of bar;
E = modulus of elasticity;
f_2 = unit stress uniformly distributed due to P,

derive the formula
$$f_1\left(I + \frac{5}{48}\frac{Pl^2}{E}\right) = M_1 e.$$

19. A horizontal eye-bar 2″×12″ is 50 ft. long centre to centre of pin-holes. The direct tensile stress f_2 on the bar is 18,000 lbs., or $P = 432,000$ lbs. The material is of steel and the weight of the bar is 81.6 lbs. per lineal foot.

(*a*) Find f_1 by the formula of the preceding problem, and obtain the maximum fibre stresses.

(*b*) Assuming that the direct bending and the tension stresses act independently, find the extreme fibre stresses in tension and compression.

Ans. (*a*) 20,168 and 15,832 lbs./sq. in. (T.); (*b*) 24,375 and 11,625 lbs./sq. in.

20. A round lateral rod 1 in. in diameter is 64 ft. long between end sup-

ports, and is subjected to a tensile stress of 8000 lbs. Find f_1 by the method of Problem 18, and also find the total maximum and minimum fibre stresses.

Ans. 471 lbs./sq. in.; 10,682 lbs. and 9718 lbs./sq. in.

21. By comparing the two preceding problems it will be seen that the bending stress for the 2″×12″ eye-bar is much greater than that of the 1-in. round bar, although the unsupported length of the rod is the greater. It is evident that for any given fibre stress f_2 there is a certain ratio of depth to length that will give a maximum value for f_1. Determine this ratio for $f_2=16,000$ lbs. *Ans.* 1 to 38.

22. A member of a truss 10 ins. long, measured on the chord joining the two panel-points, is curved to a radius of 20 ft., and carries a direct compression stress of 300,000 lbs. What is the bending moment in inch-pounds due to the curvature? *Ans.* 2,287,500 in.-lbs.

23. The end pin of a riveted span is placed 24 ins. below the centre of the bottom chord. Find the bending effect produced by a traction load of 100,000 lbs. on the truss. Is this a faulty detail? Why?

Ans. 2,400,000 in.-lbs.

24. The total reaction on a roller (cylindrical) shoe of a span is 1,500,000 lbs. The allowable bearing on the limestone masonry is 300 lbs. per square inch, and the bearing intensity on steel rollers is $p=600d$ where p = pressure per lineal inch of roller and d = the diameter. Find the size of base and the number and dimensions of rollers required. (*N.B.* The rollers should be so proportioned that the distance from the edge of the spaces they occupy to the extremity of the area required for bearing on the masonry is not less than 6 ins. at any point.)

Taking the same loading, but, instead of limestone using granite for the coping, the granite having a safe bearing strength of 550 lbs. per square inch, determine the size of base and the number and dimensions of *segmental* rollers required, the rollers to occupy a space $10\frac{1}{2}$ ins. less in each direction than the area required for bearing on the masonry.

Ans. 5000 sq. ins.; *eleven* rollers 4 ins. diam.×57 ins. long =2727 sq. ins., *eight* rollers $7\frac{1}{2}$ ins. diam.×42 ins. long.

25. The total load on the rollers of a rim-bearing draw-span is 3,000,000 lbs. The diameter of drum is 18 ft. Allowing the same intensity on the rollers as in the preceding example, determine the size and number required.

Ans. 40; 16 ins. diam.×7.81 ins. face.

26. A drum of a draw-span is 20 ft. in diameter. A load of 4,000,000 lbs. from the span is equally distributed over eight points of support on the drum. Find the moment on the drum by the formula $M=\frac{1}{10}Wl$.

Ans. 392,700 ft.-lbs.

27. A bent for a viaduct is 30 ft. high from top of masonry to base of rail. The columns are spaced at 15 ft. centres, and are subject to transverse bending due to a wind load of 20,000 lbs. The total vertical load on each column is 80,000 lbs. The base of column is 2 ft. square, and the anchor-bolts are spaced 3 ins. from the edges of shoes. Required the pull on the anchor-bolts to fix the bottoms of columns, supposing the columns to be supported transversely by a floor-beam 5 ft. deep. *Ans.* 39,000 lbs.

28. The train load on an elevated steel structure is 5000 lbs. per lineal foot; the structure is divided into bays, each consisting of six 45-ft. spans and one 30-ft. tower-span. All of the traction load is carried by the longitudinal diagonal braces of the towers, the diagonals being placed on an angle of 45 degrees. Assuming the coefficient of friction between the wheels and rails to be 0.20, find the greatest stress on the tower diagonals. There is an expansion joint between each bay. The diagonals are to be rigid sections capable of carrying either tension or compression.

If, instead of using tower bracing to provide for the longitudinal thrust from traction load, the columns are made rigid enough to resist the effect of same, determine the bending moment on each column, assuming the same loading and that the structure is divided into 50-ft. panels. The expansion joints are 300 ft. apart, the unsupported length of columns is 30 ft., and the columns are fixed top and bottom. *Ans.* 106,050 lbs. (T. or C.); 4,500,000 in.-lbs.

29. Prepare a table giving the stresses in the several members of a single-intersection through-truss of 154 ft. span, 20 ft. depth, and with eleven panels. The panel engine, live, and dead (or bridge) loads are 27,500, 17,600, and 8470 lbs., respectively.

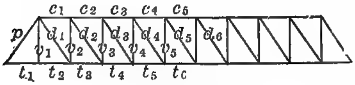


Fig. 802.

Ans.

Diag.	Mult.	2500	Mult.	1600	Sum.	Mult.	770	Sum.	sec.	Total Max. Stress.
p	10	25,000	45	72,000	97,000	55	42,350	139,350	1.22	170,007
d_1	9	22,500	36	57,600	80,100	44	33,880	113,980	1.22	139,056
d_2	8	20,000	28	44,800	64,800	33	25,410	90,210	1.22	110,057
d_3	7	17,500	21	33,600	51,100	22	16,940	68,040	1.22	83,009
d_4	6	15,000	15	24,000	39,000	11	8,470	47,470	1.22	57,914
d_5	5	12,500	10	16,000	28,500	0	....	28,500	1.22	34,770
d_6	4	10,000	6	9,600	19,600	−11	−8,470	11,130	1.22	13,579

Panel.	Mult.	3270	Mult.	2370	Sum.	tan.	Panel Stress.	Total Panel Stress.
$t_1=t_2$	10	32,700	45	106,650	139,350	$\frac{7}{10}$	97,545	97,545
t_3	−1	−3,270	45	106,650	103,380	"	72,366	169,911
t_4	−1	−3,270	34	80,580	77,310	"	54,117	224,028
t_5	−1	−3,270	23	54,510	51,240	"	35,868	259,896
t_6	−1	−3,270	12	28,440	25,170	"	17,619	277,515

30. A seven-panel single-intersection truss for a single-track bridge has a length of 105 ft. and a depth of 20 ft., the load being on the lower chord. Find the stresses in the several members (*a*) when the apex live load is 12 tons, (*b*) when the live load is produced by concentrated loads of 8, 8, 20, 20, 8, 8, 8, 8, and a uniformly distributed load of 1.5 tons per lineal foot, following each other in order over the bridge at the distances of 5.5, 9, 8, 8, 9.5, 5, 5.5, 5, and 3 ft. apart.

31. The two trusses for a 16-ft. roadway are each 100 ft. in the clear, 17 ft. 3 ins. deep, and of the type represented in the figure; under a live load of 1120 lbs. per lineal foot the greatest total stress in *AB* is 35,400 lbs. Determine the permanent load.

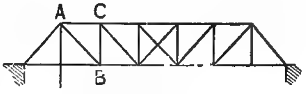


Fig. 803.

The diagonals and verticals are riveted to angle-irons forming part of the flanges. How many ¾-in. rivets are required for the connection of *AB* and *BC* at *B*? Also, how many are required between *A* and *C* to resist the tendency of the angle-irons to slip longitudinally? Working-shear stress = 10,000 lbs. per square inch. *Ans.* 708.6 lbs.; 8, 4, 7.

32. A Pratt truss with sloping end posts has a length of 150 ft. centre to centre, and a height of 30 ft. centre to centre, with panels 15 ft. long; the dead load is 3000 lbs. per lineal foot, and the live load 12,000 lbs. Determine the maximum stresses in the end posts, in the third post from one end, in the middle of the bottom chord, and in the members of the third panel met by a vertical plane. *Ans.* 158.48; 81.45; 196.875; 126; 91; 165.875 tons.

33. A ten-panel single-intersection through-bridge of 170 ft. span is 25 ft. 6 ins. in height, and has floor-beams 13 ft. in length. How many 1-in. rivets are required in the third panel from one end to connect the web with the chords, assuming the panel live load to be 30,000 lbs. and the panel dead load to be 10,000 lbs.?

Ans. 10 and 13, the shearing strength of the rivets being 10,000 lbs./sq. in.

34. Each of the two Pratt single-intersection five-panel trusses for a single-track deck-bridge is 55 ft. centre to centre of end pins and 11 ft. 6 ins. deep. Timber floor-beams are laid upon the upper chords 2¾ ft. centre to centre; the width between the chords = 10 ft. Find the proper scantling of the floor-beams for the loading given in Fig. 622, p. 683. Also determine the maximum chord and diagonal stresses in the centre panel due to the same live load.

Ans. 10 in. × 10.4 in.; max. chord stresses = 30.68 and 32.41 tons; max. diagonal stress = 8.99 tons.

35. Loads of 3¾, 6, 6, 6, and 6 tons follow each other in order over a ten-panel single-intersection truss at distances of 8, 5¾, 4½, and 4½ ft. apart. Determine the position of the loads which will give the maximum diagonal and chord stresses in the third and fourth panels. Span = 120 ft. and depth = 12 ft.

Ans. The *max. shears* in the 3d and 4th panels are 16.2875 and 13.5125 tons, respectively, and occur when the 3¾-ton wheel is *first* at the third panel-point, and *second* at the fourth panel-point; the maximum chord stresses in the 3d and 4th panels are 33.775 tons and 43.5875 tons respectively, and occur when *first* the 3¾-ton wheel is at a panel-point, and *second* when the first 6-ton wheel is at a panel-point.

36. Determine the moment of resistance of a floor-beam for the Sault Ste. Marie bridge from the following data: Floor-beams, 16 ft. 6 ins. long and 23 ft. 10¾ ins. apart; the dead weight of the flooring, stringers, etc. = 800 lbs. per lineal foot of floor-beam; the live load as given in Fig. 622, p. 683; the load is transmitted to the floor-beam by four lines of stringers so spaced as to throw two thirds of the load upon the inner pair, which are 3 ft. centre to centre. *Ans.* 296.294 ft.-tons.

37. In a truss-bridge the panels are 17 ft. and the floor-beams 13 ft. in length. Loads of 8, 12, 12, 12, 12, 10, 10, 10, and 10 tons follow each other in order over the bridge at the distances of 7½, 4½, 4½, 4½, 7½, 5½, 6½, and 5½ ft. apart. Determine the moment of resistance of the beam, taking the load due to the platform, etc., to be 500 lbs. per lineal foot. *Ans.* 82.2225 ft.-tons.

38. With the loading given by Fig. 804 design a floor-beam for a single track bridge with panels 22 ft. long, the weight of the platform being 450 lbs. per square yard, and of each longitudinal 200 lbs. per lineal yard.

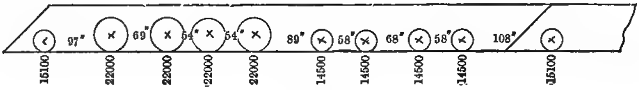


FIG. 804.

Ans. The B.M. in cross-tie is greatest when the 3d 22,000-lb. load is at a panel-point, and total max. B.M. = 139.883 ft.-tons.

39. Design a cross-girder for a five-panel through-truss bridge, 120 ft. long, for Cooper's standard loading E 50. The span of the girder is 17 ft. 6 ins. back to back, of connection angles, and there are two lines of stringers 8 ft. apart. The dead load of the stringers and floor system may be taken as 700 lbs. per foot run of bridge, and the estimated weight of the cross-girder is 3000 lbs.

40. Design the central cross-section of a plate girder having an effective span of 50 ft. and a depth of 5 ft. 6 ins. centre to centre of the flanges, for Cooper's standard loading E 50; the dead load may be assumed to be 650 lbs. per foot run of girder.

41. The stringers for a railroad span are spaced 8 ft. centres. The ties are to be spaced 5 ins. apart in the clear. Assuming that the rails are strong enough to distribute each wheel concentration equally over three ties, and that the timber will carry safely an extreme fibre stress of 2000 lbs. per square inch, including impact; determine the size of tie required for a wheel load of 25,000 lbs. Distance centre to centre of rails = 4 ft. 11 ins.; impact 80 per cent of the live load. *Ans.* $8\frac{1}{2}'' \times 10''$.

42. Design the central section of a plate girder of 45 ft. span and 5 ft. deep to carry a dead load of 500 lbs. per foot run, a live load of 3200 lbs. per foot run, and an impact load of 2400 lbs. per foot run; also determine the lengths of the flange plates.

43. Design a cross-tie for a double-track open-web bridge, the ties being 18 ft. 5 ins. centre to centre and the live load for the floor system being 8000 lbs. per lineal foot.

44. Design a stringer for a Pratt-truss bridge 20 ft. long and 3 ft. 9 ins. back to back of angles. The live load is 3200 lbs. per foot run of stringer, and the estimated weight of the dead load, including the weight of the stringer, is 400 lbs. per foot run.

45. Design the section of the top chord member of a bridge for a total dead, live, and impact load of 620,000 lbs. The length of the member is 25 ft. and the inside width is 15 ins. Use the column formula $p\left(1+\frac{l^2}{11000r^2}\right) =$ 17,000 lbs. per square inch.

46. A panel of the top chord of a deck-span is 20 ft. in length; the direct compressive stress is 320,000 lbs.; the top chord is subjected to a bending moment due to a uniform load of 4000 lbs. per lineal foot. The allowable extreme fibre stress in either tension or compression for the combined loading is 16,000 lbs. per square inch. Determine the bending moment by the formula $M=\frac{1}{10}wl^2$. Find the section required by the form shown in the diagram.

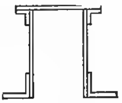


FIG. 805.

Ans. One 18″×$\frac{3}{8}$″ cover; *two* top 3″×3″×$\frac{3}{8}$″×7.2 lb. angles; *two* 20″×$\frac{1}{2}$″ webs; *two* bottom 6″×4″×$\frac{9}{16}$″×18.1 lb. angles.

47. The figure represents a counterbalanced swing-bridge, 16 ft. deep and wholly supported upon the turntable at *A* and *B*; the dead weight is 650 lbs. per lineal foot of bridge; the counterpoise is hung from *C* and *D*. Find its weight, assuming (*a*) that the whole of it is transmitted to *B*; (*b*) that a portion of it sufficient to make the reactions at *A* and *B* equal is transmitted to *A* through a member *BE*. Also, determine the stresses in the several members of the truss.

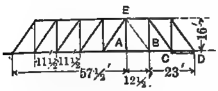


FIG. 806.

Ans. Counterpoise in case (*a*) $=26,162\frac{1}{2}$ lbs.;
in case (*b*) $=22,186\frac{41}{64}$ lbs.
Stress transmitted through *BE* in case (*b*) = 6962 lbs.

48. The figure represents a counterbalanced swing-bridge; the dead load upon the bridge is 650 lbs. per lineal foot; the counterpoise is suspended from *CD*. Find its value, the joint at *E* being so designed that the whole of the load upon the bridge is always transmitted through the main posts *EA*, *EB*, and is evenly distributed between the points of support at *A* and *B*. Find the stresses in the several members of the truss in the preceding question (*a*) when the bridge is open; (*b*) when the bridge is closed and is subjected to a live load of 3000 lbs. per lineal foot. Height of truss at *E* = 16 ft., at *F* = 8 ft.

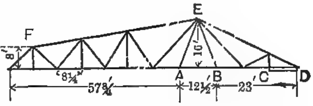


FIG. 807.

Ans. 20,694.3 lbs.

49. A swing-bridge truss of the form and dimensions shown by Fig. 808 has a panel dead load of 2 tons. Find the counterweight *C* and draw the stress diagram when the bridge is open. Also find the reactions and draw the stress diagram when the bridge is closed. Determine the effect of a panel live load of 4 tons. *Ans.* $C=6\frac{2}{3}t$.

50. A draw-span having equal arms of 100 ft. length, divided into four equal panels, Fig. 809, is subjected to a live load of 2600 lbs. per lineal foot of truss. The dead load is 1200 lbs. per lineal foot of truss. Determine the stresses in the members (*a*) when the bridge is open, (*b*) when the live

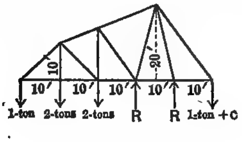


FIG. 808.

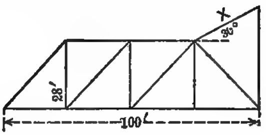


FIG. 809.

load is concentrated at the first, second, and third panel-points. Also find the stress in X when the bridge is open.

51. The wedges beneath the ends of a draw-span are each required to exert an uplift of 100,000 lbs. The slope of the wedges is 1 in. vertical to 5 ins. horizontal. The coefficient of friction on the two surfaces of the wedges is 0.30. Required the horizontal force necessary to drive and draw each wedge.

Ans. 50,000 lbs.; 10,000 lbs. In practice, however, it usually takes as much to draw as to drive the wedge if operated only at long intervals.

52. A draw-span 320 ft. in length is divided into one centre panel of 20 ft. and twelve ordinary panels of 25 ft. each. One arm of the span is subjected to an unbalanced wind load of 5 lbs. per square foot of exposed area, or 100 lbs. per lineal foot of arm. The diameter of the pitch-circle of the rack is 25 ft. There are two main driving-pinions gearing into the rack. Find (*a*) the tooth pressure on each necessary to overcome the unbalanced wind load; also (*b*) the power required to turn the span against this load through one fourth revolution in two minutes.

The total weight of the draw-span is 1,000,000 lbs. Find (*c*) the power required to turn the span through one fourth revolution in one half minute under ordinary conditions; also (*d*) against the unbalanced wind load in two minutes. Use the formula $\text{H.P.} = \dfrac{.0125Wv}{550}$ for determining the power required to overcome friction, accelerate motion, etc., where W = weight of span and v = velocity of pitch-circle on rack in feet per second.

Determine the size of main pinions for the power required in Case (*c*), assuming that the gears are to be made of cast steel capable of a safe extreme fibre stress of 16,000 lbs. per square inch. Make the face of pinions two and one half times the pitch and use the formula $p = 0.025\sqrt{\tfrac{1}{3}P}$, where p = pitch required and P the tooth pressure on each pinion.

The shafts supporting the main driving-pinions have boxes located so that the bending and torsional moments are equal; determine the size of shaft required, assuming for such conditions that steel shafting will carry safely an extreme fibre stress of 24,000 lbs. per square inch.

Take the draw-span to be moving at the rate of one revolution in four minutes. Required the force at each of the two ends necessary to bring the span to rest in a space of 6 ins. Assume the 1,000,000 lbs. of weight to be equally distributed throughout the span. Find the size of latch required for stopping the span under the conditions named, assuming the latch is of a rectangular section of steel, and that the resistance is obtained by bending on same, the unsupported length or lever-arm being 20 ins. Working stress on extreme fibre = 16,000 lbs. per square inch.

Ans. 51,200 lbs.; 30.5, 14.8, 3.7, 34.2 H.P.; 3½ ins. pitch × 8¾ ins. face × 17.8 ins. diam. × 16 teeth; diam. = 6.4 ins.; 68,200 lbs., assuming the weight of the draw-span to be uniformly distributed; the latch to be 10 ins. wide × 5.115 ins. thick, which presupposes that the latch is caught in such a way as to exert the pressure of 68,200 lbs. throughout the distance of 6 ins.

53. A bent for an elevated structure consists of two columns spaced 20 ft. apart. Between the two columns is a cross-girder AB which must carry a dead load of 4000 lbs. per foot of its length and a live load of 6000 lbs. per foot. Outside of each column are cantilever brackets AC and BD, which support the same live and dead loads per foot. Find the greatest upward and downward moments on the cross-girder; the moment in each cantilever at the columns; the total moment for which the cross-girder should be proportioned, and the maximum and minimum loads which each column will receive. The cross-girder should be designed to carry the greater moment plus three fourths of the lesser.

Ans. (*a*) L.L. on AC and BD, *net upward moment* = 300,000 ft.-lbs.; (*b*) L.L. on AB, *net downward moment* = 300,000 ft.-lbs.; (*c*) L.L. on CB, *max. load on column* = 215,000 lbs.; (*d*) *max. B.M. on cross-girder* = 525,000 ft.-lbs.; (*e*) L.L. on BD min. load on column = 65,000 lbs.

54. Find the maximum stresses in the several members of the compound A bridge truss of 160 ft. span and 40 ft. depth shown by diagram, the panel dead and live loads being 2 and 4 tons respectively.

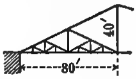


Fig. 810.

55. In a seven-panel Pegram truss for a span of 200 ft., the upper panel-points lie in a circular arc, with a chord of 160 ft. and a versed sine of 15 ft. Taking the dead load at 900 lbs. and the live load at 1800 lbs. per lineal foot per truss, determine the stresses in every member of the truss. (See Ex. 6, p. 706.)

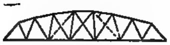


Fig. 811.

56. The figure represents the half of one of the trusses for a bridge of 120 ft. span, the panel dead and live loads being 6 and 4 tons respectively. Determine the lengths of the verticals and the stresses in the several members (*a*) so that the stress in each member of the lower chord may be 80 tons; (*b*) so that the minimum stress in each diagonal may be zero.

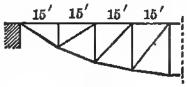


Fig. 812.

57. Determine the live-load stresses in the members of the cantilever truss shown by Fig. 813 when subjected to a panel live load of 55 tons.

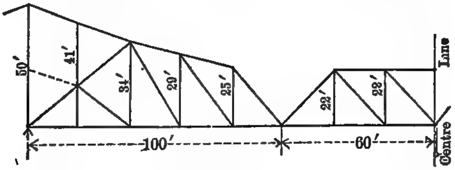


Fig. 813.

58. The accompanying truss of 240 ft. span and 30 ft. deep is to be de-

signed for a panel engine load of 24,000 lbs., a panel train load of 18,000 lbs., and a panel bridge load of 12,000 lbs. Determine graphically the maximum stresses in the members met by the vertical *MN*. Also, draw a stress diagram for the whole truss when it is covered with a uniformly distributed live load of 180,000 lbs.

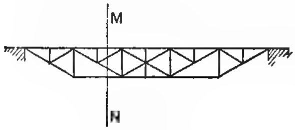


FIG. 814.

59. The compression chord of a bowstring truss is a circular arc of 80 ft. span and 10 ft. rise; the bracing is of the isosceles type, the bases of the isosceles triangles dividing the tension chord into eight equal lengths. Determine the maximum stresses in the members met by a vertical plane 28 ft. from one end. The live and dead loads are each ½ ton per lineal foot.

Ans. Chord tension = 77.39 tons; chord compression = 80.3 tons;
Diagonal stress = 7.95 tons.

60. Design a parabolic bowstring truss of 80 ft. span and 10 ft. rise for a dead load of ½ ton and a live load of 1 ton per lineal foot. The joints between the web and the tension chord are to divide the latter into eight equal divisions.

61. The compression chord of a bowstring truss is a circular arc. The depth of the truss is 14 ft. at the centre and 5 ft. at each end; the span = 100 ft.; the load upon the truss = 840 lbs. per lineal foot. Find the stresses in all the members. Determine also the maximum stresses in the members met by a vertical 25 ft. from one end when a live load of 1000 lbs. per lineal foot crosses the girder. What counterbraces are required?

62. A bowstring truss of 120 ft. span and 15 ft. rise is of the isosceles braced type, the bases of the isosceles triangles dividing the tension chord into twelve equal divisions; the dead and live loads are ½ ton and 1 ton per lineal foot respectively. Find the maximum stresses in the members met by vertical planes immediately on the right of the second and fourth joints in the tension chord.

63. Fig. 815*A* represents the riveted truss for a span of 126 ft. centre-to-centre end pins. The trusses are 17 ft. centre to centre, 28 ft. deep, and are each divided into six panels. Figs. 815*B* and *C* show the lower and upper lateral systems, the wind load on the former being 450 lbs. per lineal foot when the span is loaded and 200 lbs. per lineal foot when the span is empty. The wind load on the upper system is 150 lbs. per lineal foot. The live and dead loads are 2 tons and $\frac{4}{5}$ ton respectively per lineal foot of span. (*a*) Prepare tables of maximum stresses. (*b*) Find maximum wind stresses in bottom chords and diagonals of lateral systems. (*c*) Will there be reversion in the end panels of the bottom chord, assuming the diagonals to be in tension? (*d*) Find the effect of the transferred wind load on the inclined end posts and bottom chords. (*e*) Discuss the wind effect on the portal bracing, Fig. 815*D*.

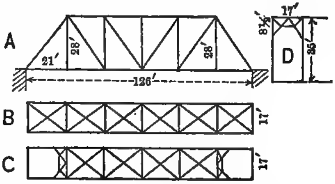


FIG. 815.

64. Determine the direct and transferred wind loads on the lower chords of the pin-connected truss-bridge shown by Fig. 816, when the wind load per foot run on the upper chord is 200 lbs. and on the lower chord is 500 lbs. per foot run. The dead-load stresses in the panels of the lower chord are 84,000, 84,000, 114,000, and 125,000 lbs., as indicated. Is there any tendency to reversion in the lower-chord members?

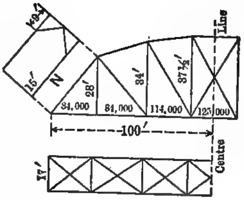


FIG. 816.

The end post of the truss is subjected to a total direct stress of 750,000 lbs. due to the dead, live, and impact loads, and the wind load on the upper-chord members is estimated at 200 lbs. per foot run. Determine the section of the member on the assumption that there is a point of inflection at N.

65. The dead and live loads for the truss shown by Fig. 817 are 6500 and 5500 lbs. per foot run respectively, and the wind load per foot run is 2000 lbs. for the upper and 500 lbs. for the lower chord. Will there be any reversion of stress in the lower chord? Determine the wind-bracing stresses.

66. A 200-ft. single-track railway span is divided into eight panels of 25 ft. each. The trusses are 33 ft. deep, centre to centre of chords, and are spaced 18 ft. centre to centre. The wind load on the upper lateral system is 200 lbs. per lineal foot of span. Find (a) the maximum bending moment in inch-pounds on each inclined end post or batter-brace due to the transverse wind, with portals as shown on Fig. 818. Also find (b) the transferred wind-load stress on the leeward batter-brace and the stresses in the different members of the portal bracing.

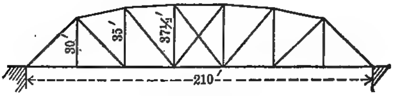


FIG. 817.

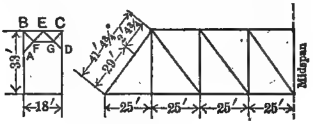


FIG. 818.

Ans. (a) 1,522,500 in.-lbs.; (b) 26,100 lbs., assuming foot of brace *fixed;* (c) $AE=32,400$ lbs. (T.); $DE=32,400$ lbs. (C.); $BE=27,850$ lbs. (C.); $CE=10,350$ lbs. (T.); $FG=0$, $BF=0$, and $CG=0$.

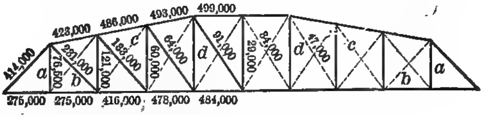


FIG. 819.

67. Fig. 819 is a skeleton diagram of the Sault Ste. Marie bridge (C. P. R.). Span = 239 ft.; there are ten panels, each of 23.9 ft., say 24 ft.; the length of the end verticals = 27 ft., of the centre verticals = 40 ft.; width of truss

centres = 17½ ft. The bridge is designed to bear the loading given by Fig. 622, p. 683. Show that

(*a*) The stresses in every panel length of each chord are greatest when the third driver is at a panel-point; and find the value of the several stresses.

(*b*) The stresses in the verticals *a* and the diagonals *b* are greatest when the third driver is at a panel-point; and find their values.

(*c*) The stresses in the remaining members of the truss are greatest when the second driver is at a panel-point; and find their values.

(*d*) The maximum stresses in the verticals *d* vary from a tension of 64,000 lbs. to a compression of 11,000 lbs.

(*e*) The stress in the counterbrace *c* is *nil.*

Ans. The values of the stresses in the several members are marked on the diagram. They are deduced from the distributions given in the table on p. 685, and are correct within a very small percentage.

68. An eight-panel bridge, Fig. 820, of 200 ft. span and 39¾ ft. rise at the centre has two main trusses 18 ft. apart. The dead load is estimated at 2000 lbs. and the live load at 4700 lbs. per foot run of bridge.

Determine the stresses in one of the trusses due to the dead load and the reversion stresses in the web members due to the live load.

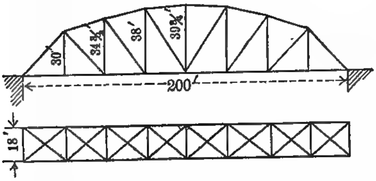


Fig. 820.

Calculate the stresses in the lower lateral bracing and the reversion stresses in the lower chord for a wind load on the upper chord of 200 lbs. and on the lower chord of 480 lbs. per foot run of bridge.

69. Determine all the stresses in the Baltimore truss shown by Fig. 821 for

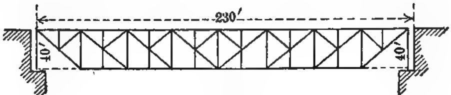


Fig. 821.

a dead load of 600 lbs. and a live load of 800 lbs. per linear foot of truss. In determining the stresses in the verticals it may be assumed that 6000 lbs. of the panel dead load is concentrated at each of the bottom-chord panel-points.

70. The figure represents one of the trusses for the Jefferson City 440-ft. draw-span, the dimensions and dead loads in tons being as indicated. In determining the stresses in the verticals it may be assumed that 2½ tons of the dead load is applied at the top-chord panel-points.

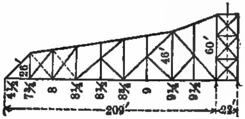


Fig. 822.

(*a*) Find the dead-load stresses in the several members when the span is open. (*b*) Find the stresses in the several members due to a live load of 0.6 ton per lineal foot per truss, assuming that one arm only is loaded and that the cantilever acts like a simple girder on two supports. (*c*) Find the stresses in the several members due to an assumed uplift of 12½ tons at the end of each truss, assuming each arm to act as a cantilever resisting said uplift. (*d*) Determine the maximum stresses in the several members for a live load of 0.6 ton per lineal foot of span, assuming the span continuous over the four points of support.

The fixed spans for the bridge are of the Petit type and of dimensions as shown. The live and dead loads are 600 and 800 lbs. per lineal foot per truss. In determining the stresses in the verticals it may be assumed that 6000 lbs. of the panel dead load are concentrated at each of the top-chord panel-points. Find (*e*) the maximum dead- and live-load stresses.

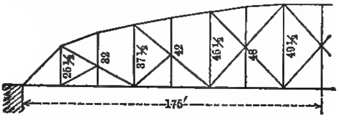


FIG. 823.

71. Figure 824 represents one half of a truss of a cantilever bridge spanning the river Agarno. The dead load for suspended span is 940 lbs. and the live load is 1678 lbs. per lineal foot per truss. (*a*) Find the dead- and live-load stresses in the members of the suspended span. (*b*) Find the erection stresses produced by cantilevering out the suspended span, the erection loads being indicated in tons on the figure. (*c*) Determine the dead-load stresses in members of cantilever proper due to total weight of suspended span, namely, 108 tons.

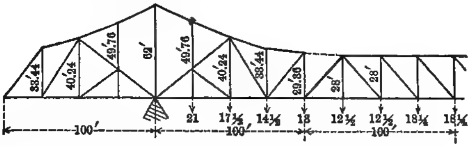


FIG. 824.

72. In a through-span 510 ft. in length the stringers are riveted together throughout the entire length of span. Assuming that the stringers are manufactured the exact length for span with no load thereon, that the dead load on span strains the metal in the bottom chords 8000 lbs. per square inch, and the live load 10,000 lbs. per square inch, find the effect on the stringer connections when the span is swung, and when the live load is on the structure. The span is divided into seventeen panels of 30 ft. each.

Ans. Each additional application of the live load produces an additional distortion of $\frac{1}{8}$ in. for each panel. How can this be remedied?

73. In a riveted joint the rivets are spaced eccentrically to the line of stress as shown in Fig. 825. The stress $P=48{,}000$ lbs. The value of each rivet is 6000 lbs., and eight are used in the connection. Find the stress in each rivet.

Ans. Total stress in $1=-2000$ lbs., in $2=2000$ lbs., in $3=2000$ lbs., in

4 = 6000 lbs., in 5 = 6000 lbs., in 6 = 10,000 lbs., in 7 = 10,000 lbs., in 8 = 14,000 lbs. per inch, showing an unequal distribution of stress and that such eccentric connections should be avoided.

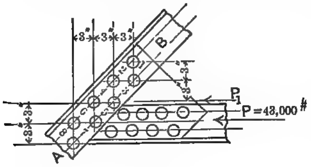


FIG. 825.

74. Given an elevated structure the columns of which are unsupported for a distance of 20 ft., and are fixed at each end, the dimensions and arrangement of longitudinal girders and towers being as shown in Fig. 826. Find the maximum fibre stress due to bending caused by a maximum variation in temperature of 100°, assuming the coefficient of expansion for 100°F to be 0.0006, and taking the distance from centre of gravity of column to extreme fibre in a longitudinal direction as $7\frac{1}{2}$ ins. *Ans.* 7550 lbs./sq. in.

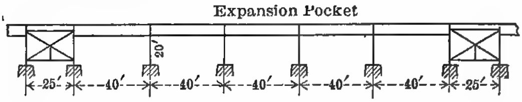


FIG. 826.

75. At a top-chord panel-point of a riveted truss-span the gravity lines of the three members assembling at this point do not have a common point of intersection. Find the bending moment about the panel-point for the conditions shown in Fig. 827.

Ans. 1,600,000 in.-lbs. *Such intersections should be avoided, as the extreme fibre stress from the bending moments is frequently greater than those produced by the actual loads. The gravity lines should all intersect in a common point.*

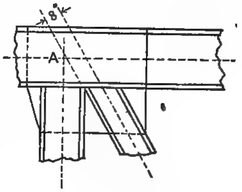


FIG. 827.

76. Determine the stresses in the several members of the three-hinged truss represented by Fig. 828 and having a panel dead load of 2 tons. If a weight of 4 tons is concentrated at each of the points A and B, find the stresses developed in the members x. Also find the positions of the load which will produce no stress in the members y and z.

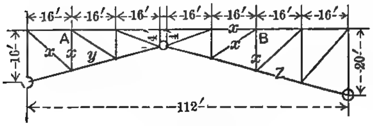


FIG. 828.

77. A three-pin arch of the form and dimensions shown by Fig. 829, and having a span of 180 ft. and a rise of 40 ft., is loaded over one half its span with a uniformly distributed load of 1200 lbs. per linear foot of truss. Determine the stresses in the members cut by the planes AB and CD.

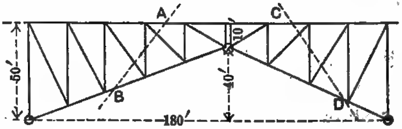


FIG. 829.

78. A three-pin arch of 160 ft. span has a rise of 32 ft. and is divided into eight panels. The live load per foot

run of truss is 3000 lbs. Determine the stress in each member when the span is fully loaded. Also determine, graphically, the line of resistance of the arch span for a live load covering three fourths of the span, and find the horizontal thrust of the arch for the given system of loading.

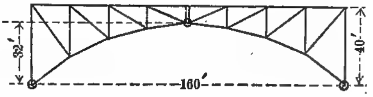


FIG. 830.

79. Figure 831 is a skeleton diagram for one of the main trusses of a bridge of 120 ft. span, 15 ft. deep at ends, and $1\frac{2}{3}$ ft. deep at centre, pivoted at the ends and centre, and of the dimensions shown. Where must the load be placed between A and B so that it may produce no stress in X? Find the maximum stresses in the several members, the panel dead and live loads being 2 and 4 tons respectively.

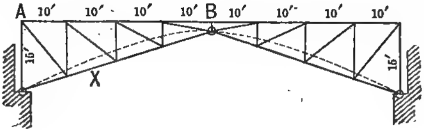


FIG. 831.

Show how the stresses are modified if the lower boom, instead of being straight, is a parabola of the same rise as the arch.

80. The figure represents a half bridge of 128 ft. span, suspended from the point P and hinged at O. The depth at P is 20 ft. and at O 4 ft. The upper ends of the verticals lie in an arc of parabola, the point O being the vertex. The bridge and train panel loads are 6 and 4 tons respectively. Find the stresses in all the members.

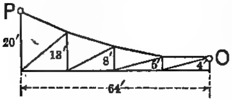


FIG. 832.

81. The accompanying figure represents a portion of a cantilever truss, the horizontal distances of the points A, B, C from the free end being l_1, l_2, l_3, respectively. The boom ABC is inclined at an angle α, and the boom XYZ at an angle β, to the horizon. Find the deflections at the end of the cantilever due to (a) an increase k_1AB in the length of AB; (2) an increase k_2BY in the length of BY; (3) a decrease k_3XY in the length of XY; (4) a decrease k_4BX in the length of BX.

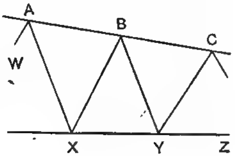


FIG. 833.

Ans. (1) $\dfrac{k_1l_1AB}{BX\sin ABX}$;

(2) $k_2\left\{\dfrac{BY\cos\beta}{\sin BYX} - l_3(\cot YBC - \cot BYX)\right\}$;

(3) $\dfrac{k_3XYl_2}{BX\sin BXY}$;

$$(4)\quad k_4\left\{\frac{BX\cos\alpha}{\sin ABX}-l_2(\cot BXY-\cot ABX)\right\}.$$

If $k_1=k_2=k_3=k_4=k$, and if AW is parallel to BX, and AX to BY, show that the angle between WX and XY after deformation

$$=2k(\cot ABX+\cot BYX).$$

Hence, also, if the truss is of uniform depth d, show that the "deviation" of the boom per unit of length is constant and equal to $\frac{2k}{d}$.

82. Determine the diameter of a steel bridge pin subjected to the stresses of which the horizontal and vertical components are shown by Figs. 834, and 835, the working coefficient of strength being 25,000 lbs. per square inch.

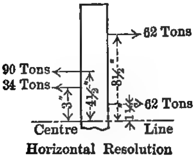


FIG. 834.

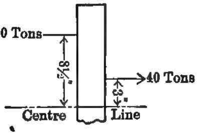


FIG. 835.

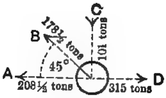


FIG. 836.

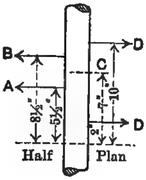


FIG. 837.

83. Calculate the size of the pin subjected to the stresses shown by Figs. 836 and 837, the extreme fibre stress is not to exceed 23,000 lbs. per square inch.

84. The figures show the magnitude and directions of the stresses concentrated at a pin-connected joint. The stresses are distributed along the pin in vertical and horizontal planes as indicated. Find the maximum bending moment to which the pin is subjected. What should be its diameter with a fibre stress of 13½ tons per square inch?

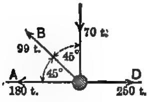


FIG. 838.

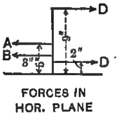


FIG. 839.

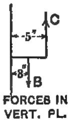


FIG. 840.

85. We have a bottom chord point L_4 at which the stresses and sections are as shown in tables and sketches below. Find the maximum bending moments in the pin for these conditions: (a) when bottom chord stresses are greatest; (b) when diagonal stress is a maximum.

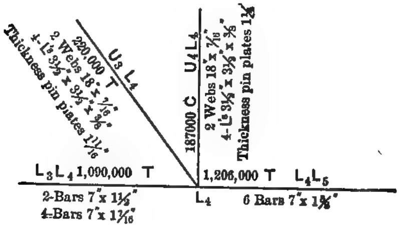


FIG. 841.

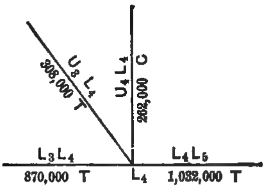


FIG. 842.

Assume the packing as follows:

Member.			Out to Out.	In to In.	C. to C.	C. to C. of Adjacent Members.
$1\frac{7}{16}''$	bar,	L_3L_4	$32\frac{3}{4}''$	$29\frac{7}{8}''$	$31\frac{5}{16}''$	
$1\frac{5}{8}''$	"	L_4L_5	$29\frac{3}{4}''$	$26\frac{1}{2}''$	$28\frac{1}{8}''$	$1\frac{19}{32}''$
$1\frac{5}{8}''$	"	L_4L_5	$25\frac{1}{4}''$	$22''$	$23\frac{5}{8}''$	$2\frac{1}{4}''$
$1\frac{7}{16}''$	"	L_3L_4	$21\frac{7}{8}''$	$19''$	$20\frac{7}{16}''$	$1\frac{19}{32}''$
$1\frac{1}{4}''$	plates,	U_4L_4	$18\frac{3}{8}''$	$15\frac{7}{8}''$	$17\frac{1}{8}''$	$1\frac{21}{32}''$
$1\frac{11}{16}''$	"	U_3L_4	$14\frac{5}{8}''$	$11\frac{1}{4}''$	$12\frac{15}{16}''$	$2\frac{3}{32}''$
$1\frac{1}{2}''$	bar,	L_3L_4	$10\frac{5}{8}''$	$7\frac{5}{8}''$	$9\frac{1}{8}''$	$1\frac{29}{32}''$
$1\frac{5}{8}''$	"	L_4L_5	$7\frac{1}{2}''$	$4\frac{1}{4}''$	$5\frac{7}{8}''$	$1\frac{5}{8}''$

Ans. 345,000 in.-lbs.; 464,000 in.-lbs.

86. A seven-panel pin-connected truss of 200 ft. span has a depth of $37\frac{1}{2}$ ft. at the centre panel and of 28 ft. at the hips, the upper-chord points lying in the arc of a parabola. The dead and live loads are 945 and 2360 lbs. per lineal foot of truss respectively. The percentage by which the live-load stresses are increased to allow for impact is given by the formula percentage $=\dfrac{400}{500+L}$, L being the length of bridge covered by the live load when the stress in any given member is greatest. Verify the *total maximum* stresses indicated on the several members of Fig. 843. If the stresses in the members of the *upper* lateral system are as indicated show that the wind load is taken at 200 lbs. per lineal foot of span, and find (a) the stress along the bottom chord due to the transferred wind load. Assuming a wind load on the *lower* lateral system of 490 lbs. per lineal foot of span when a train is crossing the bridge, verify the total maximum stresses indicated in the diagram. Assuming that the wind load, when the bridge is empty, is *one half* of that when the bridge is loaded, discuss (b) the tendency to reversion of stress in the lower chord.

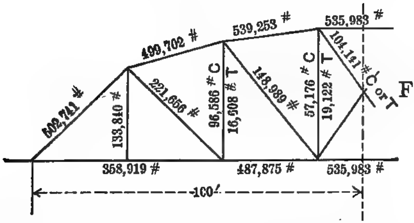


FIG. 843.

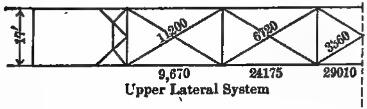


FIG. 844.

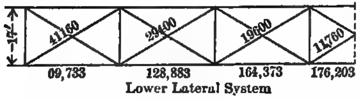


FIG. 845.

Ans. (a) 34,243 lbs.; (b) in second panel total stress due to *direct* and *transferred* wind loads is 81,563 lbs., while the dead-load stress is 82,654 lbs., so that reversion almost takes place, and it is therefore advisable to make the two end-chord panel lengths rigid members.

87. In the preceding example, the unsupported lengths of batter-braces is 30 ft., and the batter-braces may be considered as fixed at the lower ends. Find (*a*) the moment of resistance at the foot of such a brace. If the working stress is estimated by the formula, *working stress in pounds per square inch* $= 18{,}000 - 70\frac{l}{r}$, determine (*b*) whether it is safe to use a section made up of *one* $21'' \times \frac{7}{16}''$ cover, *two* $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times 24$-lb. angles, *two* bottom $5'' \times 3\frac{1}{2}'' \times 54$-lb. angles, and *two* $18'' \times \frac{3}{4}''$ side plates.

Ans. (*a*) 1.542,858 in.-lbs.; (*b*) gross $I = 2557.7$, $r = 7$, *net* $I = 2300$, max. bending stress in brace = 7857 lbs./sq. in.; required area = 50.6 sq. ins., and therefore section is ample.

88. The dead and live loads of a five-panel riveted truss 28 ft. deep and of 127 ft. span are 800 and 2100 lbs. per lineal foot of truss respectively. The impact effect increases the live-load stresses by the factor $\frac{400}{L+500}$, L being the length of bridge in feet which is loaded when any given member is subjected to a maximum live-load stress. Verify the *total maximum* stresses indicated on the several members in Fig. 846. The wind loads per lineal foot are 150 and 450 lbs. for the upper and lower lateral systems respectively. Verify the stresses in the several members of these systems as indicated in Figs. 847 and 848, and (*a*) remark upon the method of designing the wind diagonals. Also find (*b*) the chord stress due to the transferred wind load. Will there (*c*) be a reversion of stress in the chord panels?

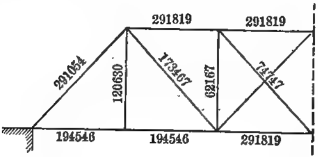


FIG. 846.

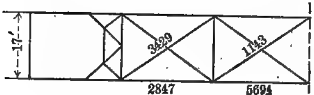


FIG. 847.

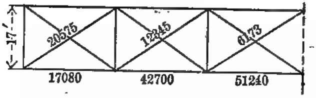


FIG. 848.

Ans. The chord and diagonal wind stresses are too small to have any appreciable effect upon the sections of the upper lateral members, which are governed by the limiting value of $\frac{l}{r}$; the $\frac{l}{r}$ for the transverse struts should be <120; (*b*) 11,384 lbs.; (*c*) in the most unfavorable case, i.e., when the bridge is empty, the combined transferred and direct wind-load stresses are less than the corresponding dead-load stresses, and there is therefore no tendency to reversion in the lower chords.

89. An A truss, Fig. 849, is 100 ft. centre to centre of end pins, and has a depth of 40 ft. at the centre, the trusses are spaced 17 ft. centre to centre; the live and dead loads are 4000 and 1400 lbs. per lineal foot of span re-

spectively; each truss has four panels of 25 ft. length. Verify the maximum stresses indicated on the members of the truss.

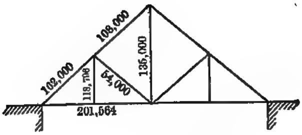


FIG. 849.

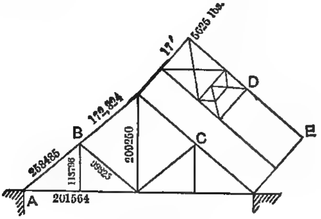


FIG. 850.

90. A four-panel A truss, Fig. 850, has a central depth of 40 ft. and its length from centre to centre of end pins is 100 ft.; the dead and live loads per lineal foot per truss are 750 and 2450 lbs. respectively. The *percentage* of increase in the live-load stresses for *impact* may be taken at $400 \div (500 + L)$ when the live load covers a length of L feet. Verify the *total maximum* stresses indicated on the several members of the truss. The wind load on the upper lateral bracing between B and C is 150 lbs. per lineal foot; find (*a*) the transferred wind load. Fig. 850 shows the sloping end members and the portal bracing. Assuming that there is a point of *inflection* at the middle point of the unsupported length DE, find (*b*) the moment of resistance at the foot of the brace. If the section of the brace is made up of *two* $15'' \times 99$-lb. channels and *one* $18'' \times \frac{3}{8}''$ cover, find (*c*) the maximum bending stress per square inch, and (*d*) the maximum stress per square inch due to the transferred wind load. If the safe working stress is $\left(18{,}000 - 80\,\dfrac{l}{r}\right)$ lbs. per square inch, determine (*a*) whether the section has sufficient area.

Ans. (*a*) 5625 lbs; (*b*) 540,192 in.-lbs.; (*c*) 5402 lbs.; (*d*) 533 lbs.; (*e*) the greatest area required is 23.8 sq. ins.

CHAPTER XI.

SUSPENSION BRIDGES.

1. Cables.—The modern suspension bridge consists of two or more cables from which the platform is suspended by iron or steel rods. The cables pass over lofty supports (piers), and are secured to anchorages upon which they exert a direct pull.

Chain or *link cables* are the most common in England and Europe, and consist of iron or steel links set on edge and pinned together. Formerly the links were made by welding the heads to a flat bar, but they are now invariably rolled in one piece, and the proportional dimensions of the head, which in the old bridges are very imperfect, have been much improved.

Hoop-iron cables have been used in a few cases, but the practice is now abandoned, on account of the difficulty attending the manufacture of endless hoop iron..

Wire-rope cables are the most common in America, and form the strongest ties in proportion to their weight. They consist of a number of parallel wire ropes or strands compactly bound together in a cylindrical bundle by a wire wound round the outside. There are usually *seven* strands, one forming a core round which are placed the remaining six. It was found impossible to employ a seven-strand cable in the construction of the Brooklyn Bridge, as the individual strands would have been far too bulky to manipulate. The same objection held against a thirteen-strand cable (thirteen is the next number giving an approximately cylindrical shape), and it was finally decided to make the cable with nineteen strands. Seven of these are pressed together so as to form a centre core, around which are placed the remaining twelve, the whole being continuously wrapped with wire.

In laying up a cable great care is required to distribute the tension uniformly amongst the wires. This may be effected either by giving each wire the same deflection or by using straight wire, i.e., wire which when unrolled upon the floor from a coil remains straight and shows no tendency to spring back. The distribution of stress is practically uniform in untwisted-wire ropes. Such ropes are spun from the wires and strands without giving any twist to individual wires.

The back-stay is the portion of the cable extending from an anchorage to the nearest pier.

The elevation of the cables should be sufficient to allow for settling, which chiefly arises from the deflection due to the load and from changes of temperature.

The cables may be protected from atmospheric influence by giving them a thorough coating of paint, oil, or varnish, but wherever they are subject to saline influence, zinc seems to be the only certain safeguard.

2. **Anchorage, Anchorage-chains, Saddles.**—The anchorage, or abutment, is a heavy mass of masonry or natural rock to which the end of a cable is made fast, and which resists by its dead weight the pull upon the cable.

The cable traverses the anchorage as in Figs. 851 to 852, passes through a strong, heavy cast-iron anchor-plate, and, if made of wire rope, has its end effectively secured by turning it round a dead-eye and splicing it to itself. Much care, however, is required to prevent a wire-rope cable from rusting on account of the great extent of its surface, and it is considered advisable that the wire portion of the cable should always terminate at the entrance to the anchorage and there be attached to a massive chain of bars, which is continued to the anchor-plate or plates and secured by bolts, wedges, or keys.

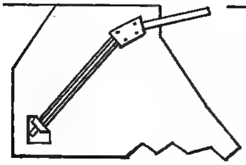


FIG. 851.

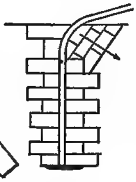


FIG. 852.

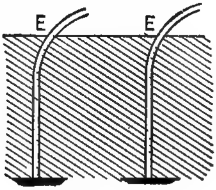


FIG. 853.

In order to reduce as much as possible the depth to which it is necessary to sink the anchor-plates, the anchor-chains are frequently curved as in Fig. 852. This gives rise to an oblique force, and the masonry in the part of the abutment subjected to such force should be laid with its beds perpendicular to the line of thrust.

The anchor-chains are made of compound links consisting alternately of an odd and an even number of bars. The friction of the link-heads on the knuckle-plates considerably lessens the stress in a chain, and it is therefore usual to diminish its sectional area gradually from the entrance *E* to the anchor. This is effected in the Niagara Suspension Bridge by varying the section of the bars, and in the Brooklyn Bridge by varying both the section and the number of the bars.

The necessity for preserving the anchor-chains from rust is of such importance that many engineers consider it most essential that the passages and channels containing the chains and fastenings should be accessible for periodical examination, painting, and repairs. This is unnecessary if the chains are first chemically cleaned and then embedded in good hydraulic cement, as they will thus be perfectly protected from all atmospheric influence.

The direction of an anchor-chain is changed by means of a saddle or knuckle-plate, which should be capable of sliding to an extent sufficient to allow for the expansion and contraction of the chain. This may be accomplished without the aid of rollers by bedding the saddle upon a four- or five-inch thickness of asphalted felt.

The chain, where it passes over the piers, rests on *saddles*, the object of which is to furnish bearings with easy vertical curves. Either the saddle may be constructed as in Fig. 854, so as to allow the cable to slip over it with comparatively little friction, or the chain may be secured to the saddle and the saddle supported upon rollers which work over a perfectly true and horizontal bed formed by a saddle-plate fixed to the pier.

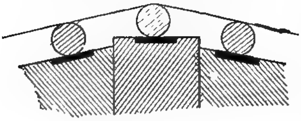


FIG. 854.

3. **Suspenders.**—The suspenders are the vertical or inclined rods which carry the platform.

In Fig. 855 the suspender rests in the groove of a cast-iron yoke which straddles the cable. Fig. 856 shows the suspender bolted to a wrought-iron or steel ring which embraces the cable. When there are more than two cables in the same vertical plane, various methods are adopted to insure the uniform distribution of the load amongst the set. In Fig. 857, for example, the suspender is fastened to the centre of a small wrought-iron lever *PQ*, and the ends of the lever are connected with the cables by the equally strained rods *PR* and *QS*. In the Chelsea bridge the distribution is made by means of an irregularly shaped plate (Fig. 858), one angle of which is supported by a joint-pin, while a pin also passes through another angle and rests upon one of the chains.

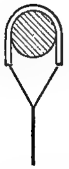


FIG. 855.

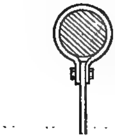


FIG. 856.

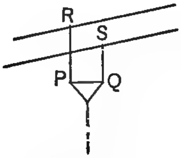


FIG. 857.

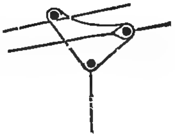


FIG. 858.

The suspenders carry the ends of the cross-girders (floor-beams) and are spaced from 5 to 20 ft. apart. They should be provided with wrought-iron screw-boxes for purposes of adjustment.

4. **Curve of Cable.**—CASE A.—An arbitrarily loaded flexible cable takes the shape of one of the catenaries, but the *true* catenary is the curve in which a cable of uniform section and material hangs under its own weight only.

Let A be the lowest point of the cable, Fig. 859, and take the vertical through A as the axis of y.

Take the horizontal through O as the axis of x, the origin O being chosen so that

$$pAO = H = mp, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

p being the weight of a unit of length of the cable and H the horizontal pull at A.

m or AO is the parameter, or modulus, of the catenary, and OG is the directrix.

Let x, y be the co-ordinates of any point P, the length of the rc AP being s.

Draw the tangent PT and the ordinate PN, and let the angle $PTN = \theta$.

The triangle PNT is evidently a triangle of forces for the portion AP, PN representing the weight of AP (viz., ps), PT the tangential pull T at P, and NT the horizontal pull H at A.

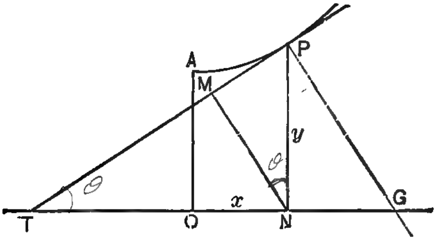


FIG. 859.

Therefore $$\frac{dy}{dx}=\tan\theta=\frac{PN}{TN}=\frac{ps}{H}=\frac{s}{m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

which gives the differential equation to the catenary.

It may be easily integrated as follows:

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{s^2}{m^2}}=\frac{1}{m}\sqrt{s^2+m^2}, \quad . \quad . \quad . \qquad (3)$$

or $$\frac{ds}{\sqrt{s^2+m^2}}=\frac{dx}{m}.$$

Therefore $$\log(s+\sqrt{s^2+m^2})=\frac{x}{m}+c,$$

c being a constant of integration.

When $x=0$, $s=0$, and $\log m=c$. Hence

$$\log\frac{s+\sqrt{s^2+m^2}}{m}=\frac{x}{m},$$

or $$s+\sqrt{s^2+m^2}=me^{\frac{x}{m}}.$$

Therefore, also, $$-s+\sqrt{s^2+m^2}=me^{-\frac{x}{m}}.$$

Hence $$\mathbf{s}=\frac{\mathbf{m}}{2}\left(\mathbf{e}^{\frac{\mathbf{x}}{\mathbf{m}}}-\mathbf{e}^{-\frac{\mathbf{x}}{\mathbf{m}}}\right)=m\frac{dy}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and integrating between 0 and x,

$$\mathbf{y}=\frac{\mathbf{m}}{2}\left(\mathbf{e}^{\frac{\mathbf{x}}{\mathbf{m}}}+\mathbf{e}^{-\frac{\mathbf{x}}{\mathbf{m}}}\right)=\sqrt{s^2+m^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Eq. (5) is the equation to the true catenary and eq. (4) gives the length of the arc AP.

Draw NM perpendicular to PT, and let the angle $PTN=PNM=\theta$. Then

$$PM=PN\sin\theta=y\frac{s}{\sqrt{s^2+m^2}}=s, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and $$MN=PN\cos\theta=y\frac{m}{\sqrt{s^2+m^2}}=m. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus, the triangle PMN possesses the property that the side PM is equal to the length of the arc AP, and the side MN is equal to the modulus m(=AO).

The area $APNO$

$$=\int_0^x ydx=\frac{m^2}{2}\left(e^{\frac{x}{m}}-e^{-\frac{x}{m}}\right)=ms=2\times\text{triangle } PMN.$$

The radius of curvature, ρ, at P

$$=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{\left(\frac{y}{m}\right)^3}{\frac{y}{m^2}}=\frac{y^2}{m}=PG, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

PG being perpendicular to PT.

At A, $y=m$, and the radius of curvature is also m. . . . (10)

Again, $$\frac{T}{ps}=\frac{PT}{NP}=\operatorname{cosec}\theta=\frac{y}{s}.$$

Therefore $$\mathbf{T}=\mathbf{py}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathbf{11})$$

and $$\mathbf{H}=\mathbf{pm}=\mathbf{p}\rho_0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathbf{12})$$

ρ_0 being the radius of curvature at A.

These catenary formulæ are of little if any use in the design and construction of suspension bridges, as they are based upon the assumption of a purely theoretical load which never occurs in practice, viz., the weight of a chain of uniform section and density.

Ex. 1. *A floating landing stage is held in position by a number of 4-in. steel-wire cables anchored to the shore, a shoreward movement being prevented by rigid iron booms, pivoted at the ends and stretching from shore to stage. The difference of level between the shore and stage attachments of the cables is 50 ft., and the horizontal distance between these points is 150 ft. The horizontal pull upon each cable is 1360 lbs. Find the length of the cable and the tensions at the points of attachment. (Weight of cable = 490 lbs. per cubic foot; form of cable a common catenary.)*

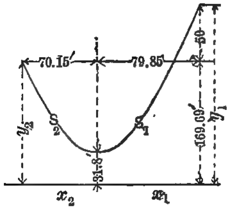


FIG. 860.

πr^2

$$1360 = \tfrac{22}{7} \cdot \tfrac{16}{4} \cdot \tfrac{1}{144} \cdot 490 \cdot m.$$

Therefore $$m = 31.8.$$

Again, $$y_1 = \frac{m}{2}\left(e^{\frac{x_1}{m}} + e^{-\frac{x_1}{m}}\right),$$

and $$y_2 = \frac{m}{2}\left(e^{\frac{x_2}{m}} + e^{-\frac{x_2}{m}}\right).$$

Therefore $$50 = y_1 - y_2 = \frac{m}{2}\left(e^{\frac{x_1}{m}} + e^{-\frac{x_1}{m}} - e^{\frac{150-x_1}{m}} - e^{-\frac{150-x_1}{m}}\right)$$

which reduces to $$e^{\frac{2x_1}{m}} - \frac{100}{m}\,\frac{e^{\frac{150}{m}}}{e^{\frac{150}{m}} - 1}\,e^{\frac{x_1}{m}} = +e^{\frac{150}{m}}.$$

But $e^{\frac{150}{m}} = 112\tfrac{1}{2}$, and the last equation becomes

$$e^{\frac{2x_1}{m}} - 3.173e^{\frac{x_1}{m}} = 112.5$$

and $$e^{\frac{x_1}{m}} = 12.31,$$

so that $$\frac{x_1}{m} = 2.511,$$

or $$x_1 = 79.85 \text{ ft.} \quad \text{and} \quad x_2 = 150 - x_1 = 70.15 \text{ ft.}$$

Again, $$s_1 = \frac{m}{2}\left(e^{\frac{x_1}{m}} - e^{-\frac{x_1}{m}}\right) = 194.46 \text{ ft.}$$

Also, $$e^{\frac{x_2}{m}}=e^{2.206}=9.08.$$

Therefore $$s_2=\frac{m}{2}\left(e^{\frac{x_2}{m}}-e^{-\frac{x_2}{m}}\right)=142.64 \text{ ft.},$$

and total length of cable $=337.1$ ft.

Also $$50(y_1+y_2)=s_1^2-s_2^2=51.82\times 337.1,$$

or $$y_1+y_2=349.38 \text{ ft.}$$

Hence $$y_1=199.69 \text{ ft.} \quad \text{and} \quad y_2=149.69 \text{ ft.}$$

Then, if T_1 and T_2 are the shore and stage tensions on the cables,

$$\frac{T_1}{1360}=\frac{y_1}{m}=6.28, \quad \text{or} \quad T_1=8541 \text{ lbs.}$$

and $$\frac{T_2}{1360}=\frac{y_2}{m}=4.71, \quad \text{or} \quad T_2=6402 \text{ lbs.}$$

CASE B. Let the platform be suspended from chains composed of a number of links, and let W be the whole weight between the lowest point O, Fig. 861, of the chain and the upper end P of any given link. Let the direction of this link intersect that of the horizontal pull (H) at O in E. Drop the perpendicular PN. The triangle PNE is evidently a triangle of forces; and if the angle $PEN=\theta$,

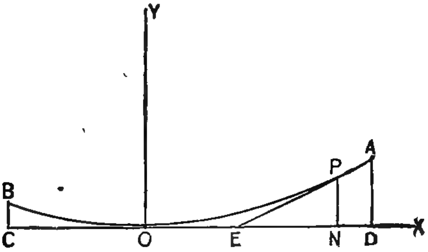


FIG. 861.

$$\tan\theta=\frac{PN}{NE}=\frac{W}{H},$$

and hence $$\tan\theta \propto W.$$

Thus, by treating each link separately, commencing with the lowest, the exact profile of the chain may be easily traced.

Ex. 2. *A light suspension bridge carries a foot-path* 8 *ft. wide over a river* 90 *ft. wide by means of eight equidistant suspending rods, the dip being* 10 *ft. Each cable consists of nine straight links. Find their several lengths. If the load upon the platform is* 120 *lbs. per square foot, and if* one sixth *of the load is borne by the piers, find the sectional areas of the several links, allowing* 10,000 *lbs. per square inch.*

Load concentrated at each of the points E, F, G, and $H=4500$ lbs. The *three* links BC, CD, DE are kept in equilibrium by the horizontal pull H at E, by the pull T_1 along BA, and by the *uniformly* distributed load between E and H. Hence L_1, the intersection of EL and AB produced, is the middle point of EH, and ALK is a triangle of forces for H, T_1 and the weight $4\times4500=18{,}000$ lbs. Therefore

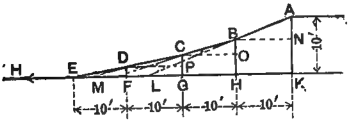


FIG. 862.

$$H=18000\frac{LK}{AK}=18000\times\frac{25}{10}=45{,}000 \text{ lbs.}$$

and
$$T_1=18000\frac{AL}{AK}=18000\sqrt{725}=4500\sqrt{116} \text{ lbs.}$$

The *two* links CD, DE are kept in equilibrium by the horizontal pull H at E, by the pull T_2 along DC, and by the uniformly distributed load between E and G. Hence the directions of H and T_2 must intersect in F the middle point of EG, and BFH is a triangle of forces for H, T_2 and the weight $3\times4500=13{,}500$ lbs.

Also
$$\frac{BH}{10}=\frac{15}{25} \quad \text{or} \quad BH=6 \text{ ft.}$$

Therefore
$$T_2=13500\,\frac{BF}{BH}=2250\sqrt{20^2+6^2}=4500\sqrt{109} \text{ lbs.}$$

The link DE is kept in equilibrium by the horizontal pull H at E, by the pull T_3 along DC, and by the uniformly distributed load between E and F. Hence the directions of H and T_3 must intersect in M the middle point of EF, and CMG is a triangle of forces for H,T_3 and the weight $2\times4500=9000$ lbs.

But
$$\frac{CG}{10}=\frac{6}{20}, \quad \text{or} \quad CG=3 \text{ ft.}$$

Therefore
$$T_3=9000\frac{CM}{CG}=3000\sqrt{234}=4500\sqrt{104}.$$

Finally, the weight of 4500 lbs. at E is equilibrated by the pull H and by he pull T_4 along ED. Hence DEF is a triangle of forces.

But $$\frac{DF}{5}=\frac{3}{15}, \quad \text{or} \quad DF=1 \text{ ft.}$$

Therefore $$T_4=4500\frac{DE}{DF}=4500\sqrt{101}.$$

The sectional areas of the several links in square inches are $\frac{T_1}{0000}$, $\frac{T_2}{10000}$, $\frac{T_3}{10000}$, $\frac{T_4}{10000}$, $\frac{H}{10000}$; i.e., 4.847, 4.698, 4.59, 4.522, and 4.5.

Case C.—Generally speaking, the distribution of the load may e assumed to be approximately uniform per *horizontal* unit of ength, the load being suspended from a number of points along ach chain or cable by means of rods. The weight upon the cable etween the lowest point A, Fig. 863, and any other point P is equilirated by the horizontal pull at A and by the tangential pull at P.

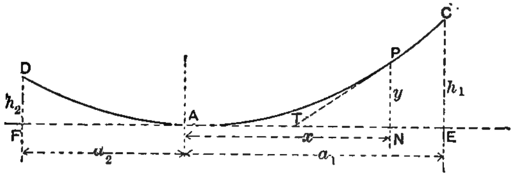


Fig. 863.

he directions of these two tensions intersect in the point T, which necessarily the middle point of AN, since the weight is uniformly istributed over AN. This is a characteristic property of the parabola.

If x, y are the co-ordinates of P with respect to A, then PTN a triangle of forces, for the horizontal tension H at A, the tangential ull T at P, and the weight wx uniformly distributed over AN, w eing the intensity of the load. Hence,

$$\frac{H}{wx}=\frac{\frac{x}{2}}{y}, \quad \text{or} \quad x^2=\frac{2H}{w}y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

e equation to a parabola with its vertex at O, its axis vertical, d its parameter equal to $\frac{2H}{w}=P$, suppose.

Also, taking the angle $PTN=\theta$,

$$\mathbf{T}\cos\theta=\mathbf{H}=\frac{\mathbf{wx}^2}{\mathbf{2y}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the horizontal pull at every point of the cable is the same as that at the lowest point.

Again, $$T=\sqrt{(wx)^2+H^2}=wx\sqrt{1+\frac{x^2}{4y^2}}.$$

The radius of curvature at P

$$=\frac{\left(1+\frac{4y^2}{x^2}\right)^{\frac{3}{2}}}{\frac{2y}{x^2}\left(=\frac{w}{H}\right)}=\frac{\left(1+2y\frac{w}{H}\right)^{\frac{3}{2}}}{\frac{w}{H}},$$

so that the radius at O is

$$\rho_0=\frac{H}{w},$$

and $$H=w\rho_0.$$

Parameter, etc.—Let h_1, h_2 be the elevations of C and D, respectively, above the horizontal line FAE.

Let $AE=a_1$, $AF=a_2$, and let $a_1+a_2=a=EF$.

By equation (1),

$$\sqrt{\frac{2H}{w}}=\frac{a_1}{\sqrt{h_1}}=\frac{a_2}{\sqrt{h_2}}=\frac{a_1+a_2}{\sqrt{h_1}+\sqrt{h_2}}=\frac{a}{\sqrt{h_1}+\sqrt{h_2}}.$$

Then $$P=\frac{2H}{w}=\left(\frac{a}{\sqrt{h_1}+\sqrt{h_2}}\right)^2.$$

Also, $$\tan\theta=\frac{2y}{x}=\frac{wx}{H}=\frac{2x}{P}=2\sqrt{\frac{y}{P}}.$$

If θ_1, θ_2 be the values of θ at C and D respectively,

$$\tan\theta_1=2\sqrt{\frac{h_1}{P}} \quad \text{and} \quad \tan\theta_2=2\sqrt{\frac{h_2}{P}}.$$

If $h_1 = h_2 = h$, then

$$a_1 = a_2 = \frac{a}{2}, \quad P = \frac{a^2}{4h},$$

and $$\tan\theta_1 = \frac{4h}{a} = \tan\theta_2.$$

Length of Arc of Cable.—Let $AP = s$.

Since $$\tan\theta = \frac{wx}{H},$$

$$\sec^2\theta d\theta = \frac{w}{H}dx = \frac{w}{H}ds\cos\theta, \quad \text{or} \quad ds = \frac{H}{w}\frac{d\theta}{\cos^3\theta}.$$

Hence

$$s = \frac{H}{w}\int_0^\theta \frac{d\theta}{\cos^3\theta} = \frac{H}{2w}\{\tan\theta\sec\theta + \log_e(\tan\theta + \sec\theta)\}$$

$$= \frac{H}{2w}\left\{\frac{w}{H}x\sqrt{1 + \frac{w^2}{H^2}x^2} + \log_e\left(\frac{w}{H}x + \sqrt{1 + \frac{w^2}{H^2}x^2}\right)\right\}.$$

An approximate value of the length of the arc which may be used in practice may be obtained as follows:

$$ds^2 = dx^2 + dy^2 = dx^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = dx^2\left(1 + \frac{w^2x^2}{H^2}\right).$$

Therefore $$ds = dx\left(1 + \frac{1}{2}\frac{w^2x^2}{H^2}\right), \text{ approximately.}$$

Integrating between O and P,

$$s = OP = x + \frac{1}{6}\frac{w^2x^3}{H^2} = x + \frac{2}{3}\frac{y^2}{x}.$$

Deflection of a Cable due to an Elementary Change in its Length.—The total approximate length (S) of the cable AOB is

$$S = a_1 + a_2 + \frac{2}{3}\frac{h_1^2}{a_1} + \frac{2}{3}\frac{h_2^2}{a_2}.$$

Now a_1 and a_2 are constant, and $h_1 - h_2$ is also constant, therefore $dh_1 = dh_2$. Hence

$$dS = \frac{4}{3}\left(\frac{h_1}{a_1} + \frac{h_2}{a_2}\right) dh_1.$$

If the alteration in length is due to a change of t° in the temperature,

$$dS = etS,$$

e being the coefficient of linear expansion.

In England the effective range of temperature is about 60° F., while in other countries it is usual to provide for a range of from 100° to 150° F.

If the alteration is due to a pull of intensity f per unit of area,

$$dS = \frac{f}{E} S,$$

E being the coefficient of elasticity of the cable material.

If $h_1 = h_2 = h$,

$$a_1 = a_2 = \frac{a}{2}, \quad \text{and} \quad dS = \frac{16}{3}\frac{h}{a} dh.$$

10. Pressure upon Piers, etc., Fig. 865.

Let T_1 be the tension in the main cable at A;
T_2 " " " " " back-stay at A;
α, β be the inclinations to the horizontal of the tangents at A to the main cable and back-stay, respectively.

The total vertical pressure upon the pier at A

$$= T_1 \sin \alpha + T_2 \sin \beta = P.$$

The total resultant horizontal force at A

$$= T_1 \cos \alpha \sim T_2 \cos \beta = Q.$$

If the cable is secured to a saddle which is free to move horizontally on the top of the pier (Fig. 862),

$Q \leqq$ the frictional resistance to the tendency to motion,

or
$$Q \leqq \mu_1 P,$$

μ_1 being the corresponding coefficient of friction.

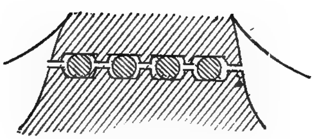


FIG. 864.

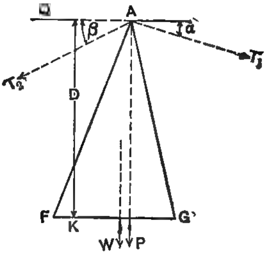


FIG. 865.

Let D, Fig. 865, be the total height of the pier, and let W be its weight.

Let FG be the base of the pier, and K the limiting position of the centre of pressure.

Let p, q be the distance of P and W, respectively, from K.

Then for *stability of position* $Q \leqq \dfrac{Pp+Wq}{D}$,

and for *stability of friction*, when the pier is of masonry,

$$\frac{Q}{P+W} \leqq \text{the coefficient of friction of the masonry}.$$

If μ_1 is sufficiently small to be disregarded, Q is approximately *nil*, and $T_1 \cos\alpha = T_2 \cos\beta = H$. The pressure upon the pier is now wholly vertical and is $= H(\tan\alpha + \tan\beta)$.

When the cable slides over smooth rounded saddles (Fig. 847), the tensions T_1 and T_2 are approximately the same.

Thus

$$P = T_1(\sin\alpha + \sin\beta) \quad \text{and} \quad Q = T_1(\cos\alpha - \cos\beta).$$

If $\alpha = \beta$, $Q = 0$, and the pressure upon the pier is wholly vertical, its amount being $2T_1 \sin\alpha$.

The piers are made of timber, iron, steel, or masonry, and allow of great scope in architectural design.

The cable should in no case be rigidly attached to the pier, unless the lower end of the latter is free to revolve through a small angle about a horizontal axis.

7. Weight of Cable.—The ultimate tenacity of iron wire is 90,000 lbs. per square inch, while that of steel rises to 200,000 lbs., and even more. The strength and gauge of cable wire may be insured by specifying that the wire is to have a certain ultimate tenacity and elastic limit, and that a given number of lineal feet of wire is to weigh *one* pound. Each of the wires for the cables of the Brooklyn Bridge was to have an ultimate tenacity of 3400 lbs., an elastic limit of 1600 lbs., and 14 lineal feet of the wire were to weigh *one* pound. A very uniform wire having a coefficient of elasticity of 29,000,000 lbs., has been the result, and the process of *straightening* has raised the ultimate tenacity and elastic limit nearly 8 per cent.

Let W_1 be the weight of a length $a_1(=AE)$ of a cable of sufficient sectional area to bear safely the horizontal tension H;

Let W_2 be the weight of the length $s_1(=AC)$ of the cable of a sectional area sufficient to bear safely the tension T_1 at C;

Let f be the safe inch-stress;

Let q be the specific weight of the cable material.

Then $$W_1=\frac{H}{f}a_1q \quad \text{and} \quad W_2=\frac{H\sec\theta_1}{f}s_1q.$$

Hence $$W_2=W_1\frac{s_1}{a_1}\sec\theta_1=\frac{W_1}{a_1}\left(a_1+\frac{2}{3}\frac{h_1^2}{a_1}\right)\left(1+\frac{2h_1^2}{a_1^2}+\ .\ .\ .\right),$$

or $$\mathbf{W_2=W_1\left(1+\frac{8}{3}\frac{h_1^2}{a_1^2}\right)}, \text{ nearly.}$$

A saving may be effected by proportioning any given section to the pull across that section. Thus at any point (x, y) the pull $=H\sec\theta$, and the corresponding necessary sectional area $=\dfrac{H\sec\theta}{f}$. The weight per unit of length $=\dfrac{H\sec\theta}{f}q$, and the total weight of the length $s_1(=AC)$ is

$$W_3 = \int_0^{a_1} \frac{H \sec \theta}{f} q \frac{ds}{dx} dx = \frac{Hq}{f} \int_0^{a_1} \sec^2 \theta dx$$

$$= \frac{Hq}{f} \int_0^{a_1} \left(1 + \frac{2y^2}{x^2} + \ldots \right)^2 dx.$$

But $$x^2 = \frac{a_1^2}{h_1} y.$$

Therefore $$W_3 = \frac{Hq}{f} \int_0^{a_1} \left(1 + 2\frac{h_1^2}{a_1^4} x^2 + \ldots \right)^2 dx$$

$$= \frac{Hq}{f} \left(a_1 + \frac{4}{3} \frac{h_1^2}{a_1}\right), \text{ nearly.}$$

Hence $$W_3 = W_1 \left(1 + \frac{4}{3} \frac{h_1^2}{a_1^2}\right),$$

and also $$2W_3 = W_1 + W_2.$$

The weight of a cubic inch of steel averages .283 lb.

The weight of a cubic inch of wrought iron averages .278 lb.

The volume in inches of the cable of weight $W_1 = 12a_1 \dfrac{H}{f}$.

Therefore $$\frac{W_1}{12a_1 \frac{H}{f}} = .283 \text{ lb. or } .278 \text{ lb.},$$

according as the cable is made of steel or iron.

Let the safe inch-stress of steel wire be taken at 33,960 lbs., of the best cable iron at 14,958 lbs., and of the best chain links at 9972 lbs. Then

$$\mathbf{W}_1 = Ha_1 \times .283 \times \frac{12}{33600} = \frac{\mathbf{Ha}_1}{10000} \text{ for steel cables;}$$

$$\mathbf{W}_1 = Ha_1 \times .278 \times \frac{12}{14958} = \frac{\mathbf{Ha}_1}{4500} \text{ for iron cables;}$$

$$\mathbf{W}_1 = Ha_1 \times .278 \times \frac{12}{9972} = \frac{\mathbf{Ha}_1}{3000} \text{ for link cables.}$$

Note.—About one eighth may be added to the net weight of a chain cable for eyes and fastenings.

Ex. 3. *A bridge* 444 *ft. long consists of a central span of* 180 *ft. and two side spans each of* 132 *ft.; each side of the platform is suspended by vertical rods from two iron-wire cables; each pair of cables passes over two masonry abutments and two piers, the former being* 24 *ft. and the latter* 39 *ft. above the surface of the ground; the lowest point of the cables in each span is* 19 *ft. above the ground surface; at the abutments the cables are connected with straight wrought-iron chains, by means of which they are attached to anchorages at a horizontal distance of* 66 *ft. from the foot of each abutment; the dead weight of the bridge is* 3500 *lbs. per lineal foot, and the bridge is covered with a proof load of* 4500 *lbs. per lineal foot.*

$$\frac{x_1}{x_2}=\sqrt{\frac{20}{5}}=2 \quad\text{and}\quad x_1=2x_2=88 \text{ ft.}$$

The load per lineal foot carried by each cable $=\dfrac{4500+3500}{4}=2000$ lbs.

Then $$H_2=T_2\frac{44}{\sqrt{44^2+20^2}}=T_3\frac{22}{\sqrt{22^2+5^2}}=88\times2000\frac{44}{20}=387{,}200 \text{ lbs.}$$

Therefore $H_2=387{,}200$ lbs., $T_2=425{,}323$ lbs. and $T_3=396{,}074$ lbs.,

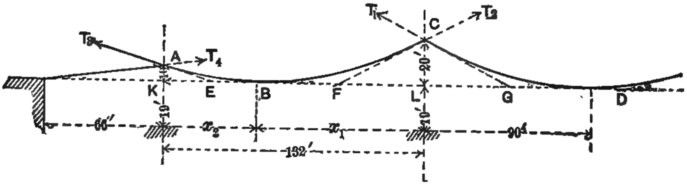


FIG. 866.

Again, $$H_1=T\frac{45}{\sqrt{45^2+20^2}}=90\times2000\times\frac{45}{20}=405{,}000 \text{ lbs.}$$

Therefore $H_1=405{,}000$ lbs. and $T_1=443{,}200$ lbs.

The length of the cable $ABC=44+\dfrac{2}{3}\dfrac{5^2}{44}+88+\dfrac{2}{3}\dfrac{20^2}{88}=135\tfrac{9}{22}$ ft.

also $$2\times \textit{length of } CD=2\left(90+\frac{2}{3}\frac{20^2}{90}\right)=185\tfrac{25}{27} \text{ ft.}$$

Change of dip corresponding to a variation of 60° F. from the mean temperature:

(*a*) *Side Span.*—Change in length of $ABC = \frac{60}{144000} 135\frac{9}{22} = .0564$ ft. Hence

$$\textit{change of dip} \times \frac{4}{3}\left(\frac{5}{44} + \frac{20}{88}\right) = .0564,$$

or $$\text{change of dip} = .1241 \text{ ft.}$$

(*b*) *Centre Span.*—Change in length of cable $= \frac{60}{144000} 185\frac{25}{27} = .0775$ ft.

Hence $$\text{change of dip} \times \frac{8}{3}\left(\frac{20}{90}\right) = .0775,$$

and $$\text{change of dip} = .1308 \text{ ft.}$$

Change of dip due to load, 15,000 lbs./sq. in., being the safe working stress:

(*a*) *Side Span.*—Sectional area of ABC (*assumed uniform*) $= \frac{T_2}{15000} =$ 28.355 sq. ins.

$$\textit{Extension of } ABC = 15{,}000 \times \frac{135\frac{9}{22}}{30000000} = .0677 \text{ ft.,}$$

and $$\text{change of dip} \times \frac{4}{3}\left(\frac{5}{44} + \frac{20}{88}\right) = .0677 \text{ ft.,}$$

or $$\textit{change of dip} = .1489 \text{ ft.}$$

(*b*) *Centre Span.*—Sectional area of CD (*assumed uniform*) $= \frac{T_1}{15000} = 29.55$ sq. ins.

$$\textit{Extension} \text{ of cable of centre span} = 15{,}000\, \frac{85\frac{25}{27}}{30000000} = .093 \text{ ft.,}$$

and $$\text{change of dip} \times \frac{8}{3}\,\frac{20}{90} = .093,$$

or $$\textit{change of dip} = .157 \text{ ft.}$$

Weight of cables:

Weight of cable for side span $= 135\frac{9}{22} \times \frac{28.355}{144} \times 1728 \times .278 = 12{,}809$ lbs.

" " " " centre span $= 185\frac{25}{27} \times \frac{29.55}{144} \times 1728 \times .278 = 18{,}329$ "

If cables are proportioned at each point to the pull, then

$$\text{wt. of side cable} = \frac{387200 \times .278}{15000}\left(88 + \frac{2}{3}\,\frac{20^2}{88} + 44 + \frac{2}{3}\,\frac{5^2}{44}\right)12 = 11{,}661 \text{ lbs.}$$

$$\text{" " centre cable} = 2 \times \frac{405000 \times .278}{15000}\left(90 + \frac{2}{3}\,\frac{20^2}{90}\right)12 = 16{,}747 \text{ lbs.}$$

Piers.—Overturning moment is greatest when the proof load covers centre span only. Total *vertical* load on pier $= 8000 \times 90 + 3500 \times 88 = 1{,}028{,}000$ lbs. The weight of the pier $= 39\frac{t+8}{2}14\frac{5}{8} \times 128$, t being the thickness of the base and $14\frac{5}{8}$ ft. its uniform width.

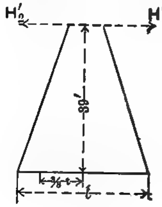


FIG. 867

Again, $H_1' = 8000 \times 90 \times \frac{45}{20}$ and $H_2' = 3500 \times 88 \times \frac{44}{20}$. Therefore the horizontal pull at the top of the pier

$$= H_1' - H_2' = 942{,}000 \text{ lbs.}$$

Hence, if centre of resistance at base is $\frac{3}{8}t$ from the middle point,

$$942400 \times 39 = \text{overturning moment}$$
$$= 39\frac{t+8}{2}14\frac{5}{8} \times 128 \times \frac{3}{8}t + 1{,}028{,}000 \times \frac{3}{8}t,$$

and $$t = 36.6 \text{ ft.}$$

Hence, too, the weight of the pier $= 39\frac{36.6+8}{2}14\frac{5}{8} \times 128 = 1{,}651{,}271$ lbs., and the total pressure on the base $= 2{,}679{,}271$ lbs.

Anchorage.—Vertical pull on anchorage

$$= H_2\tfrac{5}{66} = 387{,}200 \times \tfrac{5}{66}$$
$$= 29{,}334 \text{ lbs. } \textit{for each cable.}$$

Let W = weight of masonry required in anchorage for each cable to resist horizontal displacement.

Then $$\mu W = .76 \times W = 387{,}200 \text{ lbs.}$$

and $$W = 509{,}474 \text{ lbs.}$$

Thus the total weight required to resist the upward pull

$$= 4 \times 29{,}334 = 117{,}336 \text{ lbs,}$$

and the total weight required to resist the horizontal pull

$$= 4 \times 509{,}474 = 2{,}037{,}896 \text{ lbs.}$$

The tension in the anchorage bar $= H_2\frac{\sqrt{66^2 + 5^2}}{66} = 388{,}310$ lbs.

9. **Curve of Cable** from which the load is suspended by a series of sloping rods.

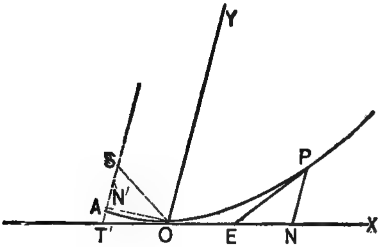


FIG. 868.

Let O be the lowest point of such a cable. Let the tangent at O, and a line through O parallel to the suspenders, be the axes of x and y respectively.

Let w' be the intensity of the oblique load. Consider a portion OP of the cable, and let the co-ordinates of P with respect to OX, OY be x and y.

Draw the ordinate PN, and let the tangent at P meet ON in E.

As before, PNE is a triangle of forces, and E is the middle point of ON. Then

$$\frac{w'x}{H}=\frac{PN}{NE}=\frac{2y}{x}, \quad \text{or} \quad x^2=\frac{2H}{w'}y,$$

the equation to a parabola with its axis parallel to OY and its focus at a point S, where $4SO=\frac{2H}{w'}$.

In bridges with sloping rods longitudinal stresses are developed which vary in intensity in different parts of the platform. Such bridges are much stiffer vertically than when the rods are vertical.

Parameter.—Let the axis meet the tangent at O in T', and let its inclination to OX be i.

Let A be the vertex, and ON' a perpendicular to the axis.

Then $$SO=ST'=SA+AT'=SA+AN'.$$

But $$4AS\cdot AN'=ON'^2=N'T'^2\tan^2 i=4AN'^2\tan^2 i.$$

Therefore $AS=AN'\tan^2 i$, and $SO=AS(1+\cot^2 i)=\dfrac{AS}{\sin^2 i}$.

Hence the parameter $=4AS=4SO\sin^2 i$.

Stresses.—Let P be the oblique load upon the cable between O and P.

Let Q be the total thrust upon the platform at E;

w " " load per horizontal unit of length;
q " " rate of increase of stress along platform;
t " " length of PE;
a = length of each bay of platform.

Then $w'=w\operatorname{cosec} i$ and *pull on each rod* $=w'a=wa\operatorname{cosec} i$.

Also $q=w\cot i$, and the horizontal component per panel $=wa\cot i$,

which represents the increment of force developed in the platform at the foot of the sloping rod.

Again, H is the horizontal force at O, and therefore the horizontal pull on the chain

$$=H-wa\cot i=\frac{wl^2}{8d}-wa\cot i,$$

if l is the span and d the dip.

The tension at $P=wx\sec\theta$, θ being the angle between the tangent at P and the horizontal.

Also, $$H=\frac{w'x^2}{2y}=2w'\cdot SO=2AS\frac{w'}{\sin^2 i}=2AS\frac{w}{\sin^3 i};$$

$$P=H\frac{2t}{x}=\frac{w'tx}{y};\text{ and}$$

$$t^2=y^2+\frac{x^2}{4}+xy\cos i.$$

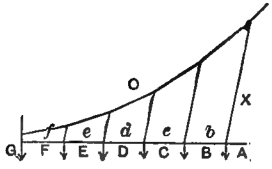


FIG. 869.

Assuming that the portions of the cable between the upper ends of consecutive rods are straight, the stresses developed may be easily determined graphically.

Fig. 869 represents the half-span and Fig. 870 is the stress diagram when the ends of the platform are not attached to the piers, the horizontal member being in tension.

This member will be in compression if its ends are attached to the piers and the stress diagram is then Fig. 871.

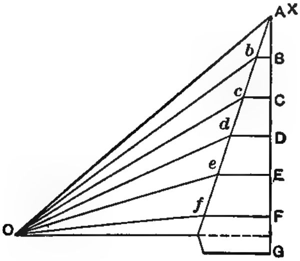


FIG. 870.

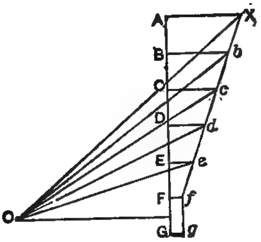


FIG. 871.

Length of Cable.—Let s be the length of OP, and let θ be the inclination of PE to OY. Then

$$s=AP-AO$$

$$=\frac{H\sin^2 i}{2w'}\Big\{\tan(90^\circ-\theta)\sec(90^\circ-\theta)$$

$$+\log_e\{\tan(90^\circ-\theta)+\sec(90^\circ-\theta)\}-\tan(90^\circ-i)\sec(90^\circ-i)$$

$$-\log_e\{\tan(90^\circ-i)+\sec(90^\circ-i)\}\Big\}$$

$$=\frac{H\sin^2 i}{2w'}\left\{\cot\theta\operatorname{cosec}\theta-\cot i\operatorname{cosec} i+\log_e\frac{\cot\theta+\operatorname{cosec}\theta}{\cot i+\operatorname{cosec} i}\right\},$$

and *approximately*,

$$s=x+y\cos i+\frac{2}{3}\frac{y^2\sin^2 i}{x+y\cos i}.$$

11. **Auxiliary or Stiffening Truss.**—The object of a stiffening truss (Fig. 872) is to distribute a passing load over the cable in such a manner that it cannot be distorted. The pull upon each suspender must therefore be the same, and this virtually assumes that the

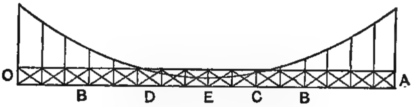


FIG. 872.

effect of the extensibility of the cable and suspenders upon the figure of the stiffening truss may be disregarded.

The ends O and A must be anchored, or held down by pins, but should be free to move horizontally.

Let there be n suspenders dividing the span into $(n+1)$ equal segments of length a.

Let P be the total weight transmitted to the cable and z the distance of its centre of gravity from the vertical through O.

Let T be the pull upon each suspender.

Taking moments about O,

$$Pz=T(a+2a+3a+\ldots+na)=Ta\frac{n(n+1)}{2}=T\frac{nl}{2},$$

l being the length of OA.

Also, if t is the *intensity of pull* per unit of span,

$$tl=nT, \quad \text{and hence} \quad Pz=t\frac{l^2}{2}.$$

Let there be a central suspender of length s. There will therefore be $\frac{n-1}{2}$ suspenders on each side of the centre.

The parameter of the parabola $=\frac{l^2}{4h}$.

Hence the total length of all the suspenders

$$=s+2\left\{\frac{n-1}{2}s+\frac{4h}{l^2}a^2\left[1^2+2^2+3^2+\ldots+\frac{(n-1)^2}{2}\right]\right\}$$

$$=ns+8h\frac{a^2}{l^2}\frac{n(n^2-1)}{24}=n\left(s+\frac{h}{3}\frac{n-1}{n+1}\right).$$

If there is no central suspender, i.e., if n is even,

$$\text{the total length}=(n-1)\left(s+\frac{h}{3}\frac{n}{n+1}\right).$$

Denote the total length of suspenders by L. Then

$$\text{the } \textit{stress length}=TL=\frac{2z}{nl}PL.$$

Let w be the uniform intensity of the dead load.

Case I. *The bridge partially loaded.*

Let w' be the maximum uniform intensity of the live load, and let this load advance from A and cover a length AB.

Let $OB = x$, and let R_1, R_2 be the pressures at O and A respectively.

For equilibrium,

$$R_1 + R_2 + tl - wl - w'(l-x) = 0; \quad . \quad . \quad . \quad . \quad (1)$$

$$R_1 l + t\frac{l^2}{2} - w\frac{l^2}{2} - \frac{w'}{2}(l-x)^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also, since the whole of the weight is to be transmitted through the suspenders,

$$tl = wl + w'(l-x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From eqs. (1), (2), and (3),

$$-R_1 = \frac{w'}{2}\frac{x}{l}(l-x) = R_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which shows that the reactions at O and A are equal in magnitude but opposite in kind. They are evidently greatest when $x = \frac{l}{2}$, i.e., when the live load covers half the bridge, and the common value is then $\frac{w'l}{8}$.

The *shearing force* at any point between O and B distant x' from O

$$= R_1 + (t-w)x' = w'\frac{l-x}{l}\left(x' - \frac{x}{2}\right), \quad . \quad . \quad . \quad . \quad (5)$$

which becomes $\frac{w'}{2}\frac{x}{l}(l-x) = -R_1 = R_2$ when x' equal x. Thus the shear at the head of the live load is equal in magnitude to the reaction at each end, and is an absolute maximum when the live load covers half the bridge. The web of the truss must therefore be designed to bear a shear of $\frac{w'l}{8}$ at the centre and ends.

Again, the *bending moment* at any point between O and B distant x' from O

$$=R_1x'+\frac{t-w}{2}x'^2=\frac{w'}{2}\frac{l-x}{l}(x'^2-xx'), \quad . \quad . \quad . \quad . \quad (6)$$

which is greatest when $x'=\frac{x}{2}$, i.e., at the centre of OB, its value then being $-\frac{w'}{8}\frac{l-x}{l}x^2$. Thus the bending moment is an *absolute maximum* when $\frac{d}{dx}(lx^2-x^3)=0$, i.e., when $x=\frac{2}{3}l$, and its value is then $-\frac{w'}{54}l^2$.

The bending moment at any point between B and A distant x' from O

$$=R_1x'+\frac{t-w}{2}x'^2-\frac{w'}{2}(x'-x)^2=\frac{w'}{2}\frac{x}{l}(x'-x)(l-x'), \quad . \quad . \quad (7)$$

which is greatest when $\frac{d}{dx'}\{(x'-x)(l-x')\}=0$, i.e., when $x'=\frac{l+x}{2}$, or at the centre of AB, its value then being $\frac{w'}{8}\frac{x}{l}(l-x)^2$. Thus the bending moment is an *absolute maximum* when $\frac{d}{dx}\{x(l-x)^2\}=0$, i.e., when $x=\frac{l}{3}$, and its value is then $+\frac{w'}{54}l^2$.

Hence *the maximum bending moments of the unloaded and loaded divisions of the truss are equal in magnitude but opposite in direction, and occur at the points of trisection* (D, C) *of* OA *when the live load covers one third* (AC) *and two thirds* (AD) *of the bridge respectively.*

Each chord must evidently be designed to resist both tension and compression, and in order to avoid unnecessary nicety of calculation, the section of the truss may be kept uniform throughout the middle half of its length.

CASE II. *A single concentrated load W at any point B of the truss.* W now takes the place of the live load of intensity w'.

The remainder of the notation and the method of procedure being precisely the same as before, the corresponding equations are

$$R_1+R_2+(t-w)l-W=0. \quad \ldots \quad (1')$$

$$R_1 l+\frac{t-w}{2}l^2-W(l-x)=0. \quad \ldots \quad (2')$$

$$t-w=\frac{W}{l}. \quad \ldots \quad (3')$$

$$-R_1=\frac{W}{l}\left(x-\frac{l}{2}\right)=R_2, \quad \ldots \quad (4')$$

which shows that the reactions at O and A are equal in magnitude but opposite in kind. They are greatest when $x=0$ and when $x=l$, i.e., when W is either at O or at A, and the common value is then $\frac{W}{2}$.

The *shearing force* at any point between O and B distant x' from O

$$=R_1+(t-w)x'=\frac{W}{l}\left(x'-x+\frac{l}{2}\right), \quad \ldots \quad (5')$$

which is a maximum when $x'=x$, and its value is then $\frac{W}{2}$.

The web must therefore be designed to bear a shear of $\frac{W}{2}$ throughout the whole length of the truss.

Again, the bending moment at any point between O and B distant x' from O

$$=R_1 x'+(t-w)\frac{x'^2}{2}=\frac{W}{l}\left\{\frac{x'^2}{2}+x'\left(\frac{l}{2}-x\right)\right\}. \quad \ldots \quad (6')$$

First, let $x<\frac{l}{2}$. The bending moment is *positive* and is a maximum when $x'=x$, its value then being

$$+\frac{W}{2l}(lx-x^2).$$

Next, let $x>\frac{l}{2}$. The bending moment is then *negative* and is a maximum when $x'=x-\frac{l}{2}$, its value then being

$$-\frac{W}{2l}\left(x-\frac{l}{2}\right)^2.$$

The bending moment at any point between B and A distant x' from O

$$=R_1x'+(t-w)\frac{x'^2}{2}-W(x'-x)=\frac{W}{l}(x'-l)\left(\frac{x'}{2}-x\right), \quad . \quad . \quad (7')$$

which is a maximum when

$$\frac{d}{dx'}\left\{(x'-l)\left(\frac{x'}{2}-x\right)\right\}=0,$$

i.e., when $x'=x+\frac{l}{2}$ and its value is then $-\frac{W}{2l}\left(x-\frac{l}{2}\right)^2$.

Note.—The stiffening truss is most effective in its action, but adds considerably to the weight and cost of the whole structure. Provision has to be made both for the extra truss and for the extra material required in the cable to carry this extra load.

Stiffening Truss Hinged at the Centre.—Provision may be made for counteracting the straining due to changes of temperature by hinging the truss at the centre E.

Let a live load of intensity w' advance from A.

First, let the live load cover a length $AB=x\left(>\frac{l}{2}\right)$.

Let R_1, R_2 be the pressures at O, A, respectively.

The equations of equilibrium are

$$R_1+R_2+(t-w)l-w'x=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$R_1\frac{l}{2}+(t-w)\frac{l^2}{8}-\frac{w'}{2}\left(x-\frac{l}{2}\right)^2=0; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$R_2\frac{l}{2}+(t-w)\frac{l^2}{8}-\frac{w'}{8}l^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eqs. (2) and (3) being obtained by taking moments about E. Hence

$$t-w=-\frac{w'}{l^2}(l^2-4lx+2x^2); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$R_1=\frac{1}{2}\frac{w'}{l}(l^2-4lx+3x^2); \quad \ldots \ldots \quad (5)$$

$$R_2=\frac{1}{2}\frac{w'}{l}(l-x)^2. \quad \ldots \ldots \quad (6)$$

Next, let the live load cover the length $BO\left(<\frac{l}{2}\right)$.

Let $AB=x$ as before, and let R_1, R_2', t' be the new values of R_1, R_2, t, respectively.

The equations of equilibrium are now

$$R_1'+R_2'+(t'-w)l-w'(l-x)=0; \quad \ldots \quad (7)$$

$$R_1'\frac{l}{2}+(t'-w)\frac{l^2}{8}-\frac{w'}{2}x(l-x)=0; \quad \ldots \quad (8)$$

$$R_2'\frac{l}{2}+(t'-w)\frac{l^2}{8}=0; \quad \ldots \quad (9)$$

and hence

$$t'-w=2\frac{w'}{l^2}(l-x)^2[=-(t-w-w')]; \quad \ldots \quad (10)$$

$$R_1'=-\frac{1}{2}\frac{w'}{l}(l^2-4lx+3x^2)(=-R_1); \quad \ldots \quad (11)$$

$$R_2'=-\frac{1}{2}\frac{w'}{l}(l-x)^2(=-R_2). \quad \ldots \quad (12)$$

Diagram of Maximum Shearing Force.—The shear at any point distant z from A in the unloaded portion BO when the live load covers AB

$$=R_1+(t-w)(l-z) \quad \ldots \ldots \quad (13)$$

$$=-\{R_1'+(t'-w-w')(l-z)\}$$

$$=-\{R_1'+(t'-w)(l-z)-w'(l-z)\}$$

=*minus* the shear at the *same* point when AB is *unloaded* and the live load covers BO.

For a given value of z the maximum shear, *positive or negative,* at any point of OB is found by making (see eq. (13))

$$dR_1+(l-z)d(t-w)=0,$$

or $$\frac{w'}{l}(-2l+3x)-\frac{w'}{l^2}(l-z)(-4l+4x)=0,$$

or $$x=l\frac{4z-2l}{4z-l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Hence, by eqs. (4), (5), (13), (14),

$$\text{the } \textit{maximum shear}=\pm\tfrac{1}{4}\mathbf{w'x}\frac{\mathbf{1-2x}}{\mathbf{1-x}}, \quad . \quad . \quad . \quad . \quad (15)$$

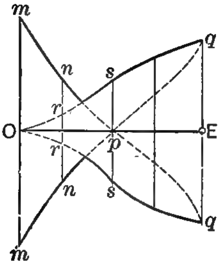


FIG. 873.

and may be represented by the ordinate (*positive or negative*) of the curve *mnpq*.

For example, at the points defined by

$$z= \quad l, \qquad \tfrac{7}{8}l, \qquad \tfrac{3}{4}l,$$

the shears are greatest when

$$x= \quad \tfrac{2}{3}l, \qquad \tfrac{3}{5}l, \qquad \tfrac{1}{2}l,$$

and their values are, respectively,

$$\mp\tfrac{1}{6}w'l, \quad \mp\tfrac{3}{40}w'l, \quad 0.$$

Again, the shear at any point distant z from A in the *loaded* portion BE when the live load covers AB

$$=R_1+(t-w)(l-z)-w'(x-z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$=R_1+(t-w-w')(l-z)+w'(l-x)$$

$$=-\{R_1'+(t'-w)(l-z)-w'(l-x)\}$$

= *minus* the shear at the *same* point when AB is *unloaded* and the live load covers BO.

Hence, by eqs. (4), (5), (16),

$$\text{the shear}=\mp\frac{1}{2}\frac{w'}{l^2}(l-4z)(l-x)^2, \quad . \quad . \quad . \quad . \quad (17)$$

increasing for a given value of z with $l-x$, and, therefore, a maximum when $x=z$. Thus

$$\text{the maximum shear}=\mp\frac{\mathbf{1}}{\mathbf{2}}\frac{\mathbf{w'}}{\mathbf{l^2}}(\mathbf{1-4x})(\mathbf{1-x})^2, \quad . \quad . \quad (18)$$

and occurs immediately *in front* of the load when it covers AB, and immediately *behind* the load when it covers *BO*. It may be represented by the ordinate (*positive or negative*) of the curve *orsq*.

For example, at the points defined by

$$z=x=l, \qquad \tfrac{7}{8}l, \qquad \tfrac{3}{4}l, \qquad \tfrac{5}{8}l, \qquad \tfrac{1}{2}l,$$

the maximum shears given by eq. (18) are, respectively,

$$0, \qquad \pm\tfrac{5}{256}w'l, \qquad \pm\tfrac{1}{16}w'l, \qquad \pm\tfrac{27}{256}w'l, \qquad \pm\tfrac{1}{8}w'l.$$

Diagram of Maximum Bending Moment.—The bending moment at any point in *BO* distant z from A when the live load covers *AB*

$$=R_1(l-z)+(t-w)\frac{(l-z)^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$=-\left\{R_1'(l-z)+(t'-w-w')\frac{(l-z)^2}{2}\right\}$$

$$=-\left\{R_1'(l-z)+(t'-w)\frac{(l-z)^2}{2}-w'\frac{(l-z)^2}{2}\right\}$$

=*minus* the bending moment at the same point when the live load covers *BO*.

Hence, by eqs. (4), (5), (19), the bending moment

$$=\pm\frac{1}{2}\frac{w'}{l}(l^2-4lx+3x^2)(l-z)\mp\frac{1}{2}\frac{w'}{l^2}(l^2-4lx+2x^2)(l-z)^2.$$

For a given value of z this is a maximum and equal to

$$\pm\frac{\mathbf{w'}}{\mathbf{2}}\,\frac{\mathbf{zl-zl-2z}}{\mathbf{(1-2z)}} \quad \text{when} \quad \mathbf{x}=\frac{\mathbf{2lz}}{\mathbf{l+2z}}.$$

Thus the maximum bending moment may be represented by the ordinate (*positive or negative*) of a curve.

For example, at the points defined by

$$z=l, \qquad \tfrac{7}{8}l, \qquad \tfrac{3}{4}l, \qquad \tfrac{5}{8}l, \qquad \frac{l}{2},$$

the bending moments are greatest when $x=$

$$\tfrac{2}{3}l, \qquad \tfrac{7}{11}l, \qquad \tfrac{3}{5}l, \qquad \tfrac{5}{9}l, \qquad \tfrac{1}{2}l,$$

their values being, respectively,

$$0, \qquad \mp\tfrac{21}{1408}w'l^2, \quad \mp\tfrac{3}{160}w'l^2, \quad \mp\tfrac{5}{384}w'l^2, \qquad 0.$$

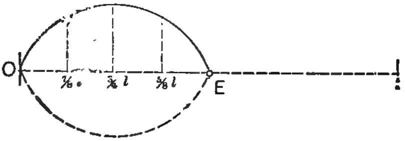


FIG. 874.

The *absolute maximum* bending moment may be found as follows:

For a given value of x the bending moment (see eq. (19)) is a maximum when

$$R_1+(t-w)(l-z)=0,$$

or

$$l-z=-\frac{R_1}{t-w}$$

Hence, the maximum bending moment

$$=\mp\frac{1}{2}\frac{R_1^2}{t-w}=\pm\frac{w'}{8}\frac{(l^2-4lx+3x^2)^2}{l^2-4lx+2x^2}.$$

It will be an absolute maximum for a value of x found by putting its differential with respect to x equal to *nil.*

This differential easily reduces to

$$3x^3-9lx^2+6l^2x-l_3=0.$$

$x=\frac{3}{5}l$ is an approximate solution of this equation, and the corresponding maximum bending moment $=\frac{8}{425}w'l^2$.

The preceding calculations show *that at every point in its length the truss may be subjected to equal, maximum shears and equal maximum bending moments of opposite signs.*

Again, it may be easily shown in a similar manner that when a single weight W travels over the truss,

the maximum *positive* shear at a distance z from A

$$=\frac{W}{l^2}(2l^2-5lz+4z^2);$$

the maximum *negative* shear either

$$=\frac{W}{l^2}(l^2-5lz+4z^2)$$

or

$$=\frac{1}{2}\frac{W}{l}(3l-4z),$$

and the maximum bending moment

$$=\pm\frac{W}{l^2}z(l-z)(l-2z).$$

12. **Suspension-bridge Loads.**—The heaviest distributed load to which a highway bridge may be subjected is that due to a dense crowd of people, and is fixed by modern French practice at 82 lbs. per square foot. Probably, however, it is unsafe to estimate the load at less than from 100 to 140 lbs. per square foot, while allowance has also to be made for the concentration upon a single wheel of as much as 36,000 lbs., and perhaps more.

A moderate force repeatedly applied will, if the interval between the blows corresponds to the vibration interval of the chain, rapidly produce an excessive oscillation (Chap. IV, Art. 3). Thus a procession marching in step across a suspension bridge may strain it far more intensely than a dead load, and will set up a synchronous vibration which may prove absolutely dangerous. For a like reason the wind usually sets up a wave motion from end to end of a bridge.

The *factor of safety* for the dead load of a suspension bridge should not be less than 2½ or 3, and for the live load it is advisable to make it 6. With respect to this point it may be remarked that the efficiency of a cable does not depend so much upon its ultimate strength as upon its limit of elasticity, and so long as the latter is not exceeded the cable remains uninjured. For example, the *breaking weight* of one of the 15-inch cables of the Brooklyn Bridge is estimated to be 12,000 tons, its *limit of elasticity* being 8118 tons; so that with 1½ only as a factor of safety, the stress would still fall below the elastic limit and have no injurious effect. The *continual* application of such a load would doubtless ultimately lead to the destruction of the bridge.

The dip of the cable of a suspension bridge usually varies from $\frac{1}{15}$ to $\frac{1}{12}$ of the span, and is rarely as much as $\frac{1}{10}$ except for small spans. Although a greater ratio of dip to span would give increased economy and an increased limiting span, the passage of a live load would be accompanied by a greater distortion of the chains and a larger oscillatory movement. Steadiness is therefore secured at the cost of economy by adopting a comparatively flat curve for the chains.

13. Modifications of the Simple Suspension Bridge.—The disadvantages connected with suspension bridges are very great. The position of the platform is restricted, massive anchorages and piers are generally required, and any change in the distribution of the load produces a sensible deformation in the structure. Owing to the want of rigidity, a considerable vertical and horizontal oscillatory motion may be caused, and many efforts have been made to modify the bridge in such a manner as to neutralize the tendency to oscillation.

(*a*) The simplest improvement is that shown in Fig. 875, where the point of the cable most liable to deformation is attached to the piers by short straight chains *AB*.

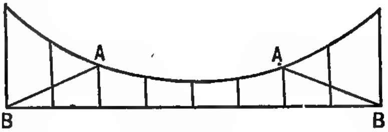


FIG. 875.

(*b*) A series of inclined stays, or iron ropes, radiating from the pier-saddles, may be made to support the platform at a number of equidistant points (Fig. 876). Such ropes were used in the Niagara Bridge, and still more recently in the Brooklyn Bridge. The lower ends of the ropes are generally made fast to the top or bottom chord of the bridge-truss, so that the corresponding chord stress is increased and the neutral axis proportionately displaced. To remedy this, it has been proposed to connect the ropes with a horizontal tie coincident in position with the neutral axis. Again, the bles of the Niagara and Brooklyn bridges do not hang in vertical ınes, but are inclined inwards, the distance between them being

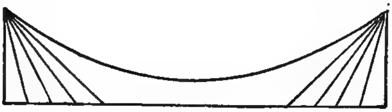


FIG. 876.

ɜatest at the piers and least at the centre of the span. This draw- ȝ in adds greatly to the lateral stability, which may be still further ɜreased by a series of horizontal ties.

(c) In Fig. 877 two cables in the same vertical plane are diag- ally braced together. In principle this method is similar to that

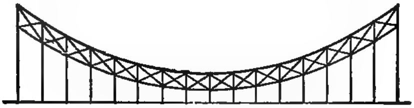


FIG. 877.

opted in the *stiffening truss* (discussed in Art. 11), but is probably s efficient on account of the flexible character of the cables, hough a slight economy of material might doubtless be realized. e braces act both as struts and ties, and the stresses to which ɜy are subjected may be easily calculated.

(d) In Fig. 878 a single chain is diagonally braced to the plat- m. The weight of the bridge must be sufficient to insure that

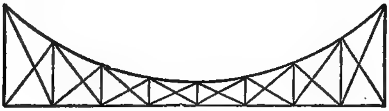


FIG. 878.

suspender will be subjected to a thrust, or the efficiency of the angement is destroyed. An objection to this as well as to the ceding method is that the variation in the curvature of the chain ɟer changes of temperature tends to loosen and strain the joints.

The principle has been adopted (Fig. 879) with greater perfec- ı in the construction of a foot-bridge at Frankfort. The girder ut at the centre, the chain is hinged, and the rigidity is obtained by means of vertical and inclined braces which act both as struts and ties.

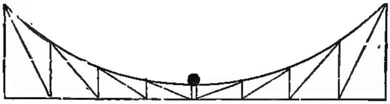


FIG. 879.

(*e*) In Fig. 880 the girder is supported at several points by straight chains running directly to the pier-saddles, and the chains are kept in place by being hung from a curved chain by vertical rods.

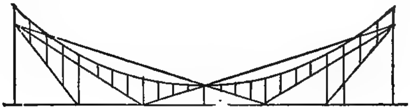


FIG. 880.

(*f*) It has been proposed to employ a stiff inverted arched rib of wrought iron instead of the flexible cable. All straining action may be eliminated by hinging the rib at the centre and piers, and the theory of the stresses developed in this tension rib is precisely similar to that of the arched rib, except that the stresses are reversed in kind.

(*g*) The platform of every suspension bridge should be braced horizontally. The floor-beams are sometimes laid on the skew in order that the two ends of a beam may be suspended from points which do not oscillate concordantly, and also to distribute the load over a greater length of cable.

EXAMPLES.

1. The span of a suspension bridge is 200 ft., the dip of the chains is 80 ft. and the weight of the roadway is 1 ton per foot run. Find the tensions at the middle and ends of each chain. *Ans.* $31\frac{1}{4}$ tons; 58.94 tons.

2. Assuming that a steel rope (or a single wire) will bear a tension of 15 tons per square inch, show that it will safely bear its own weight over a span of about one mile, the dip being one fourteenth of the span.

Ans. Maximum tension = 33,074 lbs.

3. Show that a steel rope of the best quality, with a dip of one seventh of the span, will not break until the span exceeds 7 miles, the ultimate strength of the rope being 60 tons per square inch. (1 ton = 2240 lbs.)

Ans. Maximum tension = 59,545 tons/square inch.

4. If the span $=l$, the total uniform load $=W$, and the dip $=\frac{l}{12}$, show that the maximum tension $=1.58W$, the minimum tension $=1.5W$, the length of the chain $=1.018l$, and find the increase of dip corresponding to an elongation of 1 in. in the chain.

5. A cable weighing p lbs. per lineal foot of length is stretched between supports in the same horizontal line and 20 ft. apart. If the maximum deflection is $\frac{1}{2}$ ft., determine the greatest and least tensions.

Ans. Parameter $m=100$ ft.; maximum tension $=100\frac{1}{2}p$; minimum tension $=100p$.

6. The dead weight of a suspension bridge of 1600 ft. span is $\frac{1}{2}$ ton per lineal foot; the dip $=\frac{\text{span}}{13}$. Find the greatest and least pulls upon one of the chains. The ends of the chains are attached to saddles on rollers on the top of piers 50 ft. high, and the backstays are anchored 50 ft. from the foot of each pier. Find the load upon the piers and the pull upon the anchorage.

Ans. 255 tons; $243\frac{3}{4}$ tons; $637\frac{1}{2}$ tons; 344.6 tons.

7. The platform of a suspension foot-bridge of 100 ft. span is 10 ft. wide and supports a load of 150 lbs. per square foot, including its own weight. The two suspension chains have a dip of 20 ft. Find the force acting on each chain close to the tower and in the middle, assuming the chain to hang in a parabolic curve.

8. The river span of a suspension bridge is 930 ft. and weighs 5976 tons, of which 1439 tons are borne by stays radiating from the summit of each pier, while the remaining weight is distributed between four 15-in. steel-wire cables, producing in each at the piers a tension of 2064 tons. Find the dip of the cables.

The estimated maximum traffic upon the river span is 1311 tons uniformly distributed. Determine the increased stress in the cables.

To what extent might the traffic be safely increased, the limit of elasticity of a cable being 8116 tons, and its breaking stress 12,300 tons?

Ans. 63.884 ft.; 596.4 tons to 13,303 tons uniformly distributed.

9. The uniform load on each of the cables of a suspension span 800 ft. in length is 4000 lbs. per lineal foot of span. The dip of the cables is 60 ft. Find the stress in the cables at the centre and at the towers.

Ans. 5,333,333 lbs.; 5,552,000 lbs.

10. A telegraph wire $\frac{1}{10}$ in. diameter is supported on poles 170 ft. apart and dips 2 ft. in the middle. Find the pull on the wire.

11. A trolley-wire has to be carried on poles round a curve of 1200 ft. radius. The poles are spaced 40 yards apart, and in the middle of each span the wire sags down 6 ins. below the points of support. If the wire weighs $1\frac{1}{2}$ lbs. per yard, show that the resultant horizontal pull on each pole is very nearly 180 lbs.

12. A copper telegraph-wire weighing $\frac{1}{6}$ lb. per lineal yard is suspended between poles on level ground so that the greatest dip of the wire is 2 ft. The tension at the lowest point of the wire is 100 lbs. Find the distance between the poles.

13. A flexible cable weighing $\frac{1}{10}$ lb. per lineal foot is suspended between two poles A and B, 400 ft. apart, and hangs in a catenary having a modulus of 2000 ft. The poles are of the same height as the crown level.

A second cable, of the same weight per lineal foot, is supported between B and a third pole C of the same height, the angle ABC being 150°. If the horizontal pull of this cable is 1.155 times that on the first cable, find its dip and modulus and show that the intermediate pole should be supported by a stay in a plane at right angles to B and C. If the stay slopes at 45°, find the stresses to which it is subjected.

14. A suspension bridge has a dip of 10 ft. and a span of 300 ft. Find the increase of dip due to a change of 100° F. from the mean temperature, the coefficient of expansion being .00125 per 180° F.

Also find the corresponding flange stress in the stiffening truss, which is $12\frac{1}{2}$ ft. deep, the coefficient of elasticity being 8000 tons.

Ans. 1.17 ft.; 6.24 tons.

15. The ends of a cable are attached to saddles free to move horizontally. If Δa is the horizontal movement of each saddle due to the expansion of the cables in the side spans, and if ΔS is the extension of the chain between the two saddles, show that the increment of the dip (h) is approximately

$$\frac{3}{16}\frac{a}{h}\Delta S+\Delta a\left(\frac{3}{8}\frac{a}{h}-\frac{h}{a}\right)$$

16. Show that the total extension of a cable of uniform sectional area A under a uniformly distributed load of intensity w is

$$\frac{wl^3}{8EAd}\left(1+\frac{16}{3}\frac{d^3}{l^3}\right),$$

l being the span and d the dip.

17. A suspension bridge has a dip of 30 ft. and the span is 900 ft. If the coefficient of expansion is .000007 per 1° F. and if 15,000 tons per square inch is the coefficient of elasticity, find (a) the change in dip corresponding to a fluctuation of 50° from the mean temperature, (b) the corresponding flange stress in an auxiliary truss 25 ft. deep.

18. The cables of a suspension bridge for a span of 200 ft. have a dip of 40 ft. Find (a) the length of a cable, and also find (b) the change of dip corresponding to a change of 30° C. from the mean temperature, the coefficient of linear expansion being .000012 per degree centigrade.

The suspension bridge is supplied with an auxiliary truss 4 ft. deep. Show (*c*) that the intensity of flange stress at the centre developed by the change of temperature is 717 lbs. per square inch. ($E=24{,}000{,}000$ lbs.)

If the live load is 4000 lbs. per lineal foot of span, determine (*d*) the maximum S.F. and B.M. to which the truss is subjected.

19. A suspension bridge of 240 ft. span and 20 ft. dip has 48 suspenders on each side; the dead weight = 3000 lbs. per lineal foot; the live load = 2000 lbs. per lineal foot. Find the maximum pull on a suspender, the maximum bending moment and the maximum shear on the stiffening truss. Also find the elongation in the chain due to the live load.

Ans. Max. pull = 12,500 lbs.; max. shear = 30,000 lbs.; max. B.M. = 1,066,666$\frac{2}{3}$ ft.-lbs.; elongation = $89{,}600{,}000 \div EA$, A being sectional area of a cable and E the coefficient of elasticity.

20. Each side of the platform of a suspension bridge for a span of 100 ft. is carried by nine equidistant suspenders. Design a stiffening truss for a live load of 1000 lbs. per lineal foot, and determine the pull upon the suspenders due to the live load when the load produces (1) an *absolute maximum* shear; (2) an *absolute maximum bending moment.*

Ans. Max. shear = 6250 lbs.; max. B.M. = 92,592$\frac{16}{27}$ ft.-lbs.; pull on suspender = (1) 2777$\frac{7}{9}$ lbs., (2) = 1851$\frac{23}{27}$ lbs. or 3703$\frac{19}{27}$ lbs.

21. The platform of a suspension bridge of 300 ft. span is suspended by vertical rods spaced 10 ft. apart and the platform is also provided with an auxiliary truss. Determine the pull on a suspender when one half the bridge carries a live load of half a ton per foot run. Also find the maximum bending moment and maximum shear to which the auxiliary truss is subjected.

22. A foot-path 8 ft. wide is to be carried over a river 100 ft. wide by two cables of uniform sectional area and having a dip of 10 ft. Assuming the load on the platform to be 112 lbs. per square foot, find the greatest pull on the cables, their sectional area, length, and weight. (Safe stress = 8960 lbs. per square inch; specific weight of cable = 480 lbs. per cubic foot.) Find the depression in the cables due to an increment of length under a change of 60° F. from the mean temperature. (Coefficient of expansion = $1 \div 144{,}000$.)

Ans. 56,000 lbs.; 60,312 lbs.; 6.73 sq. ins.; 102$\frac{5}{8}$ ft.; 2302.65 lbs.; .0802 ft.

23. In a suspension bridge (recently blown down) each cable was designed to carry a total load of 84 tons (including its own weight). The distance between the piers = 1270 ft.; the deflection of the cable = 91 ft. Find (*a*) the length of the cable; (*b*) the pull on the cable at the piers and at the lowest point; (*c*) the amounts by which these pulls are changed by a variation of 40° F. from the mean temperature; (*d*) the tension in the backstays, assuming them to be approximately straight and inclined to the vertical at the angle whose tangent is $\frac{2}{5}$.

The platform was hung from the cables by means of 480 suspenders (240 on each side). Find (*e*) the pull on each suspender and (*f*) the total length of the suspenders, the lowest point of a cable being 14 ft. above the platform.

Ans. (*a*) 1287.4 ft.; (*b*) 146$\frac{7}{13}$ and 152.4 tons; (*c*) 1.5 and 1.45 tons; (*d*) 394.55 tons; (*e*) .35 ton; (*f*) 10,565.6 ft.

24. In a suspension bridge of 1600 ft. span the platform is carried by two cables, one on each side. The bridge weighs $\frac{1}{2}$ ton per lineal foot of span and the dip is *one thirteenth* of the span. Find the greatest and least pulls on a cable. If the cables are attached to saddles or rollers on the top of piers 50 ft. high and the backstays are anchored at 50 ft. from the foot of each pier, find the load upon the piers and the pull on the anchorage.

25. Find the length of the cable in the preceding example, and also find the change of dip corresponding to a change of 30° C. from the mean temperature, the coefficient of linear expansion being .000012 per degree centigrade.

If an auxiliary truss 4 ft. deep is added, find the intensity of flange stress at the centre developed by the change of temperature.

26. The cables for a suspension bridge of 210 ft. clear span are suspended from piers which are 25 ft. and 4 ft. respectively above the lowest point of the cables.

The top of the lowest pier is anchored by a backstay inclined at 60° to the vertical, while the higher pier is anchored by a backstay inclined at 45° to the vertical. Determine (*a*) the length of the cable between the piers, (*b*) the horizontal pull on the cable, (*c*) the tensions in the cable at the tops of the piers when the load on each cable is half a ton per lineal foot of span.

27. At Boughton a suspension bridge of 143 ft. span and 12.3 ft. dip failed on account of serious oscillations caused by the marching of a company of soldiers across the bridge. The anchorage was a 2-in. bolt at right angles to the suspension link and having a bearing of $3\frac{1}{2}$ ins. The estimated weight of the soldiers was 4.8 tons (long), and this combined with the dead weight of the bridge produced a force of 37.2 tons on each anchor-bar. Taking 18 tons as the ultimate tenacity of good iron, determine the factor of safety and the dead weight of the bridge. *Ans.* 3; 47.8 tons.

28. The platform of a suspension bridge of 150 ft. span is suspended from the two cables by 88 vertical rods (44 on each side); the dip of the cables is 15 ft.; there are two stiffening trusses; the dead weight is 2240 lbs. per lineal foot, of which *one half* is divided equally between the two piers. Find the stresses at the middle and ends of the cables when a uniformly distributed load of 78,750 lbs. covers one half of the bridge. Also find the maximum shears and bending moments to which the stiffening trusses are subjected when a live load of 1050 lbs. per lineal foot crosses the bridge.

Ans. Pull on suspender $= 2803\frac{48}{44}$ lbs.; $H = \frac{5}{\sqrt{29}} T = 154{,}218\frac{3}{4}$ lbs.

Max. shear on each truss at centre and due to 78,750 lbs. $= 9843\frac{3}{4}$ lbs. $=$ that due to 1050 lbs. per lineal foot.

Max. B.M. due to 78,750 lbs. is at centre of loaded and unloaded halves and $= 184{,}570\frac{5}{16}$ ft.-lbs.

Abs. max. B.M. due to 1050 lbs. per lineal foot is at points of trisection and $= 218{,}750$ ft.-lbs.

29. Solve the preceding example when the trusses are hinged at the centre.

Ans. Pull on suspender $= 2803\frac{43}{44}$ lbs.; $H = \frac{5}{\sqrt{29}}T = 154{,}218\frac{3}{4}$ lbs.

Max. shear due to 78,750 lbs. $= 9843\frac{3}{4}$ lbs. at centre of span and at end of loaded half of bridge; max. shears due to 1050 lbs. per lineal foot $= 13{,}125$, $5906\frac{1}{4}$, $4921\frac{7}{8}$, $8305\frac{85}{128}$, and $9843\frac{3}{4}$ lbs. at ends of the half truss and at the points dividing the half span into four equal segments.

Max. B.M. due to 78,750 lbs. is at centre of half truss and $= 184{,}570\frac{5}{16}$ ft.-lbs. Max. B.M. due to 1050 lbs. per lineal foot $= 176{,}180\frac{265}{352}$, $221{,}484\frac{3}{8}$, and $153{,}808\frac{19}{32}$ ft.-lbs. at points dividing the half truss into four equal segments.

30. Determine the horizontal and pier tensions on a cable suspended from two piers 150 ft. and 70 ft. above the ground level, and carrying a horizontal platform loaded with one ton per lineal foot. The distance between the piers is 200 ft. and the lowest point of the cable is 60 ft. above the ground level. The platform is carried by nineteen suspenders equally spaced. Find the pull upon a suspender when a live load of one ton per lineal foot covers one half of the platform.

31. A suspension bridge 360 ft. long consists of a central span of 160 ft. and two side spans, each of 100 ft. The lowest points of the cable are in the same horizontal plane, 70 ft. above mean water level, 30 ft. below the summit of the abutments. On each side of the platform there are two cables from which the platform is suspended by vertical rods 10 ft. apart. The bridge load is 4000 lbs. per lineal foot of span. The train load is 6000 lbs. per lineal foot of span. When the train load covers the whole of the centre span, find (*a*) the maximum and minimum stresses on the cables; (*b*) the overturning moment on the pier at the mean water level. When the train load covers the bridge between the lowest points of the centre span and a side span, find (*c*) the stress on a suspender in each span. Assume that no distortion of the cables takes place.

Ans. (*a*) *Centre span,* $H = 266{,}666\frac{2}{3}$ and $T = 333{,}333\frac{1}{3}$ lbs.; *side span,* $H = 106{,}666\frac{2}{3}$ and $T = 133{,}333\frac{1}{3}$ lbs.; (*b*) 16,000,000 ft.-lbs.; (*c*) *centre span,* 17,500 lbs.; *side span,* 22,000 lbs.

32. An island divides a river into two channels. The river is crossed by a suspension bridge of two spans, the one of 216 ft., the other of 180 ft. The lowest points of the cable are in the same horizontal plane, 36 ft. below the summit of the pier erected on the island, 9 ft. below the top of the abutment at the end of the 180 ft. span and 4 ft. below the top of the abutment at the other end of the bridge. There are four cables and each cable is to carry a load of 1000 lbs. per lineal foot on the shorter span and of 2000 lbs. per lineal foot on the longer span. Find (*a*) the cable stresses at the pier and abutments, (*b*) the horizontal pull upon each cable, (*c*) the total length of the cable.

The pier is a measured prism of 12 ft. uniform width, 60 ft. high and weighing 125 ft. per cubic foot. Find (*d*) the proper thickness for the pier at the top and at the bottom, taking 10,000 lbs. per square foot as a safe load and assuming that the deviation of the centre of pressure from the middle point of the base is not to exceed 4 ft.

The platform is provided with a stiffening truss, the vertical suspenders being 12 ft. apart. If the dead load for the whole truss is 1000 lbs. per lineal foot of truss and if *one half* of each span is covered with an additional load of 1000 lbs. per lineal foot of truss, find (*e*) the stress in a suspender in each span, (*b*) the maximum B.M. and S.F. on the truss in each span, (*f*) the total length of the suspenders, the suspenders at the centre being 12 ft. in length.

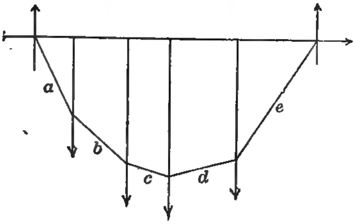
33. A bridge 444 ft. long consists of a central span of 180 ft. and two side spans each of 132 ft.; each side of the platform is suspended by vertical rods from two iron-wire cables; each pair of cables passes over two masonry abutments and two piers, the former being 24 ft. and the latter 39 ft. above the surface of the ground; the lowest point of the cables in each span is 19 ft. above the ground surface; at the abutments the cables are connected with straight wrought-iron chains, by means of which they are attached to anchorages at a horizontal distance of 66 ft. from the foot of each abutment; the dead weight of the bridge is 3500 lbs. per lineal foot, and the bridge is covered with a proof load of 4500 lbs. per lineal foot. The piers are wrought-iron oscillating columns, and if equilibrium under an unequally distributed load is maintained by connecting the heads of the columns with each other and with the abutments by iron-wire stays, determine the proper dimensions of the stays, assuming them approximately straight. Assume that the proof load covers (*a*) a side span; (*b*) two side spans; (*c*) the centre span.

Ans. (*a*) Pull on stays in centre span = 840,050 lbs.
(*b*) " " " " " " = double that in (*a*).
(*c*) " " " " side span = 948,466 lbs.

CHAPTER XII.

ARCHES AND ARCHED RIBS.

1. **Arches.**—Fig. 881 represents a string stretched over a number smooth pegs and carrying certain specified loads, *ab, bc, cd, de.* Tak



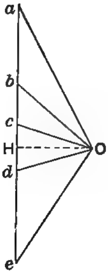


FIG. 881. FIG. 882.

ing *ae* as the line of loads and *O* as the pole (Chapter I) in Fig. 88 the horizontal component of the tension along each element betwee consecutive pegs is constant and equal to *OH*.

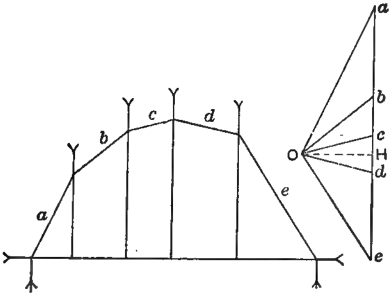


FIG. 883. FIG. 884.

Invert Fig. 881 and assume that each element remains rigi The several forces remain the same in magnitude but are reversed i kind, and each element is now acted upon by a constant horizontal thrust, OH. With a new set of forces the frame will assume a new shape and there will be a new constant horizontal thrust. This is a general property and the rigid portions between consecutive pegs form a *line of resistance*. Again, KL in Fig. 885 is the closing line of the funicular polygon for a number of given vertical loads acting

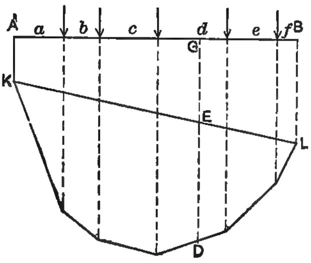


FIG. 885.

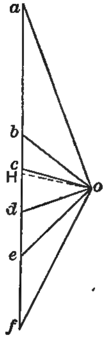


FIG. 886.

upon the beam AB, and by Chapter I the B.M. at any point G is $=OH\times DE$, where OH is parallel to the closing line. Taking OH to be *unity* and inverting the funicular polygon so that the closing line is horizontal, it becomes the actual B.M. diagram for the loads in question. The axis AEB of an arch, Fig. 887, rises to meet the

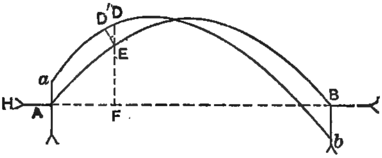


FIG. 887.

B.M. diagram aDb, and the B.M. on the arch, as distinguished from that on the horizontal beam, is $OH\times DE$. The following is a simple general proof of this result:

In Fig. 887 AEB is the axis of the arch, aDb is the B.M. curve, and DEF is a vertical ordinate at any point E of the axis.

Let H, V be respectively the horizontal and vertical components of the reaction at A;

Let M_x be the B.M. at E of the vertical load on the arch betwee A and F.

Then $$\text{B.M. at } E = V \cdot AF - H \cdot EF - M_x.$$

But $$0 = V \cdot AF - H \cdot DF - M_x.$$

Therefore $$\textbf{B.M. at E} = \mathbf{M} = \mathbf{H \cdot DE}.$$

Again, let the normal at E meet the B.M. curve in D' and let be the thrust along the axis at E. Then

$$T \cos DED' = H = T\frac{D'E}{DE}, \text{ approximately,}$$

or $$\mathbf{H \cdot DE = T \cdot D'E = M}.$$

The great difficulty in developing the analysis of the arch is due t the uncertainty as to the true positions of the ends of the B.M. curve If it is possible to introduce hinges at the ends of the arch or at an other point, as, e.g., its centre, the B.M. curve must pass throug such points. From the nature of the construction it is impracticabl to use hinges with masonry arches, although attempts have bee made to provide for a partial rotation by introducing flat iro plates at the skewbacks and at the key. These arches necessaril belong more or less to what may be called an indeterminate clas

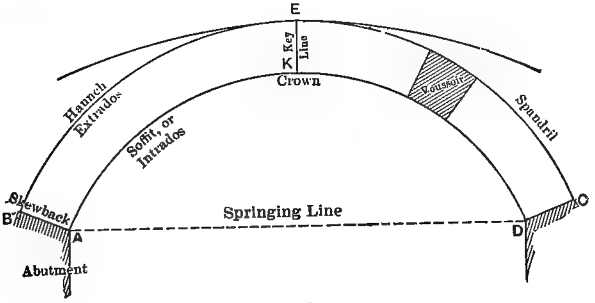


FIG. 888.

The different parts of an arch are indicated by the terms in Fi 888. The rise of the arch is the vertical distance between the sprin ing and the crown.

In the figure, *ABCD* represents the profile of an arch. The under surface *AD* is called the *soffit* or *intrados*. The upper surface *BC* is sometimes improperly called the *extrados*. The highest point *K* of the soffit is the *crown* or *key* of the arch. The *springings* or *skewbacks* are the surfaces *AB*, *DC* from which the arch springs, and the *haunches* are the portions of the arch about half-way between the springings and the crown. Upon each of the arch faces stands a *spandrel* wall, and the space between these two *external* spandrels may be occupied by a series of *internal* spandrels spaced at definite distances apart, or may be filled up to a certain level with masonry (i.e., *backing*) and above that with ordinary ballast or other rough material (i.e., *filling*).

A *masonry* arch consists of courses of wedge-shaped blocks with the bed-joints perpendicular, or nearly so, to the soffit. The blocks are called *voussoirs*, and the *voussoirs* at the crown are the *keystones* of the arch.

A *brick* arch is usually built in a number of rings.

Consider the portion of the arch bounded by the vertical plane *KE* at the key and by the plane *AB*.

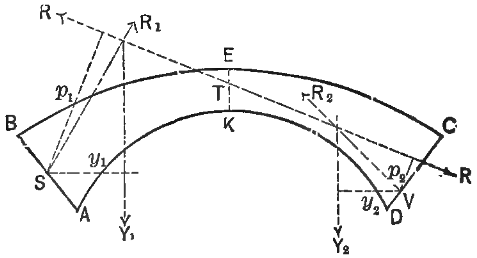


FIG. 889.

It is kept in equilibrium by the reaction R at KE, the reaction R_1 at AB, and the weight Y_1 of the portion under consideration and its superincumbent load.

Let S and T be the points of application of R_1 and R respectively.

Let the directions of R_1 and R intersect in a point. The direction of Y_1 must also pass through the same point.

Taking moments about S,

$$Rp_1 = Y_1 y_1,$$

p_1 and y_1 being respectively the perpendicular distances of the directions of R and Y_1 from S.

Similarly, the portion $KECD$ of the arch gives the equation

$$Rp_2 = Y_2 y_2,$$

Y_2 being the weight to which it is subjected, and p_2, y_2 the perpendicular distances of the directions of R and Y_2 from the point of application V of the reaction at the plane DC.

If the arch and the loading are symmetrical with respect to the plane KE,

$$Y_1 = Y_2, \quad y_1 = y_2, \quad \text{and therefore} \quad p_1 = p_2.$$

Hence the direction of R will be horizontal, which might have been inferred by reason of the symmetry.

The magnitudes of the reactions are indeterminate, as the positions of the points of application (S, T, V) are arbitrary, and can only be fixed by a knowledge of the law of the variation of the stress in the material at the bounding planes AB, KE.

Suppose the arch to be divided into a number of elementary portions ke', $k'e''$ . . . (e.g., the voussoirs of a masonry arch) by a series of joints ke, $k'e'$ . . .

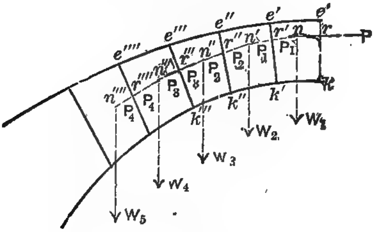


FIG. 890.

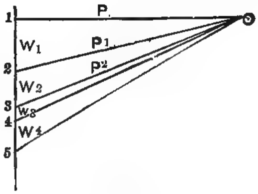


FIG. 891.

Let W_1, W_2, . . . be the loads directly supported by the severa portions. These loads generally consist of the weight of a portion (e.g., ke') + the weight of the superincumbent mass + the load upon the overlying roadway; the lines of action of the loads are, therefore nearly always vertical.

Each elementary portion may be considered as acted upon and kept in equilibrium by *three* forces, viz., the external load and the pressures at the joints. If the pressure and its point of application at any given joint have been determined, the pressures and the corresponding points of application at the other joints may also be found.

For, let 1234 . . . be the line of loads, so that $12 = W_1$, $23 = W_2$, . . .

Assume that the pressure P and its point of application r at any given joint ke are known.

Draw $O1$ to represent P in direction and magnitude.

Then $O2$ evidently represents the resultant of P and W_1 in direction and magnitude, and this resultant must be equal and opposite to the pressure P_1 at the joint $k'e'$.

Hence a line $n'n$ drawn through n, the intersection of P and W_1, parallel to $O2$, is the direction of the pressure P_1, and intersects $k'e'$ in the point of application r' of P_1.

Again $O3$ represents the resultant of P_1 and W_2 in direction and magnitude, and this resultant must be equal and opposite to the pressure P_2 at the joint $k''e''$.

The line $n''n'$ drawn through n', the intersection of P_1 and W_2, parallel to $O3$, is the direction of the pressure P_2 and intersects $k''e''$ in the point of application r'' of P_2.

Proceeding in this manner, a series of points of application or *centres of resistance* r', r'', r''', . . . may be found, the corresponding pressures being represented by $O2$, $O3$, $O4$, . . .

The *polygon of pressures* formed by the lines of action of P, P_1, P_2, . . . is termed an *equilibrated polygon*, and is a funicular polygon of the loads upon the several portions.

The polygon formed by joining the points r, r', r'', . . . successively, is called the *line of resistance.*

In the limit, when the joints are supposed indefinitely near, these polygons become curves, the curve in the case of the equilibrated polygon being known as a *linear arch.*

The two curves may, without sensible error, be supposed identical, and they will exactly coincide if the joints (of course imaginary in such a case) are made parallel to the lines of action of the external loads. This may be easily proved as follows:

Let the figure represent a portion of an arch bounded by the joints (imaginary) *KE*, *MN* parallel to the lines of action of the external loads, whic will be assumed vertical.

Reduce the superincumbent loads to an equivalent mass of arch materia

Let h, e.g., be the depth of material of specific weight w_1, overlying th arch at any given point, and let Q be the load per unit of area of roadway.

Also, let w be the specific weight of the arch material.

Then x, the equivalent depth, is given by

$$wx = w_1h + Q.$$

If the value of x is determined at different points along the arch, a profile *en* may be obtained defining a mass *ENne* of arch material which may be substituted for the superincumbent load. Denote the weight of the mass *MKen* by W.

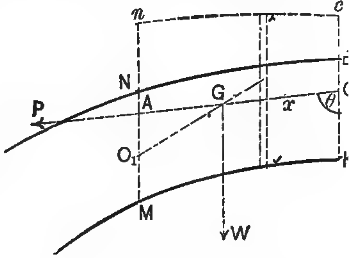


FIG. 892.

Let the pressure P and its point of application O at the joint KE be giver

Take O as the origin, the line OA in the direction of P as the axis of x, an the vertical through O as the axis of y, and let θ be the angle between the tw axes.

Let the lines of action of P and W intersect in G. The line of action c their resultant will intersect MN in the centre of resistance O_1.

Let X, Y be the co-ordinates of O_1.

Let z be the depth of an elementary slice of thickness dx, parallel to Ol at any abscissa x. Its weight $= wzdx \sin\theta$.

Then
$$W \cdot OG = \int_0^X wzdx \sin\theta \cdot x = W(X - AG).$$

But $\dfrac{P}{W} = \dfrac{AG}{AO_1} = \dfrac{AG}{Y}$, since the triangle AGO_1 is evidently a triangle of force for the forces acting upon the mass under consideration.

Also,
$$W = \int_0^X wzdx \cdot \sin\theta.$$

Therefore
$$\int_0^X wzxdx \sin\theta = WX - W\frac{P}{W}Y = X\int_0^X wz \sin\theta dx - PY.$$

This is the equation to the line of resistance.

Taking the differential of this equation,

$$wz'X \sin\theta dX = Xwz' \sin\theta dX + WdX - PdY,$$

z' being the depth corresponding to the abscissa X.

Therefore
$$\frac{dY}{dX} = \frac{W}{P} = \frac{AO_1}{AG} = \frac{\sin AGO_1}{\sin AO_1G} = \tan AGO_1 \text{ if } \theta = 90°.$$

Thus the tangents to the curve of pressures and to the curve of centres of pressure at any given point coincide, and the curves must also coincide.

2. Conditions of Equilibrium.—Let the Fig. 893 represent a portion of an arch of thickness, *unity*, between any two bed-joints (*real* or *imaginary*) MN, PQ.

Let W be its weight together with that of the superincumbent load. Let the direction of the reaction R' at the joint MN intersect MN in m and the direction of W in n. For equilibrium, the reaction R'' at the joint PQ must also pass through n. Let its direction intersect PQ in O. In order that the equilibrium may be stable, three conditions must be fulfilled, viz.:

First. The point O must lie between P and Q, so that there may be no tendency to turn about the edges P and Q.

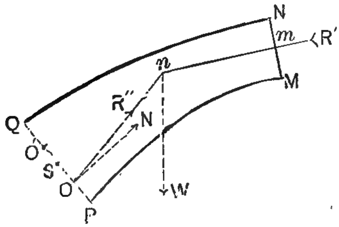


Fig. 893.

Second. There must be no sliding along PQ, and therefore the angle between the direction of R'' and the normal to PQ must not exceed the angle of friction of the material of which the arch is composed.

(*N.B.*—The angle of friction for stone upon stone is about 30°.)

Third. The maximum intensity of stress at any point in PQ must not exceed the safe resistance of the material.

Further, the stress should not change in *character*, in the case of masonry and brick arches, but should be a compression at every point, as these materials are not suited to withstand tensile forces.

The best position for O would be the middle point of PQ, as the pressure would then be uniformly distributed over the area PQ. It is, however, impracticable to insure such a distribution, and it has been sometimes assumed that the stress varies uniformly.

With this assumption, let N be the normal component of R''.

Let f be the maximum compressive stress, i.e., the stress at the most compressed edge, e.g., P.

Let $OS = qPQ$, S being the middle point of PQ, and q a coefficient whose value is to be determined.

Then if $PO < \frac{PQ}{3}$,

$$N = \tfrac{3}{2} fPO = \tfrac{3}{2} fPQ(\tfrac{1}{2} - q);$$

if $PO > \frac{PQ}{3}$,

$$N = \frac{fPQ}{1+6q};$$

and in the limit when $PO = \frac{PQ}{3}$, i.e., when the intensity of stress varies uniformly from f at P to *nil* at Q,

$$q = \tfrac{1}{6} \quad \text{and} \quad N = \frac{fPQ}{2}.$$

(See Art. 12, Chap. V.)

Similarly, if Q is the most compressed edge, the limiting position of O, the *centre of resistance or pressure,* is at a point O' defined by $QO' = \frac{PQ}{3}$.

Hence, as there should be no tendency on the part of the joints to open at either edge, it is inferred that PO or QO' should be $> \frac{PQ}{3}$, i.e., that the point O should lie within the *middle third* of the joint.

Experience, however, shows that the "middle-third" theory cannot be accepted as a solution of the problem of arch stability, and that its chief use is to indicate the proper dimensions of the abutments. Joint cracks are to be found in more than 90 per cent of the arches actually constructed, and cases may be instanced in which the joints have opened so widely that the whole of the thrust is transmitted through the edges. In Telford's masonry arch over the Severn, of 150 ft. span, Baker discovered that there had been a settlement (15 ins.) sufficient to induce a slight *reverse* curvature at the crown of the soffit. Again, the *position* of the centre of pressure at a joint is indeterminate, and it is therefore impossible as well as useless to make any calculations as to the maximum *intensity* of stress due to the pressure at the joint. What seems to happen in practice is, that the straining at the joints generally exceeds the limit of elasticity, and that the pressure is uniformly distributed for a certain distance on each side of the curve of pressures. Thus, the proper dimensions of a stable arch are usually determined by empirical rules which have been deduced as the results of experience. For example, Baker makes the following statement:

Let T be the thrust in tons or pounds per lineal foot of width of arch.

Let f be the safe working stress in tons or pounds per square foot.

An arch will be stable if an ideal arch, with its bounding surfaces at a minimum distance of $\frac{1}{2}\frac{T}{f}$ from the curve of pressures, can be traced so as to lie within the actual arch. An advance would be made towards a more correct theory if it were possible to introduce into the question the elasticity and compressibility of the materials of construction. These elements, however, vary between such wide limits that no reliance can be placed upon the stresses derivable from their values.

Joint of Rupture.—As already shown, the B.M. at any point in the axis of an arch is $H \cdot DE$, H being the horizontal thrust and DE the vertical intercept at the point between the axis and the B.M. curve. For a constant B.M., H diminishes as DE increases, and for a minimum thrust the B.M. curve should be as high as possible, consistent, of course, with stability. It is evident that in a masonry or brick arch the line of resistance should fall *within the thickness* of the arch, or failure may occur by the opening of the joints, as in Fig. 894. To avoid such a result it is a common practice to require that the *line of resistance* shall fall within the *middle third* of the thickness of the arch, with the object also of insuring that the stress at every point shall be a compression. Thus, for a *minimum thrust*

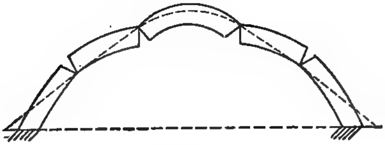


FIG. 894.

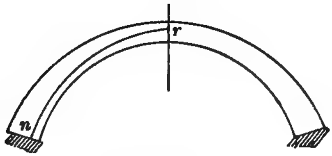


FIG. 895.

e line of resistance should take a position *nr* where the centre of sistance *r* is as high as possible consistent with stability, coinciding, r example, with the upper end of the *middle third* of the depth of e arch. The point *n* should not fall below the lower end of the iddle third of the arch thickness at the skew-back.

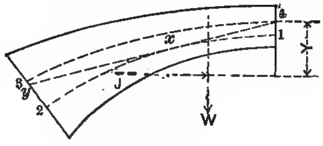


FIG. 896.

In general, let 12, 34 be the bound-g surfaces between which the curve pressures must lie and let 4 be the ntre of resistance at the crown. A ries of curves of pressure may be rawn for the same given load, but ith different values of the horizontal thrust h.

Let $4xy$ be that particular curve which for a value H of the hori-ontal thrust is tangent to the surface 12 at x; the joint at x is called he *joint of rupture.*

The position of this joint in any given arch may be tentatively und as follows:

Let J be any joint in the surface 12.

Let W be the weight upon the arch between J and 1.

Let X be the horizontal distance between J and the centre of ravity of W.

Let Y be the vertical distance between J and 4.

It will also be assumed that the thrust at 4 is horizontal.

If the curve of pressure is now supposed to pass through J, the orresponding value of the horizontal thrust h is given by

$$hY = WX.$$

By means of this equation, values of h may be calculated for a umber of joints in the neighborhood of the haunch, and the greatest f these values will be the horizontal thrust H for the joint x. This evident, as the curve of pressure for a smaller value of h must ecessarily fall *below* $4xy$.

When this happens, the joints will tend to open at the lower dge of the joint 14 and at the upper edges of the joints at x and 3, so that the arch may sink at the crown and spread, unless the butments and the lower portions of the arch are massive enough o counteract this tendency.

If the curve of pressure fall *above* $4xy$ an amount of backing sufficient to transmit the thrust to the abutments must be provided. The same result may be attained by a uniform increase in the thickness of the arch ring, or by a gradual increase from the crown to the abutments.

For example, the upper surface (extrados) of the ring for an arch with a semicircular soffit AKB, having its centre at O, may be delineated in the following manner:

Let x define the joint of rupture in the soffit.

The angle AOx is approximately 30° for a semicircular soffit and 45° for an elliptical soffit. In the present case take $AOx=30°$, and in Ox produced take $xx'=2\times KD$, KD being the thickness at the crown.

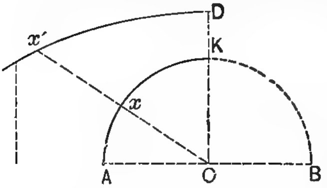


FIG 897.

The arc Dx' of a circle struck from a centre in DO produced may be taken as a part of the upper boundary of the ring, and the remainder may be completed by the tangent at x' to the arc Dx'.

Minimum Thickness of Abutment.—Let T be the resultant thrust at the horizontal joint BC of a rectangular abutment $ABCD$.

Let y be the distance of its point of application from B.

Let H and V be the horizontal and vertical components of T.

Let w be the specific weight of the material in the abutment.

Let h be the height AB of the abutment.

Let t be the width AD of the abutment.

In order that there may be no tendency to turn about the toe D, the moment of the weight of the abutment with respect to D plus the moment of V with respect to D must be greater than the moment of H with respect to D. Or,

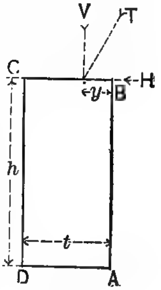


FIG. 898.

$$wht\frac{t}{2}+V(t-y)>Hh,$$

or

$$t+\frac{V}{wh}>\sqrt{\frac{2H}{w}+\frac{2V}{wh}y+\frac{V^2}{w^2h^2}}.$$

This relation must hold good whatever the height of the abutment may be; and if h is made equal to ∞,

$$t>\sqrt{\frac{2H}{w}},$$

which defines a minimum limit for the thickness of the abutment.

3. Empirical Formulæ.—In practice the thickness t at the crown is often found in terms of s, the span, or in terms of ρ, the radius of curvature at the crown, from the formulæ

$$t=\sqrt{s}, \quad \text{or} \quad t=\sqrt{c\rho},$$

t, s, and ρ being all in feet, and c being a constant.

According to Dupuit, $t=.36\sqrt{s}$ for a full arch;

$$t=.27\sqrt{s} \text{ for a segmental arch.}$$

According to Rankine, $=\sqrt{.12\rho}$ for a single arch;

$$t=\sqrt{.17\rho} \text{ for an arch of a series.}$$

Ex. 1. *A masonry arch of* 90 *ft. span and* 30 *ft. rise, with a parabolic intrados and a horizontal extrados, springs from abutments with vertical faces and* 10 *ft. thick, the outside faces being carried up to meet the extrados. The depth of the keystone is* 3 *ft. The centre of resistance at the springing is the middle of the joint, and at the crown* 12 *ins. below the extrados. The specific weight of the masonry may be taken at* 150 *lbs. per cubic foot. Determine* (*a*) *the resultant pressure in the vertical joint at the crown;* (*b*) *the resultant pressure in the horizontal joint at the springing;* (*c*) *the maximum stress in the vertical joint aligning with the inside of an abutment.*

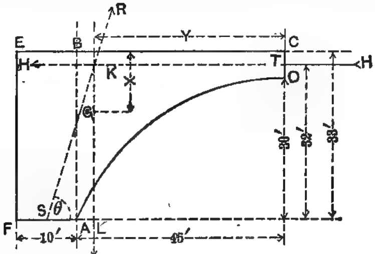


FIG. 899.

Let X, Y be the vertical and horizontal distances, respectively, from the point C, of the C. of G. of portion of the structure bounded by the surfaces CO, CE, EF, FA, and AO. Then

$$Y(45\times3+33\times10+\tfrac{1}{3}\times30\times45)=Y\times915$$

$$=45\times3\times22\tfrac{1}{2}+33\times10\times50+\tfrac{1}{3}\times30\times45\times33\tfrac{3}{4}$$

and

$$X(4\times53+33\times10+\tfrac{1}{3}\times30\times45)=X\times915$$

$$=45\times3\times1\tfrac{1}{2}+33\times10\times16\tfrac{1}{2}+\tfrac{1}{3}\times30\times45\times12.$$

Therefore

$$Y=37'\cdot95 \quad \text{and} \quad X=12'\cdot08.$$

The weight W of the portion under consideration $=915\times 150$
$=137{,}250$ lbs.

The resultant force R at the centre of resistance S, the middle point of the horizontal plane AF, passes through the point of intersection K of the line of action of W and of the horizontal thrust H at the centre of resistance T, CO being the key line and $CT=\frac{1}{3}CO=1$ ft. The triangle KLS is therefore a triangle of forces and if θ is the angle KSL,

$$\cot\theta=\frac{SL}{KL}=\frac{50-37.95}{32}=.3766.$$

Therefore $$\theta^\circ=69^\circ\cdot 22 \quad\text{and}\quad \operatorname{cosec}\theta=1.069.$$

Hence $$H=W\frac{SL}{KL}=137{,}250\cot\theta=51{,}689\text{ lbs.}$$

and $$T=W\frac{SK}{KL}=137{,}250\ \operatorname{cosec}\theta=146{,}721\text{ lbs.}$$

Again let $x'y,'$ be the vertical and horizontal distances of G', the C. of G. of the portion between AB and CO, from C. Then

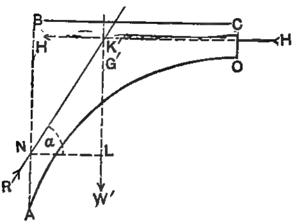


FIG. 900.

$$y'(45\times 3+\tfrac{1}{3}\times 30\times 45)=y'\times 585=45\times 3\times 22\tfrac{1}{2}$$
$$+\tfrac{1}{3}\times 30\times 45\times 33\tfrac{3}{4}=18{,}225$$

and $$x'\times 585=45\times 3\times 1\tfrac{1}{2}+\tfrac{1}{3}\times 30\times 45\times 12$$
$$=5602\tfrac{1}{2}.$$

Therefore $y'=31'\frac{2}{13}$ and $x'=9'\frac{15}{26}$.

Also w', the weight of $ABCO$
$$=585\times 150=87{,}750\text{ lbs.}$$

The resultant of H and W' acts along $K'N$ and must be equal and opposite to the resultant pressure R' upon the face AB, so that N is the centre of resistances in this plane. Thus $K'NL'$ is a triangle of forces and if α is the angle $K'NL'$,

$$\cot\alpha=\frac{H}{W'}=\frac{51689}{146721}=.3523 \quad\text{and}\quad \alpha=70^\circ\,35'.$$

Hence $$R'=W'\operatorname{cosec}\alpha=93{,}042\text{ lbs.}$$

If the intensity of the pressure varies from a maximum f at A to *nil* at B, then

$$f=\frac{2}{AB}H=\frac{2}{33}\times 51{,}689=3133\text{ lbs./sq. ft.}$$

4. **Moseley's Principle.**—If the forces, which act upon or within a body or structure, are in eqiliubrium they may be classified as *active* and *passive* forces, standing to each other in the relation of cause and effect. The passive forces are then the *least* which are capable of balancing the active forces consistently with the physical condition of the body or structure.

For the passive forces are due to the application of the active forces to the structure and do not increase after they have balanced the active forces. They will, consequently, not increase beyond the smallest amount capable of balancing the active forces. It may, therefore, be concluded that as the force which one member of a structure exerts on another is a minimum with any given specified loading, then the horizontal component of the thrust in the arch ring must be a minimum with that loading, and hence the line of resistance will be that which, consistent with stability, gives the horizontal component a minimum value.

To draw a force polygon through any three points *K*, *L*, and *M*.

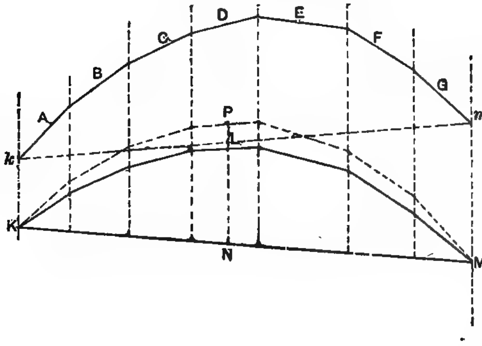


Fig. 901.

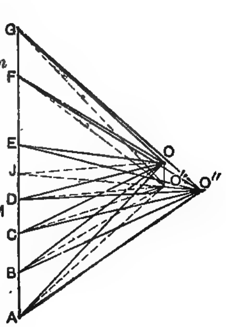


Fig. 902.

First. Draw *any* force polygon with a closing-line *km*, *O* being the pole and *OJ* parallel to *km*.

Second. Draw *JO'* parallel to *KM* and with pole *O'* draw a new polygon which will pass through a point *P*. It will not in general pass through *L*, but will be above or below this point. If above, increase the polar distance in the proportion of $\frac{PN}{LN}$, and take a new pole O'' at the proper horizontal distance from the line of loads, along the line JO'.

Of course it is only necessary to draw the first trial polygon.

5. Fuller's Method.—The following is the Fuller method of drawing the line of least resistance for an unsymmetrically loaded arch:

Let W_1, W_2, . . . W_9 be the loads on the several voussoirs. Draw the load-line and also the corresponding funicular polygon *Jklmno*.

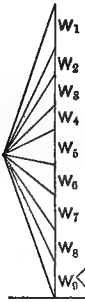


FIG. 904.

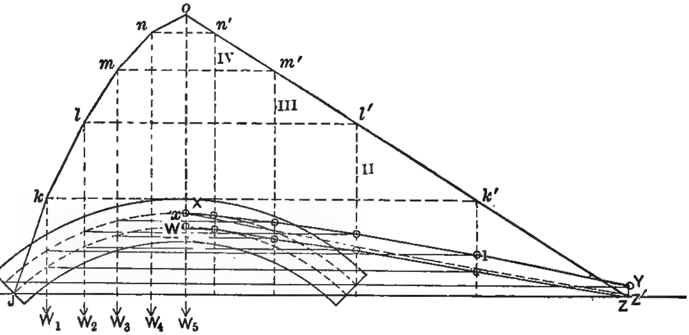


FIG. 903.

From the point o draw any line oZ intersecting the base-line in Z. Let horizontal lines through k, l, m . . . intersect the line oZ, and from the points of intersection drop the verticals I, II, III . . .

From the points of intersection of the lines of action of W_1, W_2, W_3 . . . with the upper and lower boundaries of the middle third, draw horizontals intersecting the corresponding verticals I, II, III, etc.

Join the points so obtained and complete the irregular figure $WXYZ$. Within the area $WXYZ$ select a straight line xZ' containing the smallest possible angle at x—and contained wholly within the boundaries of the area $WXYZ$.

The straight line xZ' bears the same relations to the polygon required to be drawn within the central third of the arch ring as the straight line oZ does to the polygon *Jklmno*.

From the points of intersection of the line $Z'x$ with the verticals I, II, etc., draw horizontal lines intersecting the lines of action of W_1, W_2, W_3 . . . By joining the points thus obtained, the required polygon lying within the central third of the arch ring is drawn.

A second force diagram may now be drawn from the polygon last obtained, giving a polar distance H equal to the minimum horizontal thrust of the arch required to meet all the conditions.

If it is not found possible to draw any straight line within the boundaries of the area $WXYZ$ intersecting the middle third line at x, the depth of the arch ring must be increased until this condition is satisfied.

The above method is of course equally applicable to an arch with a symmetrical load, and it avoids the necessity of drawing repeated trial lines of least resistance, which is very tedious.

6. Examples of Linear Arches or Curves of Resistance.—*Linear Arch in the Form of a Catenary.* If the cable in Art. 4, Chap. XI, Case A, is inverted and stiffened so as to resist distortion, a linear arch is obtained suitable for a real arch which has to support a load distributed in such a manner that the weight upon any portion AP is proportional to the length of AP, and is in fact $= ps$, the area $OAPN$ being ms.

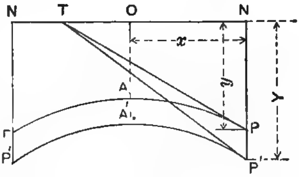


FIG. 905.

Thus, a lamina of thickness unity and specific weight w, bounded by the curve AP, the directrix ON, and the verticals AO, PN, weighs wms, and may be taken to represent the load upon the arch if $wms = ps$, i.e., if $wm = p$, i.e., if the weight of m units of the lamina is w.

The horizontal thrust at the crown $= H = wm = w\rho$, the radius of curvature (ρ) at the crown being equal to m.

A disadvantage attached to a linear arch in the form of a catenary lies in the fact that only *one* catenary can pass through *two* given points, while, in practice, it is often necessary that an arch shall pass through *three* points in order to meet the requirements of a given rise and span. This difficulty may be obviated by the use of the *transformed catenary*.

Upon the lamina $PAPNN$ as base erect a solid, with its horizontal sections all the same, and, for simplicity, with its generating line perpendicular to the base.

Cut this solid by a plane through NN inclined at any required angle to the base. The intersection of the plane and solid will define a *transformed catenary* $P'A'P'$, or a new linear arch, and the shape of a new lamina $P'A'P'NN$, under which the arch will be balanced. This is evident, as the new arch and lamina are merely parallel projections of the original.

The projections of horizontal lines will remain the same in length.

The projection of the vertical lines will be c times the length of the lines from which they are projected, c being the secant of the angle θ made by the cutting-plane with the base. Thus

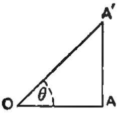


FIG. 906.

$$c = \sec\theta = \frac{OA'}{OA} = \frac{NP'}{NP}.$$

Let x, Y be the co-ordinates of any point P' of the *transformed catenary;*

" x, y be the co-ordinates of the corresponding point P in the catenary proper;

" $A'O = M(>m)$.

Then

$$\frac{Y}{y} = \frac{P'N}{PN} = c = \frac{A'O}{AO} = \frac{M}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equation to the catenary proper is

$$y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Therefore

$$Y = \frac{M}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is the equation to the transformed catenary.

With this form of linear arch the depths M over the crown and Y over the springings, for a span $2x$, may be assumed, and the corresponding value of m determined from eq. (3), which may be more conveniently written in the form

$$\frac{x}{m} = \log_e\left\{\frac{Y}{M} \pm \sqrt{\frac{Y^2}{M^2} - 1}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The slope i' at P' is given by

$$\tan i' = \frac{dY}{dx} = \frac{M}{2m}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right) = \frac{Ms}{m},$$

s being the length AP of the catenary proper, corresponding to the length $A'P'$ of the transformed catenary.

$$\text{The area } OA'P'N=\int_0^x Y\,dx=\frac{Mm}{2}\left(e^{\frac{x}{m}}-e^{-\frac{x}{m}}\right)=\mathbf{Ms}.$$

The tangents at P and P' necessarily intersect in the point T, and $P'TN$, PTN are triangles of forces for the portion AP' and AP, respectively.

Let H', H be the horizontal thrusts at A' and A, respectively.

P', P " " weights upon $A'P$ and AP, respectively.
R' " " tangential thrust at P'.

Then
$$\frac{P'}{P}=\frac{\text{area } OA'P'N}{\text{area } OAPN}=\frac{Ms}{ms}=\frac{M}{m},$$

and therefore
$$\mathbf{P'}=\frac{M}{m}P=\frac{M}{m}wms=\mathbf{wMs}.$$

Also,
$$\mathbf{H'}=P'\cot i'=wMs\frac{m^2}{Ms}=\mathbf{wm^2}=\mathbf{H},$$

and
$$\mathbf{R'}=H'\sec i'=wm^2\sqrt{1+\frac{M^2s^2}{m^4}}=\mathbf{w\sqrt{m^4+M^2s^2}}.$$

The radius of curvature ρ' at the crown $=\dfrac{m^2}{M}$. Therefore

$$H'=wM\rho'=H=w\rho,$$

and the radius of the "catenary proper" is M times the radius of the transformed catenary.

The term "equilibrated arch" has generally been applied to a linear arch with a horizontal extrados.

Ex. 2. *Determine the transformed catenary for an arch of* 30 *ft. span and* 22½ *ft. rise, the height of masonry over the crown being* 13½ *ft.; weight of the masonry* = 125 *lbs. per cubic foot. Also find the thrust at the springing and the curvature at the crown and the springing.*

$$Y=22\tfrac{1}{2}+13\tfrac{1}{2}=36'; \quad M=13\tfrac{1}{2}'; \quad \frac{15}{m}=\log_e\left(\frac{8}{3}+\sqrt{\left(\frac{8}{3}\right)^2-1}\right)=1.6369;$$

therefore $m=9.1637$, $e^{\frac{15}{m}}=5.1387$ and $e^{-\frac{15}{m}}=.1946.$

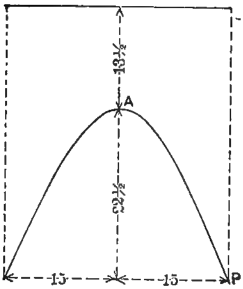


FIG. 907.

Hence

$$s=AP=\frac{9.1637}{2}(5.0387-.1946)=22'.653.$$

Also

$$\tan i'=\frac{Ms}{m^2}=\frac{13\frac{1}{2}\times 22.653}{(9.1637)^2}=3.6418 \quad\text{and}\quad i'=74°.39';$$

$$H'=wm^2=125(69.1637)^2=10{,}497 \text{ lbs};$$

$$R'=H'\sec i'=39{,}653 \text{ lbs}.$$

At the crown $\rho'=\dfrac{m^2}{M}=6.22$ ft.

Again, $Y=6\frac{3}{4}\left(e^{\frac{x}{m}}+e^{-\frac{x}{m}}\right).$

Therefore $\dfrac{dY}{dx}=\dfrac{6\frac{3}{4}}{m}\left(e^{\frac{x}{m}}-e^{-\frac{x}{m}}\right)$ and $\dfrac{d^2Y}{dx^2}=\dfrac{6\frac{3}{4}}{m^2}\left(e^{\frac{x}{m}}+e^{-\frac{x}{m}}\right).$

Hence *at the springing*

$$\frac{dY}{dx}=\frac{6\frac{3}{4}\times 4.9441}{9.1637}=3.6418, \quad \left(\frac{dY}{dx}\right)^2=13.263,$$

and

$$\frac{d^2Y}{dx^2}=\frac{6\frac{3}{4}\times 5.3333}{(9.1637)^2}=.4287.$$

Therefore *at the springing,*

$$\rho'=\frac{(1+13.263)^{\frac{3}{2}}}{.4287}=125.65 \text{ ft}.$$

(*b*) *Linear Arch in the Form of a Parabola.*—Suppose that the cable in Art. 4, Chap. XI, Case C, is exactly inverted, and that it is stiffened in such a manner as to resist distortion. Suppose also that the load still remains a uniformly distributed weight of intensity w per horizontal unit of length. A *thrust* will now be developed at every point of the *inverted* cable equal to the tension at the corresponding point of the original cable. Thus the inverted parabola is a linear arch suitable for a real arch which has to support a load of intensity w per horizontal unit of length.

The horizontal thrust at the crown $=H=w\rho$,

ρ being the radius of curvature at the crown.

(*c*) *Circular and Elliptic Linear Arches.*—A linear arch which has to support an external normal pressure of uniform intensity should be circular.

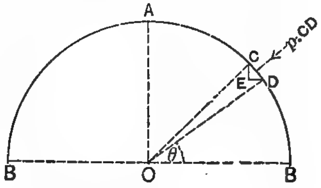


Fig. 908.

Consider an indefinitely small element CD, which may be assumed to be approximately straight.

Let the direction of the resultant pressure upon CD, viz., $p \cdot CD$, make an angle θ with OB.

Let CE, DE be the vertical and horizontal projections of CD.

The angle $DCE = \theta$.

The horizontal component of $p \cdot CD = p \cdot CD \cos\theta = p \cdot CE$.

This is distributed over the vertical projection CE, and the horizontal intensity of pressure $= p \cdot CE \div CE = p$.

Similarly, it may be shown that the vertical intensity of pressure $= p$.

Thus, at any point of the arch,

the horizontal intensity of pressure = vertical intensity = normal intensity $= p$.

Again, the total horizontal pressure on one half of the arch

$$= \Sigma(p \cdot CE) = p\Sigma(CE) = pr = H;$$

and the total vertical pressure on one half of the arch

$$= \Sigma(p \cdot DE) = p\Sigma(DE) = pr = P.$$

Hence at any point of the arch the tangential thrust = **pr.**

Next, upon the semicircle as base erect a semi-cylinder. Cut the latter by an inclined plane drawn through a line in the plane of the base parallel to OA. The intersection of the cutting-plane and the semi-cylinder is the semi-ellipse $B'AB'$, in which the vertical lines are unchanged in length, while the lengths of the horizontal lines are c times the lengths of the corresponding lines in the semicircle, c being the secant of the angle made by cutting-plane with the base. A semi-elliptic arch is thus obtained, and the forces to which it is subjected are parallel projections of the forces acting upon the semicircular arch.

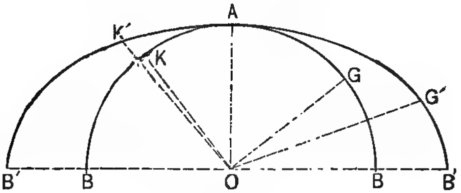


Fig. 909.

These new forces are in equilibrium.

Let P' = the total vertical pressure upon one half of the arch;

H' = the total horizontal pressure upon one half of the arch;

p'_y = vertical intensity of pressure $=\dfrac{P'}{OB'}$;

p'_x = horizontal intensity of pressure $=\dfrac{H'}{OA}$.

Then
$$P'=P=H=pr; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$p'_y=\frac{P'}{OB'}=\frac{P}{cOB}=\frac{pr}{cr}=\frac{p}{c}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$H'=cH=cP=cP'; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$p'_x=\frac{H'}{OA'}=\frac{cH}{OA}=\frac{cpr}{r}=cp. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence, by eq. (3),
$$\frac{H'}{P'}=c=\frac{OB'}{OA};$$

or *the total horizontal and vertical thrusts are in the ratio of the axes to which they are respectively parallel, and, by eqs.* (2) *and* (4),
$$\frac{p'_y}{p'_x}=\frac{1}{c^2}=\frac{OA'^2}{OB'^2};$$

or *the vertical and horizontal intensities of pressure are in the ratio of the squares of the axes to which they are respectively parallel.*

Any two rectangular axes OG, OK in the circle will project into a pair of conjugate radii OG', OK' in the ellipse.

Let $OG'=r_1$, $OK'=r_2$;

Q = total thrust along elliptic arch at K;

R = " " " " " " G.

Then
$$\frac{H}{Q}=\frac{r}{r_1}, \quad \frac{H}{R}=\frac{r}{r_2}, \quad \text{and} \quad \frac{Q}{R}=\frac{r_1}{r_2};$$

or *the total thrusts along an elliptic arch at the extremities of a pair of conjugate radii are in the ratio of the radii to which they are respectively parallel.*

The preceding results show that an elliptic linear arch is suitable for a load distributed in such a manner that the vertical and horizontal intensities, eqs. (2) and (4), at any point of the arch are unequal, but are uniform in direction and magnitude.

Again, it can be easily shown that the projected forces acting upon the elliptic arch are in equilibrium.

The equations of equilibrium for the forces acting upon the circular arch may be written
$$d\left(T\frac{dx}{ds}\right)+Xds=0,$$

$$d\left(T\frac{dy}{ds}\right)+Yds=0,$$

T being the thrust along the arch at the point xy, and X, Y the forces acting upon the arch parallel to the axes of x and y respectively.

If T', X', Y' be the corresponding projected forces,

$$\frac{T'}{ds'}=\frac{T}{ds}, \quad Xds=cX'ds', \quad Yds=Y'ds'.$$

Hence the above equations may be written

$$d\left(\frac{T'}{ds'}cdx'\right)+cX'ds'=0,$$

and

$$d\left(\frac{T'}{ds'}dy'\right)+Y'ds'=0;$$

or

$$d\left(T'\frac{dx'}{ds'}\right)+X'ds'=0,$$

and

$$d\left(T'\frac{dy'}{ds'}\right)+Y'ds'=0.$$

Hence the forces T', X', and Y' are also in equilibrium

(*d*) *Hydrostatic Arch.*—Let the figure represent a portion of a linear arch suited to support a load which will induce on it a normal pressure at every point. The pressure being normal has no tangential component, and the thrust (T) along the arch must therefore be everywhere the same.

Consider any indefinitely small element CD.

It is kept in equilibrium by the equal thrusts (T) at the extremities C and D, and by the pressure $p\cdot CD$. The intensity of pressure p being assumed uniform for the element CD, the line of action of the pressure $p\cdot CD$ bisects CD at right angles.

Let the normals at C and D meet in O_1, the centre of curvature.

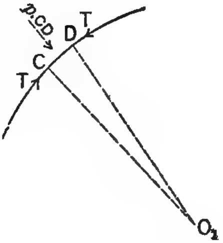


FIG. 910.

Take $O_1C=O_1D=\rho$, and the angle $CO_1D=2\Delta\theta$.

Resolving along the bisector of the angle CO_1D,

$$2T\sin\Delta\theta=p\cdot CD=p\rho\cdot 2\Delta\theta,$$

or

$$2T\Delta\theta=p\rho\cdot 2\Delta\theta;$$

and hence

$$\mathbf{T=p\rho=}\textbf{a constant}. \qquad (1)$$

Thus, a series of curves may be obtained in which ρ varies *inversely* as p, and the hydrostatic arch is that curve for which the *pressure p at any point is directly proportional to the depth of the point below a given horizontal plane.*

Denote the depth by y, and let w be the specific weight of the substance to which the pressure p is due. Then

$$p = wy, \qquad (2)$$

and

$$T = p\rho = wy\rho = \text{a constant}. \qquad (3)$$

The curve may be delineated by means of the equation

$$y\rho = \text{const}. \qquad (4)$$

It may be shown, precisely as in case (c), that the horizontal intensity of pressure (p_x)

$$= \text{the vertical intensity } (p_y) = p. \qquad (5)$$

Take as the origin of coordinates the point O, Fig. 911, vertically above the crown of the arch in the given horizontal plane.

Let the horizontal line through O be the axis of x;
" vertical " " "" " " " y.

Any portion AM of the arch is kept in equilibrium by the equal thrusts (T) at A and M, and by the resultant load P upon AM, which must necessarily act in a direction bisecting the angle ANM.

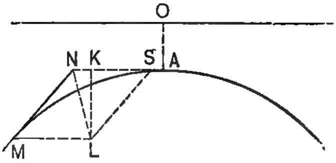


FIG. 911.

Complete the parallelogram AM, and take $SN = NM$ to represent T.

The diagonal NL will therefore represent P.

Let θ be the inclination of the tangent at M to the horizontal.

The vertical load upon AM = vertical component of P

$$= LK = T\sin\theta = p\rho\sin\theta = wy\rho\sin\theta = wy_0\rho_0\sin\theta, \qquad (6)$$

y_0, ρ_0 being the values of y, ρ, respectively, at A.

The horizontal load upon AM = horizontal component of P

$$= NK = SN - KS = T - T\cos\theta = 2T\left(\sin\frac{\theta}{2}\right)^2$$

$$= 2p\rho\left(\sin\frac{\theta}{2}\right)^2 = 2wy\rho\left(\sin\frac{\theta}{2}\right)^2 = 2wy_0\rho_0\left(\sin\frac{\theta}{2}\right)^2. \qquad (7)$$

Again, the vertical load upon AM

$$= \int_0^x p\,dx = w\int_0^x y\,dx = wy_0\rho_0\sin\theta; \qquad (8)$$

the horizontal load upon AM

$$= \int_{y_0}^{y} p\,dy = w\int_{y_0}^{y} y\,dy = \frac{w}{2}(y^2 - y_0^2) = 2wy_0\rho_0\left(\sin\frac{\theta}{2}\right)^2. \qquad (9)$$

Eq. (8) also shows that the area bounded by the curve *AM*, the verticals through *M* and *A*, and the horizontal through *O* is equal to $y_0\rho_0 \sin\theta$, and is therefore proportional to $\sin\theta$. At the points defined by $\theta=90°$ the tangents to the arch are vertical, and the portion of the arch between these tangents is alone available for supporting a load. The vertical and horizontal loads upon one half the arch are each equal to $wy_0\rho_0$.

The relation given in eq. (1) holds true in any arch for elements upon which the pressure is wholly normal.

This has been already proved for the parabola and catenary, in cases (*a*) and (*b*).

At the point *A′* of the elliptic arch,

$$\rho=\frac{OB'^2}{OA'}=\frac{c^2r^2}{r}=c^2r.$$

Hence, the horizontal thrust at *A′*

$$=p_y\rho=\frac{p}{c}\rho=pcr=cH.$$

(*e*) *Geostatic Arch.*—The *geostatic arch,* Fig. 912, is a parallel projection of the *hydrostatic* arch.

The vertical forces and the lengths of vertical lines are unchanged.

The horizontal forces and lengths of horizontal lines are changed in a given ratio *c* to 1.

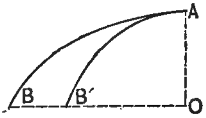


FIG. 912.

Let *B′A* be the half-geostatic curve derived from the half-hydrostatic curve *BA*.

The vertical load on *AB′*

$$=P'=P=\text{thrust along arch at } B'. \quad . \quad . \quad (1)$$

The horizontal load on *AB′*

$$=H'=cH=\text{thrust along arch at } A. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The new vertical intensity

$$=p_y'=\frac{P'}{OB'}=\frac{P}{cOB}=\frac{p_y}{c}=\frac{p}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The new horizontal intensity

$$=p_x'=\frac{H'}{OA}=\frac{cH}{OA}=cp_x=cp. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus, the geostatic arch is suited to support a load so distributed as to produce at any point a pair of conjugate pressures; pressures, in fact, similar to those developed according to the theory of earth-work.

Let R_1, R_2 be the radii of curvature of the geostatic arch at the points A, B', respectively, and let r_1, r_2 be the radii of curvature at the corresponding points A, B of the hydrostatic arch.

The load is wholly normal at A and B'. Thus,

$$H' = p_y' R_1 = \frac{p}{c} R_1 = cH = cpr_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$R_1 = c^2 r_1 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Also,

$$P' = p_x' R_2 = cpR_2 = P = pr_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$cR_2 = r_2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

(*f*) *General Case.*—Let Fig. 913 represent *any* linear arch suited to support a load which is symmetrically distributed with respect to the crown A, and which produces at every point of the arch a pair of conjugate pressures, the one horizontal and the other vertical.

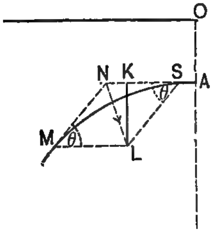


FIG. 913.

Take as the axis of y the vertical through the crown, and as the axis of x the horizontal through an origin O at a given distance from A.

Any portion AM of the arch is kept in equilibrium by the horizontal thrust H at A, the tangential thrust T at M, and the resultant load upon AM, which must necessarily act through the point of intersection N of the lines of action of H and T.

Since the load at A is wholly vertical, H at A is given by

$$H_0 = p_0 \rho_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

p_0 and ρ_0 being, respectively, the vertical intensity of pressure and the radius of curvature at A.

Let $MN = T$, and take $NS = H_0$.

Complete the parallelogram SM; the diagonal NL is the resultant load upon AM in direction and magnitude.

The vertical (KL) and the horizontal (KN) projections of NL are, therefore, respectively, the *vertical* and *horizontal* loads upon AM.

Denote the vertical load by V, the horizontal by H. Then

$$T \sin \theta = KL = V = \int_0^x p_y dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{and} \quad H = KN = SN - SK = H_0 - V \cot \theta, \quad . \quad . \quad . \quad (3)$$

being the angle between MN and the horizon.

$$p_y, \text{ the vertical intensity of pressure} = \frac{dV}{dx}. \quad . \quad . \quad . \quad (4)$$

$_x$, the horizontal intensity of pressure

$$= \frac{dH}{dy} = -\frac{d}{dy}(V \cot \theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Ex. 3. *A semicircular arch of radius r, with a horizontal extrados at a vertical stance R from the centre.*

The angle between the radius to M and the vertical $= \theta$. Then,

$$x = r \sin \theta, \quad y = R - r \cos \theta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$dx = r \cos \theta d\theta \quad dy = r \sin \theta d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$p_y = wy = w(R - r \cos \theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

being the specific weight of the load. Hence

$$V = w \int_0^\theta (R - r \cos \theta) r \cos \theta d\theta$$

$$= wr\left(R \sin \theta - \frac{r\theta}{2} - \frac{r \sin 2\theta}{4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Eqs. (3) and (4) give H; for

$$p_0 = w(R - r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

d hence

$$H_0 = wr(R - r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

, the horizontal intensity of pressure,

$$= -\frac{d}{dy}(V \cot \theta) = w\left(R - \frac{r}{2}\frac{\theta - \sin \theta \cos \theta}{\sin \theta} - r \cos \theta\right). \quad . \quad . \quad . \quad (7)$$

Rankine gives the following method of determining whether inear arch may be adopted as the intrados of a real arch. At the own *a* of a linear arch *ab* measure on the normal a length *ac*, so that *c* may fall within the limits required for stability (e.g., within the middle third).

At *c* two equal and opposite forces, of the same magnitude as the horizontal thrust *H* at *a*, and acting at right angles to *ac*, may be introduced without altering the equilibrium.

Thus the thrust at *a* is replaced by an equal thrust at *c*, and a right-handed couple of moment $H \times ac$.

Similarly, the tangential thrust *T* at any point *d* of *ab* may be replaced by an equal and parallel thrust at *e*, and a couple of moment $T \times de$.

The arch will be stable if the length of *de*, which is normal to *ab* at *d*, is fixed by the condition $T \times de = H \times ac$, and if the line which is the locus of *e* falls within a certain area (e.g., within the middle third of the arch ring.)

7. Arched Ribs in Iron, Steel, or Timber.—The term arched rib is applied to arches constructed of iron, steel, or timber. The coefficients of elasticity are quantities which are found to lie between certain not very wide limits, and their values may be introduced into the calculations with the result of giving to them greater accuracy. There are other considerations, however, involved in the problem of the stability of arched ribs which still render its solution more or less indeterminate.

The total stress at any point is made up of a number of subsidiary stresses, of which the most important are: (1) a direct compressive stress; (2) a stress due to flexure; (3) a stress due to a change of temperature. Each of these may be investigated separately, and the results superposed.

8. Rib with Hinged Ends; Invariability of Span.—Let *ABC* be the axis of a rib supported at the ends on pins or on cylindrical bearings. The resultant thrusts at *A* and *C* must necessarily pass

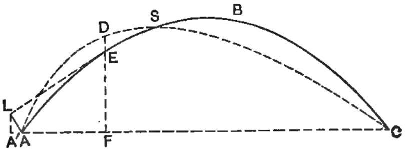


FIG. 914.

through the centres of rotation. The vertical components of the thrusts are equal to the corresponding reactions at the ends of a girder of the same span and similarly loaded, and H is to be determined as in Art. 1 when DE has been found.

Let ADC be the linear arch for any arbitrary distribution of the load, and let it intersect the axis of the rib at S. The curvature of the more heavily loaded portion AES will be flattened, while that of the remainder will be sharpened.

The bending moment at any point E of the axis tends to change the inclination of the rib at that point.

Let the vertical through E intersect the linear arch in D and the horizontal through A in F.

Let θ be the inclination of the tangent at E to the horizontal.

Let I be the moment of inertia of the section of the rib at E.

Let ds be an element of the axis at E.

Change of inclination at $E = d\theta = \frac{M ds}{EI} = \frac{H \cdot DE \cdot ds}{EI}$.

If this change of curvature were effected by causing the whole curve on the left of E to turn about E through an angle $d\theta$, the horizontal displacement of A would be

$$AA' = AL \cos LAA' = AEd\theta \sin EAF = AEd\theta \frac{EF}{AE}$$

$$= EF \cdot d\theta = \frac{H}{EI} DE \cdot EF \cdot ds.$$

This is evidently equal to the horizontal displacement of E, and the algebraic sum of the horizontal displacements of all points along the axis is

$$\Sigma \frac{H \cdot DE \cdot EF \cdot ds}{EI} = \int \frac{H \cdot DE \cdot EF \cdot ds}{EI} = 0, \quad . \quad . \quad . \quad (1)$$

since the length AC is assumed to be invariable.

Thus, the actual linear arch must fulfil the condition expressed by eq. (1), which may be written

$$\int \frac{\mathbf{DE} \cdot \mathbf{EF} \cdot \mathbf{ds}}{\mathbf{I}} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

since H and E are constant.

If the rib is of uniform section, I is also constant, and eq. (2) becomes

$$\int \mathrm{DE}\cdot\mathrm{EF}\cdot ds = o. \qquad (3)$$

Also, since DE is the difference between DF and EF,

$$\int (\mathrm{DF}\sim\mathrm{EF})\mathrm{EF}\cdot ds = o = \int \mathrm{DF}\cdot\mathrm{EF}\cdot ds - \int \mathrm{EF}^2 ds. \qquad (4)$$

It must be remembered that the integration extends from end to end of the arch.

Eq. 1 expresses the fact that the span remains invariable when a series of bending moments, $H\cdot DE$, act at points along the rib. These, however, are accompanied by a thrust along the arch, and the axis of the rib varies in length with the variation of thrust.

Let H_0 be the horizontal thrust for that symmetrical loading which makes the linear arch coincide with the axis of the rib.

Let T_0 be the corresponding thrust along the rib at E.

The shortening of the element ds at E of unit section

$$= \frac{T - T_0}{E} ds.$$

Let a number of weights W_1, W_2, W_3 . . . be concentrated at different points along the arched rib.

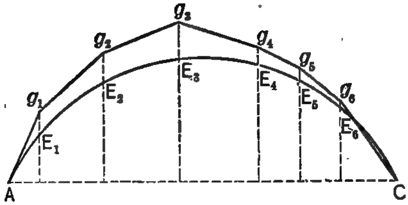


FIG. 915.

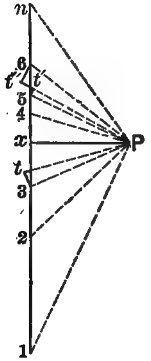


FIG. 916.

Take 1234 . . . n as the line of loads, W_1 being represented by 12, W_2 by 23, W_3 by 34, etc., and let the segments $1x$, nx, respectively, represent the vertical reaction at A and C. Take the horizontal length xP to represent H, and draw the radial lines $P1$, $P2$, $P3$. . .

The equilibrium polygon $Ag_1g_2g_3$ . . . must be the funicular polygon of the forces with respect to the pole P, and therefore the directions of the resultant thrusts from A to E_1, E_1 to E_2, E_2 to E_3, . . . are respectively parallel to $P1$, $P2$, $P3$, . . .

The tangential (axial) thrust and shear at any point p of the rib, e.g., between E_2 and E_3, may be easily found by drawing Pt parallel to the tangent at p, and $3t$ perpendicular to Pt. The direct tangential thrust is evidently represented by Pt, and the normal shear at the same point by $3t$. The latter is borne by the web.

If p is a point at which a weight is concentrated, e.g., E_5, draw $Pt't''$ parallel to the tangent at E_5, and $5t'$, $6t''$ perpendicular to $Pt't''$.

Pt' represents the axial thrust immediately on the left of E_5, and $5t'$ the corresponding normal shear, while Pt'' represents the axial thrust immediately on the right of E_5 and $6t''$ the corresponding normal shear.

A vertical line through P can only meet the line of loads at infinity.

Thus, it would require the loads at A and C to be infinitely great in order that the thrusts at these points might be vertical. Practically, no linear arch will even approximately coincide with the axis of a rib rising vertically at the springings, and hence neither a semicircular nor a semi-elliptical axis is to be recommended.

Ex. 4. *Let the axis of a rib or uniform section and hinged at both ends be a semicircle of radius r.*

Let a single weight W be placed upon the rib at a point whose horizontal distance from O, the centre of the span, is a.

The *linear arch* (or *bending-moment diagram*) consists of two straight lines DA, DC.

Draw any vertical line intersecting the axis, the linear arch and the springing-line AC in E', D', F', respectively.

Let $OF'=x$, and let dx be the horizontal projection upon AC of the element ds at E'.

Then

$$\frac{ds}{dx}=\text{cosec } E'OF'=\frac{r}{E'F'}, \quad \text{or} \quad E'F'\cdot ds=r\cdot dx.$$

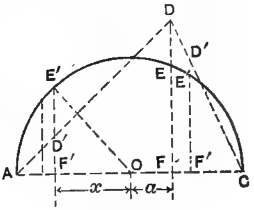


FIG. 917.

Applying condition (4),

$$\int_{-a}^{r} D'F'rdx+\int_{a}^{r} D'F'rdx=\int_{-r}^{r} E'F'rdx,$$

or

$$\int_{-a}^{r} D'F'dx+\int_{a}^{r} D'F'dx=\int_{-r}^{r} E'F'dx,$$

or area of triangle ADC = area of semicircle.

Therefore, if z be the vertical distance of D from AC,

$$zr=\frac{\pi r^2}{2} \quad \text{or} \quad z=\frac{\pi r}{2}=\text{one-half of length of rib.}$$

Therefore

$$DE=DF-EF=\frac{\pi r}{2}-\sqrt{r^2-a^2},$$

and if h is the horizontal thrust on the arch due to W.

$$h\times DE=\text{B.M.}=\frac{W(r^2-a^2)}{2r}-h\times DF$$

or

$$h\times DF=W\frac{r^2-a^2}{2r}$$

Similarly, if there are a number of weights W_1, W_2, W_3, . . . upon the rib, and if h_1, h_2, h_3, . . . are the corresponding horizontal thrusts, the total horizontal thrust H will be the sum of these separate thrusts, i.e.,

$$H=h_1+h_2+\ldots \qquad (5)$$

It will be observed that the apices (D_1, D_2, D_3, . . .) of the several linear arches (triangles) lie in a horizontal line at the vertical distance $\frac{\pi r}{2}$ from the springing-line.

Ex. 5. *A semicircular rib of 28 ft. span carries a weight of ¼ ton at 4 ft. (measured horizontally) from the centre. Find the thrust and shear at the centre of the rib and at the point at which the weight is concentrated.*

$$z=DF=\tfrac{22}{7}\times 7=22'.$$

Therefore

$$H\times 22=\frac{1}{4}\,\frac{18\times 10}{28}=\frac{45}{28}\text{ tons}; \quad \text{or} \quad H=\frac{45}{616}t.$$

The vertical reaction at $A=\frac{5}{56}t$ and at $B=\frac{9}{56}t$.

Let θ be the angle between the tangent at E and the horizontal. Then

$$\operatorname{cosec}\theta=\frac{OE}{OF}=\frac{14}{4}=3.5, \quad \text{and} \quad \theta=16°.36'.$$

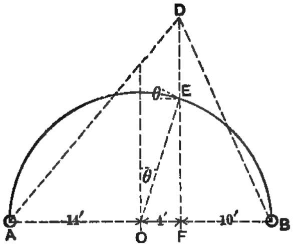


FIG. 9_8

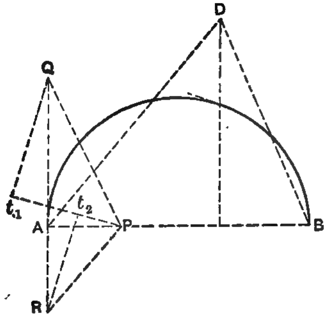


FIG. 919.

In Fig. 919 QR is the line of loads, QA being $\frac{9}{56}t$ and RA $\frac{5}{56}t$. Take $AP=\frac{45}{616}t$ and draw Pt_2t_1 parallel to the tangent at E. Also let fall the perpendiculars Qt_1 and Rt_2 upon Pt_2t_1

Then, *below the weight*

$$\text{the axial thrust} = Pt_1 = .116t\left(=H\cos\theta+\tfrac{9}{56}\sin\theta\right),$$

$$\text{“ shear} = Q_1t = .1332t\left(=-H\sin\theta+\tfrac{9}{56}\cos\theta\right),$$

and, *above the weight*

$$\text{the axial thrust} = Pt_2 = .0445t\left(=H\cos\theta-\tfrac{5}{56}\sin\theta\right),$$

$$\text{“ shear} = Rt_2 = .1064t\left(=H\sin\theta+\tfrac{5}{56}\cos\theta\right).$$

PQ and PR are of course parallel to the lines DB, DA, respectively, of the linear arch.

Ex. 6. *Let the axis be a parabola of span 2l and rise k.* From the properties of the parabola,

$$E'F'=k\left(1-\frac{x^2}{l^2}\right),\quad D'F'=\frac{z(l-x)}{l\pm a}.$$

Also, $$ds^2=dx^2\left(1+4\frac{k^2}{l^2}x^2\right),\quad\text{or}\quad ds=dx\left(1+2\frac{k^2}{l^2}x^2\right),\ \text{approximately.}$$

Applying condition (4),

$$\int_{-l}^{l}k^2\left(1-\frac{x^2}{l^2}\right)^2dx\left(1+2\frac{k^2}{l^4}x^2\right)$$

$$=\int_{-a}^{l}\frac{z(t-x)}{l+a}k\left(1-\frac{x^2}{l^2}\right)\left(1+2\frac{k^2}{l^4}x^2\right)dx$$

$$+\int_{a}^{l}\frac{z(l-x)}{l-a}k\left(1-\frac{x^2}{l^2}\right)\left(1+2\frac{k^2}{l^4}x^2\right)dx,$$

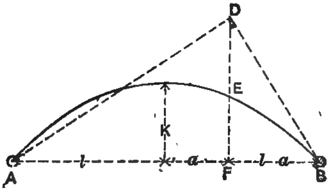


FIG. 920.

which easily reduces to

$$z\left\{1+\frac{1}{6}\frac{l^2+a^2}{l^2}\left(\frac{2k^2}{l^2}-1\right)-\frac{2}{15}\frac{k^2}{l^6}(l^4+l^2a^2+a^4)\right\}=k\left(\frac{16}{15}+\frac{32}{105}\frac{k^2}{l^2}\right),$$

an equation giving z or DF.

If the arch is flat, so that ds may be considered as approximately equal to dx, the term $2\frac{k^2}{l^4}x^2$ in the above equation may be disregarded, and it may be easily shown that

$$z\left\{1-\frac{1}{6}\frac{l^2+a^2}{l^2}\right\}=k\frac{16}{15},$$

or

$$z=\frac{32}{5}\frac{kl^2}{5l^2-a^2}.$$

Ex. 6. *Draw the equilibrium polygon for a parabolic arch of* 100 *ft. span and* 20 *ft. rise when loaded with weights of* 3, 2, 4, *and* 2 *tons, respectively, at the end of the third, sixth, eighth, and ninth divisions from the left support, of ten equal horizontal divisions.* (*Neglect the weight of the rib.*)

If the rib consist of a web and of two flanges $2\frac{1}{2}$ *ft. from centre to centre, determine the maximum flange stress.*

First to find H. $\dfrac{20-E_3F_3}{20^2}=\dfrac{20}{50^2}=\dfrac{20-E_6F_6}{10^2}=\dfrac{20-E_8F_8}{30^2}=\dfrac{20-E_9F_9}{40^2}.$

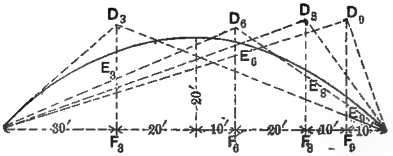


FIG. 921

Therefore $E_3F_3=16'.8$, $E_6F_6=19'.2$, $E_8F_8=12'.8$, and $E_9F_9=7'.2$.

Applying the last formula,

$$z_3=\frac{3200}{121}\text{ ft.},\qquad z_6=\frac{800}{31}\text{ ft.},$$

$$z_8=\frac{800}{29}\text{ ft., and }\quad z_9=\frac{3200}{109}\text{ ft.}$$

Hence $\quad h_3\frac{3200}{121}=3\frac{30\times70}{100},\quad h_6\frac{800}{31}=2\frac{60\times40}{100}=48,\quad h_8\frac{800}{29}=4\frac{80\times20}{100}=64,$

$$\text{and}\quad h_9\frac{3200}{109}=2\frac{90\times10}{100}=18.$$

Therefore $\quad h_3=2.382t,\quad h_6=1.86t,\quad h_8=2.32t,\quad h_9=0.613t,$

and $$H=h_3+h_6+h_8+h_9=7.175\text{ tons.}$$

In Fig. 922 take the vertical through A as the line of loads.

The vertical reaction at the left support $=3.9$ tons.

The vertical reaction at the right support $=7.1$ tons.

The polar distance $AO=7.175$ tons.

Draw the radial lines Op, Oq, Or, Os, Ot. Then $Ad_3d_6d_8d_9\ B$ is the linear arch, with its sides parallel to the radial lines from O. Denoting by α, β, γ, δ, ϵ the inclinations of these lines to the horizontal,

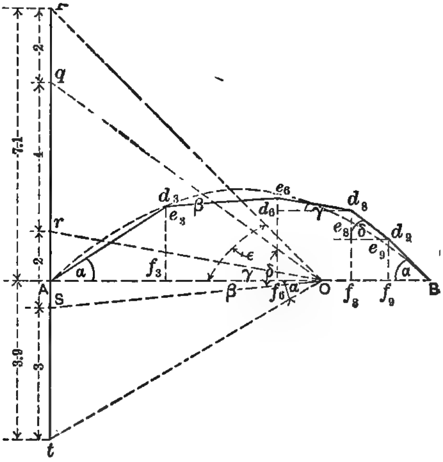


FIG. 922.

$$\tan\alpha=\frac{3.9}{H},\quad \tan\beta=\frac{0.9}{H},$$

$$\tan\gamma=\frac{1.1}{H},\quad \tan\delta=\frac{5.1}{H},$$

$$\tan\epsilon=\frac{7.1}{H}.$$

Therefore $\quad d_3f_3=30\tan\alpha=\dfrac{117}{H},\quad d_6f_6=30(\tan\alpha+\tan\beta)=\dfrac{144}{H},$

$$d_8f_8=d_6f_6-20\tan\gamma=\frac{122}{H},\quad\text{and}\quad d_9f_9=10\tan\epsilon=\frac{71}{H}.$$

The vertical intercept between the linear arch and the axis of the rib is greatest at e_8 and therefore the B.M. at this point is a maximum. Hence

$$\text{max. B.M.}=H\cdot d_8e_8=7.175\left(\frac{122}{H}-12.8\right)=30.156\text{ ft.-tons.}$$

The maximum *bending stress* at the same point, I being the moment of inertia of the rib,

$$=\pm\frac{15}{I}\times30.156\times12=\pm\frac{5428}{I}\text{ tons per sq. in.}$$

Draw Ot_2t_1 parallel to the tangent at e_8, and drop the perpendiculars qt_1, rt_2. If θ is the inclination of the tangent to the horizontal,

$$\tan\theta=\frac{2(20-12.8)}{30}=0.48,\quad\text{and}\quad\theta=25°\ 39'.$$

Hence, *just below* the weight,

$$\text{the axial thrust} = Ot_1 = 8.676 \text{ tons } (= H \cos\theta + 5.1 \sin\theta)$$

and the shear $= qt_1$.

Just above the weight,

$$\text{the axial thrust} = Ot_2 = 6.945 \text{ tons } (= H \cos\theta + 1.1 \sin\theta)$$

and the shear $= rt_2$.

Therefore the maximum stress in tons per square inch $= \dfrac{8.676}{A} \pm \dfrac{5428}{I}$, A being the sectional area of the rib at e_8.

Again, if F is the total flange stress,

$$F \times 2\tfrac{1}{2} = 30.156 \quad \text{and} \quad F = 12.063 \text{ tons.}$$

Ex. 7. *Let the axis of the rib be a circular arc of span $2l$ and radius r, subtending an angle 2α at the centre N.*

Let the angles between the radii NE, NE', and the vertical be β and θ respectively.

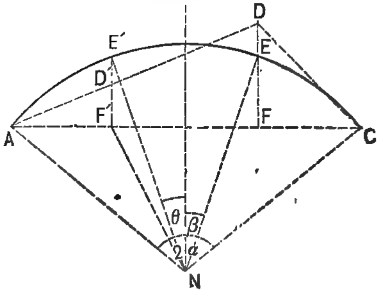


FIG. 923.

The element ds at $E' = rd\theta$. Also,

$$E'F' = r(\cos\theta - \cos\alpha); \quad AF' = l - r\sin\theta;$$

$$D'F' = \frac{z}{l \pm a}(l - r\sin\theta).$$

Applying condition (4),

$$\int_{-\alpha}^{\alpha} r^2(\cos\theta - \cos\alpha)^2 r d\theta$$

$$= \int_{-\beta}^{\alpha} \frac{z}{l+a}(l - r\sin\theta) r(\cos\theta - \cos\alpha) r d\theta$$

$$+ \int_{\beta}^{\alpha} \frac{z}{l-a}(l - r\sin\theta) r(\cos\theta - \cos\alpha) r d\theta,$$

which easily reduces to

$$r\{\alpha(\cos 2\alpha + 2) - \tfrac{3}{2}\sin 2\alpha\}$$

$$= \frac{2z}{l^2 - a^2}\Big\{l^2(\sin\alpha - \alpha\cos\alpha) + \frac{rl}{4}(\cos 2\alpha - \cos 2\beta)$$

$$- rl\cos\alpha(\cos\alpha - \cos\beta) - la(\sin\beta - \beta\cos\alpha)\Big\},$$

an equation giving z or DF. Also,

$$DE = DF - EF,$$

and the corresponding horizontal thrust may be found, as before, by the equation

$$h \cdot DF = W\frac{l^2 - a^2}{l}.$$

Note.—If $\alpha^\circ = 90^\circ$,

$$r\frac{\pi}{2} = \frac{2z}{l^2 - a^2}\left(\frac{l^2 - a^2}{2}\right), \quad \text{or} \quad z = \frac{\pi r}{2} \text{ as in Ex. 3.}$$

9. Rib with Ends Absolutely Fixed.—Let ABC be the axis of the rib. The fixture of the ends introduces two unknown moments

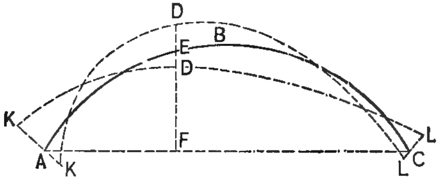


FIG. 924.

at these points, and since H is also unknown, three conditions must be satisfied before the strength of the rib can be calculated.

Represent the linear arch by the dotted lines KL; the points K, L may fall above or below the points A, C.

Let a vertical line EF intersect the linear arch in D, the axis of the rib in E, and the horizontal through A in F.

As before, the change of inclination at E, or $d\theta$, $= \frac{Mds}{EI}$. But the total change of inclination of the rib between A and C must be *nil*, as the ends are fixed.

Therefore $$\int \frac{Mds}{EI} = 0 = \int \frac{H \cdot DE \cdot ds}{EI}, \quad \ldots \ldots \quad (1)$$

which may be written

$$\int \frac{\mathbf{DE}}{\mathbf{I}} \mathbf{ds} = \mathbf{o}, \quad \ldots \ldots \ldots \quad (2)$$

since H and E are constant.

If the section of the rib is uniform, I is constant and eq. (2) becomes

$$\int DE \cdot ds = 0. \quad \ldots \ldots \ldots \quad (3)$$

Again, the total *horizontal* displacement between A and C will be *nil* if the abutments are immovable. If they yield, the amount of the yielding must be determined in each case, and may be denoted by an expression of the form μH, μ being some coefficient.

As before, the total horizontal displacement

$$=\int\frac{H\cdot DE\cdot EF\cdot ds}{EI}.$$

Therefore $$\int\frac{H\cdot DE\cdot EF\cdot ds}{EI}=0 \quad\text{or}\quad =\mu H. \quad .\ .\ .\ .\ .\ . \quad (4)$$

But H and E are constant, so that

$$\int\frac{\mathbf{DE\cdot EF\cdot ds}}{\mathbf{I}}=\mathrm{o} \quad\text{or}\quad =\mu\mathbf{E} \quad .\ .\ .\ .\ .\ . \quad (5)$$

If the section of the rib is uniform, I is also constant, and hence

$$\int DE\cdot EF\cdot ds=0 \quad\text{or}\quad =\mu EI; \quad .\ .\ .\ .\ . \quad (6)$$

and since DE is the difference between DF and EF, this last may be written

$$\int DF\cdot EF\cdot ds\sim\int EF^2ds=0 \quad\text{or}\quad =\mu EI. \quad .\ .\ . \quad (7)$$

Again, the total *vertical* displacement between A and C must be *nil.*

The vertical displacement of E (Fig. 914)

$$=A'L=AL\cos EAF=AEd\theta\frac{AF}{AE}$$

$$=AFd\theta=AF\frac{Mds}{EI}.$$

Hence the total vertical displacement

$$=\int\frac{H\cdot DE\cdot AF}{EI}ds=0, \quad .\ .\ .\ .\ .\ . \quad (8)$$

which may be written

$$\int \frac{\mathrm{DE}\cdot\mathrm{AF}}{\mathrm{I}}\mathrm{ds}=\mathrm{o}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

since H and E are constant. If the section of the rib is also constant,

$$\int \mathrm{DE}\cdot\mathrm{AF}\cdot\mathrm{ds}=\mathrm{o}=\int \mathrm{DF}\cdot\mathrm{AF}\cdot\mathrm{ds}-\int \mathrm{EF}\cdot\mathrm{AF}\cdot\mathrm{ds}. \quad . \quad . \qquad (10)$$

Eqs. (2), (5), and (9) are the three equations of condition.

In eq. (9) AF must be measured from same abutment throughout the summation.

The integration extends from A to C.

Ex. 8. *Let the axis be a parabola of span 2l and rise k* (Fig. 925).

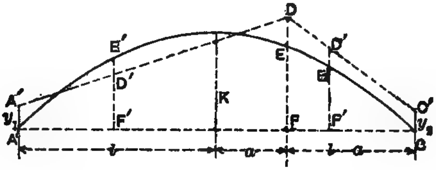


Fig. 925.

$$E'F'=k\left(1-\frac{x^2}{l^2}\right); \quad ds'=\left(1+\frac{2k^2}{l^4}x^2\right)dx.$$

Also, $$D'F'=y_1+(l-x)\frac{z-y_1}{l+a} \text{ on the left of } DF,$$

or $$=y_2+(l-x)\frac{z-y_2}{l-a} \text{ on the right of } DF;$$

and $$AF'=l\mp x.$$

The equations of condition become

$$\int_{-a}^{l}\left\{y_1+(l-x)\frac{z-y_1}{l+a}\right\}\left(1+2\frac{k^2}{l^4}x^2\right)dx+\int_{a}^{l}\left\{y_2+(l-x)\frac{z-y_2}{l-a}\right\}\left(1+2\frac{k^2}{l^4}x^2\right)dx$$

$$=\int_{-l}^{l}k\left(1-\frac{x^2}{l^2}\right)\left(1+2\frac{k^2}{l^4}x^2\right)dx,$$

$$\int_{-a}^{l}\left\{y_1+(l-x)\frac{z-y_1}{l+a}\right\}k\left(1-\frac{x^2}{l^2}\right)\left(1+2\frac{k^2}{l^4}x^2\right)dx$$

$$+\int_{a}^{l}\left\{y_2+(l-x)\frac{z-y_2}{l-a}\right\}\left(1-\frac{x^2}{l^2}\right)k\left(1+2\frac{k^2}{l^4}x^2\right)dx$$

$$=\int_{-l}^{l}k^2\left(1-\frac{x^2}{l^2}\right)^2\left(1+2\frac{k^2}{l^4}x^2\right)dx,$$

and

$$\int_{-a}^{l}\left\{y_1+(l-x)\frac{z-y_1}{l+a}\right\}(l-x)\left(1+2\frac{k^2}{l^4}x^2\right)dx$$

$$+\int_{a}^{l}\left\{y_2+(l-x)\frac{z-y_2}{l-a}\right\}(l+x)\left(1+2\frac{k^2}{l^4}x^2\right)dx$$

$$=\int_{-a}^{l}k\left(1-\frac{x^2}{l^2}\right)(l-x)\left(1+2\frac{k^2}{l^4}x^2\right)dx+\int_{a}^{l}k\left(1-\frac{x^2}{l^2}\right)(l+x)\left(1+2\frac{k^2}{l^4}x^2\right)dx.$$

These equations may be at once integrated, and the resulting equations will give the values of y_1, y_2, z.

If the arch is very flat, so that ds may be taken to be approximately the same as dx, it may be easily shown that

$$y_1=\frac{2k}{15}\frac{l+5a}{l+a},\quad y_2=\frac{2k}{15}\frac{l-5a}{l-a},\quad \text{and}\quad z=\frac{6}{5}k.$$

Let h', h'', h''' . . . be the horizontal thrusts due to loads p', p'', p''' . . . . , respectively.

Let y_1', y_1'', y_1''' . . . , y_2', y_2'', y_2''' . . . , be the corresponding values of y_1, y_2.

Let $\overline{y_1}$, $\overline{y_2}$ be the resultant values for the total thrust H. Then

$$\mathbf{H}\cdot\overline{\mathbf{y}_1}=h'y_1'+h''y_1''+h'''y_1'''+\ldots=\Sigma(\mathbf{hy}_1)$$

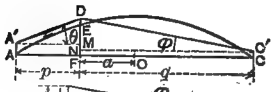
and

$$\mathbf{H}\cdot\overline{\mathbf{y}_2}=h'y_2'+h''y_2''+h'''y_2'''+\ldots=\Sigma(\mathbf{hy}_2),$$

Σ denoting *algebraic* sum. If $\overline{y_1}$ or $\overline{y_2}$ is negative, it merely indicates that A' or C' falls below AC.

10. Value of H.—In the case of the flat parabolic arch AC, fixed at both ends and carrying a weight W at E, distant a from the middle point O, measured horizontally, simple expressions for H and the vertical reactions (R_A, R_C) at A and C can easily be found.

FIG. 926.



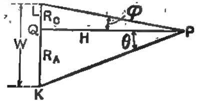


FIG. 927.

Let the segments $AF=l-a=p$,
$CF=l+a=q$.

Take the verticals

$$AA'=y_1=\frac{2}{15}k\frac{l-5a}{l-a}=\frac{2}{15}k\frac{3p-2q}{p},$$

$$CC'=y_2=\frac{2}{15}k\frac{l+5a}{l+a}=\frac{2}{15}k\frac{3q-2p}{q},$$

and $$FD=z=\frac{6}{5}k\ (\text{Ex. } 8).$$

The linear arch is composed of the straight lines $A'D$, DC'.

Take the vertical $LK=W$ and draw KP, LP parallel to $A'D$, $C'D$, respectively. Also draw the horizontal lines PQ, $A'M$, and $C'N$.

Then $$PQ=H,\quad KQ=R_A,\quad \text{and}\quad LQ=R_C.$$

Hence $$\frac{W}{H}=\frac{R_A}{H}+\frac{R_C}{H}=\tan\theta+\tan\phi=\frac{DM}{A'M}+\frac{DN}{C'N}$$

$$=\frac{z-y_1}{p}+\frac{z-y_2}{q}=\frac{8}{15}k\left(\frac{l+p}{p^2}+\frac{l+q}{q^2}\right)=\frac{32}{15}\frac{kl^3}{p^2q^2},$$

and $$\mathbf{H}=\frac{\mathbf{15}}{\mathbf{32}}\mathbf{W}\frac{\mathbf{p^2q^2}}{\mathbf{kl^3}}.$$

Hence, also,

$$\mathbf{R_A}=H\tan\theta=\frac{15}{32}W\frac{p^2q^2}{kl^3}\frac{8}{15}k\frac{l+p}{p^2}=\frac{\mathbf{W}}{\mathbf{4}}\frac{\mathbf{q^2(l+p)}}{\mathbf{l^3}},$$

and $$\mathbf{R_C}=H\tan\phi=\frac{15}{32}W\frac{p^2q^2}{kl^3}\frac{8}{15}k\frac{l+p}{p^2}=\frac{\mathbf{W}}{\mathbf{4}}\frac{\mathbf{p^2(l+q)}}{\mathbf{l^3}}.$$

Ex. 9. *Draw the linear arch and determine the maximum flange stresses for an arched rib of* 80 *ft. span,* 16 *ft. rise, and loaded with five weights each of* 2 *tons at the end of the first, second, third, fourth, and fifth divisions, of eight equal horizontal divisions. The rib is of double-tee section and* 30 *ins. deep. Also find the shears and the axial thrusts at the fifth point of division.*

$$\frac{16-E_1F_1}{30^2}=\frac{16}{40^2}=\frac{16-E_2F_2}{20^2}$$
$$=\frac{16-E_3F_3}{10^2}=\frac{16-E_5F_5}{10^2}.$$

Therefore $E_1F_1=7'$, $E_2F_2=12'$, $E_3F_3=15'=E_5F_5$.

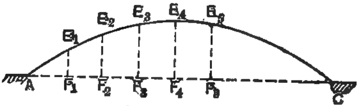


FIG. 928.

Applying the formulæ obtained in Ex. 8 and in Art. 9,

At E_1, $y_1' = -\frac{352}{15}$, $y_2' = \frac{608}{105}$, $h' = \frac{3675}{8192}t$, $R_A' = \frac{245}{128}t$, $R_C' = \frac{11}{128}t$,
" E_2, $y_1'' = -\frac{32}{5}$, $y_2'' = \frac{224}{45}$, $h'' = \frac{1350}{1024}t$, $R_A'' = \frac{27}{16}t$, $R_C'' = \frac{5}{16}t$,
" E_3, $y_1''' = -\frac{32}{45}$, $y_2''' = \frac{96}{25}$, $h''' = \frac{16875}{8192}t$, $R_A''' = \frac{175}{128}t$, $R_C''' = \frac{81}{128}t$,
" E_4, $y_1'''' = +\frac{32}{15}$, $y_2'''' = \frac{32}{15}$, $h'''' = \frac{75}{32}t$, $R_A'''' = 1-t$, $R_C'''' = 1-t$,
" E_5, $y_1''''' = +\frac{96}{25}$, $y_2''''' = -\frac{32}{45}$, $h''''' = \frac{16875}{8192}t$, $R_A''''' = \frac{81}{128}t$, $R_C''''' = \frac{175}{128}t$.

Therefore the total $H = h' + h'' + h''' + h'''' + h''''' = 8.2306t$
" " " $R_A = R_A' + R_A'' + \ldots + R_A''''' = \frac{845}{128}t$
" " " $R_C = R_C' + R_C'' + \ldots + R_C''''' = \frac{435}{128}t$.

Again, if y_1, y_2 are the true ordinates of the linear arch at A and C respectively,

$$Hy_1 = \frac{3675}{8192} \times -\frac{352}{15} + \frac{1350}{1024} \times -\frac{32}{5} + \frac{16875}{8192} \times -\frac{32}{45} + \frac{75}{32} \times \frac{32}{15} + \frac{16875}{8192} \times \frac{96}{25}$$

and $$Hy_2 = \frac{3675}{8192} \times \frac{608}{105} + \frac{1350}{1024} \times \frac{224}{45} + \frac{16875}{8192} \times \frac{96}{25} + \frac{75}{32} \times \frac{32}{15} + \frac{16875}{8192} \times -\frac{32}{45}.$$

Hence $y_1 = -0.9136$ ft. $= AA'$

and $y_2 = 2.5035$ ft. $= CC'$.

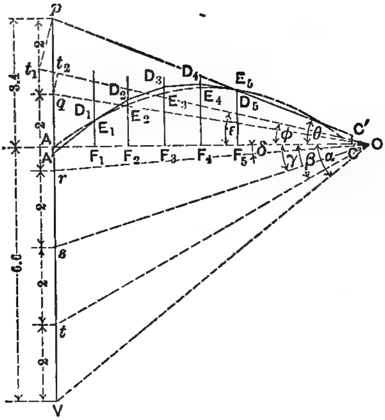


FIG. 929.

Taking $pqrstv$ as the line of loads, then

$$Ap = R_C = 3.4 \text{ tons}$$

and $$Av = R_A = 6.6 \text{ tons}.$$

Also, the polar distances $OA = 8.2306t$.

Join Op, Oq . . . Ov and draw the linear arch $A'D_1D_2 \ldots D_5C'$, with its sides $A'D_1$, D_1D_2 . . . parallel to the corresponding lines Ov, Ot, . . . , respectively. Designating by α, β, γ . . . the slopes of these lines,

$$\tan\alpha = \frac{6.6}{H}, \quad \tan\beta = \frac{4.6}{H},$$

$$\tan\gamma = \frac{2.6}{H}, \quad \tan\delta = \frac{0.6}{H}, \quad \tan\phi = \frac{1.4}{H}, \quad \text{and} \quad \tan\theta = \frac{3.4}{H}.$$

Hence

$$\begin{array}{llll} D_1F_1 = 10\tan\alpha - y_1 & = 7'.1072 & \text{and} & D_1E_1 = D_1F_1 - E_1F_1 = 0.1072 \text{ ft.} \\ D_2F_2 = D_1F_1 + 10\tan\beta & = 12'.698 & \text{``} & D_2E_2 = D_2F_2 - E_2F_2 = 0.698 \text{ ``} \\ D_3F_3 = D_2F_2 + 10\tan\gamma & = 15'.859 & \text{``} & D_3E_3 = D_3F_3 - E_3F_3 = 0.8588 \text{ ``} \\ D_4F_4 = D_3F_3 + 10\tan\delta & = 16'.5897 & \text{``} & D_4E = D_4F_4 - E_4F_4 = 0.5897 \text{ ``} \\ D_5F_5 = 30\tan\phi + y_5 & = 14'.8906 & \text{``} & D_5E_5 = D_5F_5 - E_5F_5 = 0.1094 \text{ ``} \end{array}$$

At E_3, $H \times 0.8588 = \text{B.M.} = 7.0685$ ft.-tons, and therefore the max. bending stress in tons per sq. in. $= \pm\frac{15}{I}\times 7.0685 \times 12 = \pm\frac{1272.33}{I}$

If ϵ is the slope at E_3, $\tan\epsilon = \frac{2}{10} = 0.2$ and $e = 11° 19'$.

Therefore $\sec\epsilon = 1.02$ and $H\sec\epsilon = 8.3952t$.

Hence the total maximum skin stress per square inch in tons at E

$$= \frac{8.3952}{A} \pm \frac{1272.33}{I}.$$

Draw Ot_2t_1 parallel to the tangent at E_5, and let ζ be the slope at E_5. Then $\tan\zeta = 0.2$.

Just *above the weight, at* E_5,

the axial thrust $= Ot_2 = 8.3486t$,

and the shear $= pt_2 = 0.243t$,

and just *below the weight, at* E_5,

the axial thrust $= Ot_1 = 8.741t$,

and the shear $= qt_2 = 1.719t$.

The *absolute maximum* B.M. is at C and is

$$= Hy_2 = 20.6054 \text{ ft.-tons.}$$

The slope at $C = \tan^{-1} 0.8 = 38° 40'$, and $H \sec 38° 40' = 10.434t$. Hence the absolute maximum flange stress per square inch

$$= \frac{10.434}{A} \pm \frac{15 \times 20.654 \times 12}{I} = \frac{10.434}{A} \pm \frac{3709}{I}.$$

The B.M. changes sign at the points where the linear arch intersects the axis of the rib.

Ex. 10. *Let the axis of the rib be a circular arc of span $2l$, subtending an angle 2α at the centre N.*

Let a weight W be concentrated on the rib at a point E whose horizontal distance from the middle point of the span is a.

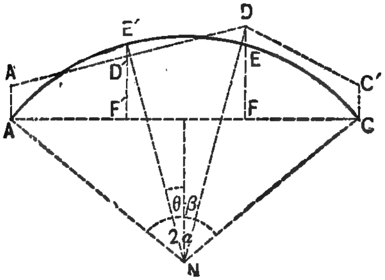


FIG. 930.

Let the radius NE make an angle β with the vertical.

The "linear arch" consists of two straight lines DA', DC'.

Let $AA' = y_1$, $DF = z$, $CC' = y_2$.

Draw any ordinate $E'F'$ intersecting the linear arch in D'.

Let the radius NE' make an angle θ with the vertical. Then

$$E'F' = r(\cos\theta - \cos\alpha).$$

Also, $AF' = l - r\sin\theta$ and $D'F' = (l - r\sin\theta)\dfrac{z-y_1}{l+a} + y_1$,

if F' is on the left of F,

while $AF' = l + r\sin\theta$ and $D'F' = (l - r\sin\theta)\dfrac{z-y_2}{l-a} + y_2$,

if F' is on the right of F.

Also, $ds = r\,d\theta$.

Applying condition (2),

$$\int_{-\beta}^{\alpha}\left\{(l-r\sin\theta)\frac{z-y_1}{l+a}+y_1\right\}d\theta+\int_{\beta}^{\alpha}\left\{(l-r\sin\theta)\frac{z-y_2}{l-a}+y_2\right\}d\theta$$
$$= r\int_{-\alpha}^{\alpha}(\cos\theta-\cos\alpha)\,d\theta. \quad (1)$$

Applying condition (5), and assuming $\mu = 0$,

$$\int_{-\beta}^{\alpha}(\cos\theta-\cos\alpha)\left\{(l-r\sin\theta)\frac{z-y_1}{l+a}+y_1\right\}d\theta$$
$$+\int_{\beta}^{\alpha}(\cos\theta-\cos\alpha)\left\{(l-r\sin\theta)\frac{z-y_2}{l-a}+y_2\right\} = r\int_{-\alpha}^{\alpha}(\cos\theta-\cos\alpha)^2 d\theta. \quad (2)$$

Applying condition (9),

$$\int_{-\beta}^{\alpha}(l-r\sin\theta)\left\{(l-r\sin\theta)\frac{z-y_1}{l+a}+y_1\right\}d\theta$$
$$+\int_{\beta}^{\alpha}(l+r\sin\theta)\left\{(l-r\sin\theta)\frac{z-y_2}{l-a}+y_2\right\}d\theta$$
$$= r\int_{-\beta}^{\alpha}(\cos\theta-\cos\alpha)(l-r\sin\theta)\,d\theta + r\int_{\beta}^{\alpha}(\cos\theta-\cos\alpha)(l+r\sin\theta)\,d\theta. \quad (3)$$

Eqs. (1), (2), (3) may be easily integrated, and the resulting equations will give the values of y_1, z, and y_2.

The corresponding horizontal thrust, h, may now be obtained from the equation $h \cdot DE = M = h(z - EF)$.

Note.—If the axis is a semicircle, and if W is at the crown,

$$a=0, \quad \alpha=90^\circ, \quad \beta=0,$$

and eqs. (1), (2), (3) reduce to

$$z(\pi-2)+y_1+y_2=2r;$$

$$2z+y_1+y_2=\pi r,$$

$$z(\pi-2)+y_1\left(\frac{\pi}{4}-1\right)-y_2\left(\frac{\pi}{4}+1\right)=2r.$$

Therefore $$z=\tfrac{4}{3}r \quad \text{and} \quad y_1=\tfrac{10}{21}r=y_2.$$

11. Effect of a Change of Temperature.—The variation in the span $2l$ of an arch for a change of t° from the mean temperature is approximately $=2\epsilon tl$, ϵ being the coefficient of expansion.

Hence, if H_t is the horizontal force induced by a change of temperature, the condition that the length AC is invariable is expressed by the equation

$$H_t\int\frac{DE \cdot EF \cdot ds}{EI} \pm 2\epsilon tl=0.$$

If the rib is of uniform section, I is constant; and since E is also constant, the equation may be written

$$\frac{H_t}{EI}\int DE \cdot EF \cdot ds \pm 2\epsilon tl=0.$$

Ex. 11. *Let the axis AEC of a rib of uniform section be a parabola of span 2l and rise k.*

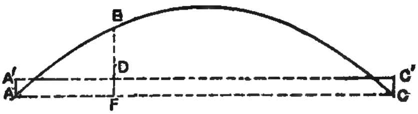


FIG. 931.

First, let the rib be *hinged* at both ends.

The straight line AC is the linear arch. Then

$$\int DE\cdot EF\cdot ds=\int_{-l}^{l}EF^2\left(1+2\frac{k^2}{l^4}x^2\right)dx$$

$$=\int_{-l}^{l}k^2\left(1-\frac{x^2}{l^2}\right)^2\left(1+\frac{2k^2}{l^4}x^2\right)dx=\frac{16}{15}kl^2+\frac{32}{105}\frac{k^4}{l},$$

and hence

$$\frac{H_t}{EI}k^2l\left(\frac{16}{15}+\frac{32}{105}\frac{k^2}{l^2}\right)\pm2\epsilon tl=0.$$

Second, let the rib be *fixed* at both ends.

The linear arch is the line $A'C'$ at a distance $z(=DF)$ from AC given by the equation

$$\int DE\cdot ds=0=\int(DF\backsim EF)ds,$$

or

$$DF\int ds=\int EF\cdot ds.$$

Hence
$$z\int_{-l}\left(1+\frac{2k^2}{l^4}x^2\right)dx=\int_{-l}^{l}k\left(1-\frac{x^2}{l^2}\right)\left(1+2\frac{k^2}{l^4}x^2\right)dx,$$

and integrating,
$$2zl\left(1+\frac{2}{3}\frac{k^2}{l^2}\right)=2kl\left(\frac{2}{3}+\frac{4}{15}\frac{k^2}{l^2}\right),$$

or
$$z\left(1+\frac{2}{3}\frac{k^2}{l^2}\right)=\frac{2}{3}k\left(1+\frac{2}{5}\frac{k^2}{l^2}\right).$$

Also,

$$\int DE\cdot EF\cdot ds=\int DF\cdot EF\cdot ds\backsim\int EF^2ds$$

$$=z\int EF\cdot ds-\int EF^2ds=2klz\left(\frac{2}{3}+\frac{4}{15}\frac{k^2}{l^2}\right)\backsim\left(\frac{16}{15}k^2l+\frac{32}{105}\frac{k^4}{l}\right).$$

Hence
$$\frac{H_t}{EI}\frac{4kl}{3}\left\{z\left(1+\frac{2}{5}\frac{k^2}{l^2}\right)\backsim\frac{4k}{5}\left(1+\frac{2}{7}\frac{k}{l^2}\right)\right\}\pm2\epsilon tl=0.$$

Remark.—The coefficient of expansion per degree of Fahrenheit is 0.0000062 and 0.0000067 for cast- and wrought-iron beams respectively. Hence the corresponding total expansion or contraction in a length of 100 ft., for a range of 60° F. from the mean temperature, is 0.0372 ft. $(=\frac{9}{20}'')$ and 0.0402 ft. $(=\frac{1}{2}'')$.

In practice the actual variation of length rarely exceeds *one half* of these amounts, which is chiefly owing to structural constraint.

Ex. 12. *Let the axis AEC of a rib of uniform section be the arc of a circle of radius r subtending an angle 2α at the centre.*

First, let the rib be *hinged* at both ends.

It is evident that the straight line AC is the "linear arch." Then

$$\int DE\cdot EF\cdot ds=\int EF^2ds=r^3\int_{-\alpha}^{\alpha}(\cos\theta-\cos\alpha)^2d\theta=r^3\{\alpha(2+\cos 2\alpha)-\tfrac{3}{2}\sin 2\alpha\}.$$

Also, $l=r\sin\alpha$. Therefore

$$\frac{H_t}{EI}\frac{l^3}{\sin^3\alpha}\{\alpha(2+\cos 2\alpha)-\tfrac{3}{2}\sin 2\alpha\}\pm 2\epsilon tl=0.$$

Note.—If the axis is a semicircle, $\alpha=90°$, and

$$\frac{H_t}{EI}\frac{\pi l^3}{2}\pm 2\epsilon tl=0.$$

Second, let the rib be *fixed* at both ends.

The "linear arch" is now a straight line $A'C'$ at a distance z $(=DF)$ from AC given by the equation

$$\int DE\cdot ds=0.$$

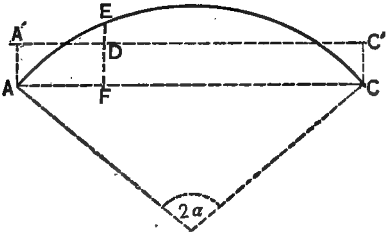


FIG. 932.

Then
$$\int DF\cdot ds=\int EF\cdot ds,$$

or
$$z\int ds=r^2\int_{-\alpha}^{\alpha}(\cos\theta-\cos\alpha)d\theta,$$

or
$$\alpha z=r(\sin\alpha-\alpha\cos\alpha).$$

Also,

$$\int DE\cdot EF\cdot ds=\int(DF\cdot EF\backsim EF^2)ds=z\int EFds\backsim\int EF^2ds$$

$$=2zr^2(\sin\alpha-\alpha\cos\alpha)\backsim r^3\{\alpha(2+\cos 2\alpha)-\tfrac{3}{2}\sin 2\alpha\}.$$

Hence

$$\frac{H_t}{EI}\left\{2zr^2(\sin\alpha-\alpha\cos\alpha)-r^3\{\alpha(2+\cos 2\alpha)-\tfrac{3}{2}\sin 2\alpha\}\right\}\pm 2\epsilon tl=0,$$

and

$$l=r\sin\alpha.$$

12. Deflection of an Arched Rib.

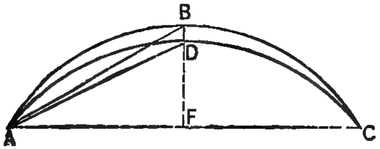


Fig. 933.

Let the abutments be immovable.

Let ABC be the axis of the rib in its normal position.

Let ADC represent the position of the axis when the rib is loaded.

Let BDF be the ordinate at the centre of the span; join AB, AD. Then

$$DF^2=AD^2-AF^2=AB^2\left(\frac{\text{arc } AD}{\text{arc } AB}\right)^2-AF^2.$$

But

$$\frac{\text{arc } AB=\text{arc } AD}{\text{arc } AB}=\frac{f}{E},$$

f being the intensity of stress due to the change in the length of the axis. Then

$$DF^2=AB^2\left(1-\frac{f}{E}\right)^2-AF^2=BF^2-AB^2\left\{2\frac{f}{E}-\left(\frac{f}{E}\right)^2\right\},$$

and

$$AB^2\left\{2\frac{f}{E}-\left(\frac{f}{E}\right)^2\right\}=BF^2-DF^2=(BF-DF)(BF+DF)$$

$$=2BF(BD), \text{ approximately.}$$

$\left(\frac{f}{E}\right)^2$ is also sufficiently small to be disregarded. Hence

$$BD, \textbf{ the deflection,}=\frac{AB^2}{BF}\frac{f}{E}=\frac{\mathrm{k}^2+\mathrm{l}^2}{\mathrm{k}}\,\frac{\mathrm{f}}{\mathrm{E}}, \text{ approximately.}$$

13. Elementary Deformation of an Arched Rib.

The arched rib represented by Fig. 934 springs from two abutments and is under a vertical load. The neutral axis PQ is the locus of the centres of gravity of all the cross-sections of the rib, and may be regarded as a linear arch, to which the conditions governing the equilibrium of the rib are equally applicable.

Let AA' be any cross-section of the rib. The segment $AA'P$ is kept in equilibrium by the external forces which act upon it, and by the molecular action at AA'.

The external forces are reducible to a single force at C and to a couple of which the moment M is the algebraic sum of the moments with respect to C of all the forces on the right of C.

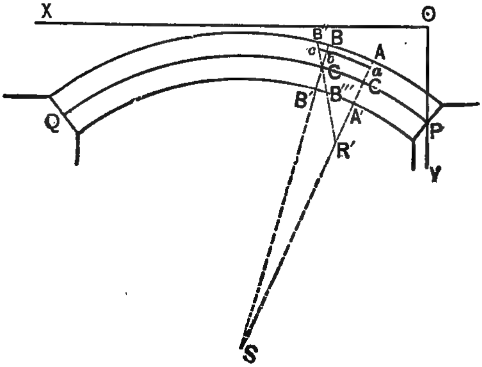


FIG. 934.

The single force at C may be resolved into a component T along the neutral axis, and a component S in the plane AA'. The latter has very little effect upon the curvature of the neutral axis, and may be disregarded as compared with M.

Before deformation let the consecutive cross-sections BB' and AA' meet in R; R is the centre of curvature of the arc CC' of the neutral axis.

After deformation it may be assumed that the plane AA' remains unchanged, but that the plane BB' takes the position $B''B'''$. Let AA' and $B''B'''$ meet in R'; R' is the centre of curvature of the arc CC' *after* deformation.

Let abc be any layer at a distance z from C.

Let $CC'=ds$, $CR=R$, $CR'=R'$, and let da be the sectional area of the layer abc.

By similar figures,

$$\frac{ac}{ds}=\frac{R'+z}{R'} \quad \text{and} \quad \frac{ab}{ds}=\frac{R+z}{R}.$$

Therefore $$bc = ac - ab = ds \cdot z\left(\frac{1}{R'} - \frac{1}{R}\right).$$

The tensile stress in *abc*

$$= Eda\frac{bc}{ab} = Eda\frac{z}{ab}\left(\frac{1}{R'} - \frac{1}{R}\right)ds$$

$$= z\left(\frac{1}{R'} - \frac{1}{R}\right)Eda, \text{ very nearly.}$$

The moment of this stress with respect to C

$$= z^2\left(\frac{1}{R'} - \frac{1}{R}\right)Eda.$$

Hence the moment of resistance at AA'

$$= \int z^2\left(\frac{1}{R'} - \frac{1}{R}\right)Eda = E\left(\frac{1}{R'} - \frac{1}{R}\right)\int z^2 da,$$

the integral extending over the whole of the section.

Therefore $$M = EI\left(\frac{1}{R'} - \frac{1}{R}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, the effect of the force T is to lengthen or shorten the element CC', so that the plane BB' will receive a motion of translation, but the position of R' is practically unaltered.

Corollary 1. Let A be the area of the section AA'.

The total unit stress in the layer *abc*

$$= p = \frac{T}{A} \pm \frac{Mz}{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the sign being *plus* or *minus* according as M acts towards or from the edge of the rib under consideration.

From this expression may be deduced (1) the position of the point at which the intensity of the stress is a maximum for any given distribution of the load; (2) the distribution of the load that makes the intensity an absolute maximum; (3) the value of the intensity.

Cor. 2. Let w be the total intensity of the vertical load per horizontal unit of length.

Let w_1 be the portion of w which produces only a direct compression.

Let H be the horizontal thrust of the arch.

Let P be the total load between the crown and AA' which produces compression.

Refer the rib to the horizontal OX and the vertical OPY as the axes of x and y respectively.

Let x, y be the coordinates of C.

Then $$P=H\frac{dy}{dx}; \quad \text{but} \quad dP=w_1dx.$$

Therefore $$w_1=H\frac{d^2y}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and also $$T=H\frac{ds}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

14. General Equations.

Let l be the span of the arch.

Let x, y be the coordinates of the point C *before* deformation.

Let x', y' be the coordinates of the point C *after* deformation.

Let θ be the angle between tangent at C and OX *before* deformation.

Let θ' be the angle between tangent at C and OX *after* deformation.

Let ds be the length of the element CC' *before* deformation.

Let ds' be the length of the element CC' *after* deformation.

Effect of flexure. $\frac{d\theta'}{ds'}=\frac{1}{R'}$ and $\frac{d\theta}{ds}=\frac{1}{R}$. Hence

$$\frac{M}{EI}=\frac{1}{R'}-\frac{1}{R}=\frac{d\theta'}{ds'}-\frac{d\theta}{ds}=\frac{d\theta'-d\theta}{ds}, \text{ very nearly.}$$

Let i be the change of slope at C. Then

$$di=d\theta-d\theta'=\frac{Mds}{EI}=\frac{M}{EI}\frac{ds}{dx}dx.$$

Integrating, $$i'=\theta-\theta'=i_0+\int_0^x \frac{M}{EI}\frac{ds}{dx}dx, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

i_0 being the *change of slope* at P, and a quantity whose value has yet to be determined.

Again, the general equations of equilibrium at the plane AA' are

$$\frac{d^2M}{dx^2}=\frac{dS}{dx}=-(w-w_1)=-\left(w-H\frac{d^2y}{dx^2}\right) \quad . \quad . \quad . \quad . \quad (6)$$

for the portion w_1, Cor. 2, Art. 13, produces compression only and no shear. Therefore

$$S=S_0-\int_0^x wdx+H\left(\frac{dy}{dx}-\frac{dy_0}{dx_0}\right), \quad . \quad . \quad . \quad . \quad . \quad (7)$$

S_0 being the still undetermined vertical component of the shear at P, and $\frac{dy_0}{dx_0}$ the slope at P. Also,

$$M=M_0+S_0x-\int_0^x\int_0^x wdx^2+H\left(y-y_0-x\frac{dy_0}{dx_0}\right), \quad . \quad . \quad (8)$$

M_0 being the still undetermined bending moment at P.

Equations (5), (6), (7), and (8) contain the *four* undetermined constants H, S_0, M_0, i_0.

Let M_1, S_1, and i_1 be the values of M, S, and i, respectively, at Q.

Equations of Condition.—In practice the ends of the rib are either *fixed* or *free*.

If they are fixed, $i_0=0$; if they are free, $M_0=0$. In either case the number of undetermined constants is reduced to *three*.

If the abutments are immovable, $x_1-l=0$. If the abutments yield, x_1-l must be found by experiment. Let $x_1-l=\mu H$, μ being some coefficient. The *first* equation of condition is

$$x_1-l=0, \quad \text{or} \quad x_1-l=\mu H. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Again, Q is immovable in a vertical direction, and the *second* equation of condition is

$$y_1-y_0=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Again, if the end Q is fixed, $i_1=0$; and if free, $M_1=0$; and the *third* equation of condition is

$$i_1=0, \quad \text{or} \quad M_1=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Substituting in equations (7) and (8) the values of the three constants as determined by these conditions, the shearing force and bending moment may be found at any section of the rib.

Again, $\cos\theta'=\cos(\theta-i)=\cos\theta+i\sin\theta,$

$$\sin\theta'=\sin(\theta-i)=\sin\theta-i\cos\theta.$$

Therefore $\dfrac{dx'}{ds'}=\dfrac{dx}{ds}+i\dfrac{dy}{ds}$ and $\dfrac{dy'}{ds'}=\dfrac{dy}{ds}-i\dfrac{dx}{ds}$. . . . (12)

Hence, approximately,

$$\frac{d}{ds}(x'-x)=i\frac{dy}{ds} \quad \text{and} \quad \frac{d}{ds}(y'-y)=-i\frac{dx}{ds}.$$

Thus, if X and Y are respectively the horizontal and vertical displacements,

$$\frac{dX}{ds}=i\frac{dy}{ds} \quad \text{and} \quad \frac{dY}{ds}=-i\frac{dx}{ds},$$

or

$$\frac{dX}{dy}=i=-\frac{dY}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

15. Effect of T and of a Change of $t°$ in the Temperature.

$$ds'=ds\left(1-\frac{T}{EA}\right).$$

Also, if there is a change from the mean of $t°$ in the temperature, the length $ds\left(1-\dfrac{T}{EA}\right)$ must be multiplied by $(1\pm\epsilon t)$, ϵ being the coefficient of linear expansion. Hence

$$ds'=ds\left(1-\frac{T}{EA}\right)(1\pm\epsilon t,$$

$$=ds\left(1-\frac{T}{EA}\pm\epsilon t\right), \text{ approximately. } . \qquad (14)$$

By equations (12),

$$dx' = (dx + idy)\frac{ds'}{ds} = (dx + idy)\left(1 - \frac{T}{EA} \pm \varepsilon t\right)$$

and

$$dy' = (dy - idx)\frac{ds'}{ds} = (dy - idx)\left(1 - \frac{T}{EA} \pm \varepsilon t\right).$$

Therefore

$$dX = d(x' - x) = idy - \left(\frac{T}{EA} \mp \varepsilon t\right) dx,$$

and

$$dY = d(y' - y) = -idx - \left(\frac{T}{EA} \mp \varepsilon t\right) dy, \text{ approximately.}$$

Hence

$$X = x' - x = \int_0^x i\frac{dy}{dx}dx - \int_0^x \left(\frac{T}{EA} \mp \varepsilon t\right) dx \quad . \quad . \quad . \quad (15)$$

and

$$Y = y' - y = -\int_0^x idx - \int_0^x \left(\frac{T}{EA} \mp \varepsilon t\right)\frac{dy}{dx}dx. \quad . \quad . \quad (16)$$

Note.—A nearer approximation than is given by the preceding results may be obtained as follows:

Let $x+dx$, $y+dy$ be the coordinates of a point very near C *before* deformation.

Let $x'+dx'$, $y'+dy'$ be the coordinates of a point very near C *after* deformation.

Then $$ds^2 = dx^2 + dy^2 \quad \text{and} \quad ds'^2 = dx'^2 + dy'^2$$

and $$ds'^2 - ds^2 = dx'^2 - dx^2 + dy'^2 - dy'^2,$$

or $$(ds' - ds)(ds' + ds) = (dx' - dx)(dx' + dx) + (dy' - dy)(dy' + dy).$$

Hence $$(ds' - ds)ds = (dx' - dx)dx + (dy' - dy)dy, \text{ approximately,}$$

or $$dx' - dx = (ds' - ds)\frac{ds}{dx} - (dy' - dy)\frac{dy}{dx}$$

and $$dy' - dy = (ds' - ds)\frac{ds}{dx}\frac{dx}{dy} - (dx' - dx)\frac{dx}{dy}.$$

Hence, by equations (12) and (14),

$$dx' - dx = i\frac{dy}{dx}dx - \frac{T}{EA}\left(\frac{ds}{dx}\right)^2 dx \pm \varepsilon t\left(\frac{ds}{dx}\right)^2 dx$$

and $$dy'-dy=-idx-\frac{T}{EA}\left(\frac{ds}{dx}\right)^2\frac{dx}{dy}dx\pm\epsilon t\left(\frac{ds}{dx}\right)^2\frac{dx}{dy}dx.$$

Integrating,

$$x'-x=\int_0^x i\frac{dy}{dx}dx-\int_0^x\frac{T}{EA}\left(\frac{ds}{dx}\right)^2dx\pm\epsilon t\int_0^x\left(\frac{ds}{dx}\right)^2dx$$

and $$y'-y=-\int_0^x idx-\int_0^x\frac{T}{EA}\left(\frac{ds}{dx}\right)^2\frac{dx}{dy}dx\pm\epsilon t\int_0^x\left(\frac{ds}{dx}\right)^2\frac{dx}{dy}dx$$

These equations are to be used instead of eqs. (15) and (16), the remainder of the calculations being computed precisely as before.

The following problems are in the main the same as those given in Rankine's *Civil Engineering*, 20th edition.

Ex. 12. *Rib of Uniform Stiffness.*—Let the *depth* and *sectional form* of the rib be uniform, and let its breadth at each point vary as the *secant* of the inclination of the tangent at the point to the horizontal.

Let A_1, I_1 be the sectional area and moment of inertia at the crown.

Let A, I be the sectional area and moment of inertia at any point C, Fig. 934.

Then

$$A=A_1\sec\theta=A_1\frac{ds}{dx}. \quad \ldots\ldots\ldots\ldots \quad (17)$$

Also, since the moments of inertia of similar figures vary as the breadth and as the cube of the depth, and since the depth in the present case is constant,

$$I=I_1\sec\theta=I_1\frac{ds}{dx}. \quad \ldots\ldots\ldots\ldots \quad (18)$$

Again, $\frac{T}{A}=\frac{H\sec\theta}{A_1\sec\theta}=\frac{H}{A_1}$, and the intensity of the thrust is constant throughout.

Hence eqs. (5), (15), and (16), respectively, become

$$i=i_0=\frac{1}{EI_1}\int_0^x Mdx; \quad \ldots\ldots\ldots\ldots \quad (19)$$

$$x'-x=\int_0^x i\frac{dy}{dx}dx-\frac{H}{EA_1}x\pm\epsilon tx; \quad \ldots\ldots\ldots \quad (20)$$

$$y'-y=-\int_0^x idx-\left(\frac{H}{EA_1}\mp\epsilon t\right)(y-y_0). \quad \ldots\ldots \quad (21)$$

Eq. (19) shows that the deflection at each point of the rib is the same as that at corresponding points of a straight horizontal beam of a uniform section equal to that of the rib at the crown, and acted upon by the same bending moments.

Ribs of uniform stiffness are not usual in practice, but the formulæ deduced in the present article may be applied without sensible error to flat segmental ribs of uniform section.

Ex. 13. *Parabolic rib of uniform depth and stiffness, with rolling load; the ends fixed in direction; the abutments immovable.*

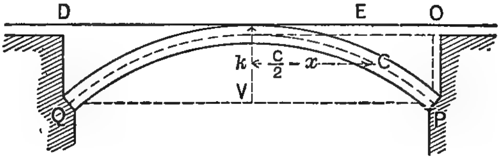


FIG. 935

Let the axis of x be a tangent to the neutral curve at its summit. Let k be the rise of the curve. Let x, y be the coordinates at any point C with respect to O. Then

$$y = \frac{4k}{l^2}\left(\frac{l}{2} - x\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and

$$\frac{dy}{dx} = -\frac{8k}{l^2}\left(\frac{l}{2} - x\right), \quad \frac{dy_0}{dx_0} = -\frac{4k}{l}, \quad \frac{dy_1}{dx_1} = \frac{4k}{l}, \quad \frac{d^2y}{dx^2} = \frac{8k}{l^2}. \quad . \quad . \quad (23)$$

Let w be the dead load per horizontal unit of length.
" w' " " live " " " " " "

Let the live load cover a length DE, $=rl$, of the span.

Denote by (A) formulæ relating to the unloaded division OE, and by (B) formulæ relating to the loaded division DE.

Eqs. (7) and (8), respectively, become

(A) $$S = S_0 + \left(\frac{8kH}{l^2} - w\right)x; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

(B) $$S = S_0 + \left(\frac{8kH}{l^2} - w\right)x - w'\{x - (1-r)l\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

(A) $$M = M_0 + S_0 x + \left(\frac{8kH}{l^2} - w\right)\frac{x^2}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

(B) $$M = M_0 + S_0 x + \left(\frac{8kH}{l^2} - w\right)\frac{x^2}{2} - \frac{w'}{2}\{x - (1-r)l\}^2. \quad . \quad . \quad . \quad (27)$$

Since the ends are fixed,

$$i = 0 = i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Hence by eqs. (19) and (26),

$$\text{(A)} \qquad i=-\frac{1}{EI_1}\left\{M x+S_0\frac{x^2}{2}+\left(\frac{8kH}{l^2}-w\right)\frac{x^3}{6}\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

and by eqs. (19) and (27),

$$\text{(B)} \qquad i=-\frac{1}{EI_1}\left\{M_0x+S_0\frac{x^2}{2}+\left(\frac{8kH}{l^2}-w\right)\frac{x^3}{6}-\frac{w'}{6}\{x-(1-r)l\}^3\right\}. \quad . \quad . \qquad (30)$$

When $x=l$, $i=i_1=0$, and therefore, by the last equation,

$$0=M_0+S_0\frac{l}{2}+\left(\frac{8kH}{l^2}-w\right)\frac{l^2}{6}-\frac{w'}{6}r^3l^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

Again, let $i=\frac{dv}{dx}$. Then

$$\int_0^l i\frac{dy}{dx}dx=\int_0^l\frac{dv}{dx}\frac{dy}{dx}dx=i_1\frac{dy}{dx}-\int_0^l v\frac{d^2y}{dx^2}dx.$$

But $i_1=0$, and $\frac{d^2y}{dx^2}=\frac{8k}{l^2}$. Hence,

$$\int_0^l i\frac{dy}{dx}dx=-\frac{8k}{l^2}\int_0^l vdx=-\frac{8k}{l^2}\int_0^l\int_0^x idx^2. \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

By the conditions of the problem $x'-x$ and $y'-y$ are each zero at Q. Hence equations (20) and (21), respectively, become

$$0=-\frac{8k}{l^2}\int_0^l\int_0^x idx^2-\left(\frac{H}{EA}\mp\varepsilon t\right)l; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (33)$$

$$0=-\int_0^l idx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

Substitute in eqs. (33) and (34) the value of i given by eq. (30), and integrate between the limits 0 and l. Then

$$0=-\frac{8k}{l^2}\frac{1}{EI_1}\left\{M_0\frac{l^3}{6}+S_0\frac{l^4}{24}+\left(\frac{8kH}{l^2}-w\right)\frac{l^5}{120}-w'r^5\frac{l^5}{120}\right\}-\left(\frac{H}{EA_1}\mp\varepsilon t\right)l,$$

and

$$0=-\frac{1}{EI_1}\left\{\frac{M_0l^2}{2}+\frac{S_0l^3}{6}+\left(\frac{8kH}{l^2}-w\right)\frac{l^4}{24}-w'r^4\frac{l^4}{24}\right\},$$

which may be written

$$0=M_0+S_0\frac{l}{4}+\left(\frac{8kH}{l^2}-w\right)\frac{l^2}{20}-w'r^5\frac{l^2}{20}+\frac{3}{4}\left(\frac{H}{EA}\mp\varepsilon t\right)\frac{EI}{k} \quad . \quad . \quad . \quad . \quad . \qquad (35)$$

and

$$0=M_0+S_0\frac{l}{3}+\left(\frac{8kH}{l^2}-w\right)\frac{l^2}{12}-w'r^4\frac{l^2}{12}. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Hence, by eqs. (31), (35), (36),

$$S_0=\frac{wl}{2}+w'lr^2\left(1-\frac{r}{2}\right)-4\frac{k}{l}H; \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$M_0=-\frac{wl^2}{12}-\frac{w'l^2r^3}{3}\left(1-\frac{3}{4}r\right)+\frac{2}{3}kH; \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$H=\frac{l^2\left\{\frac{w}{8}+w'\left(\frac{5}{4}r^3-\frac{15}{8}r^4+\frac{3}{4}r^5\right)\pm\frac{45}{4}et\frac{EI_1}{k}\right\}}{k\left(1+\frac{45}{4}\frac{I_1}{A_1k^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

When $x=l$, $M=M_1$, and $S=S_1$.

Hence, by eqs. (25) and (27),

$$S_1=S_0+\left(\frac{8kH}{l^2}-w\right)l-w'rl,$$

and

$$M_1=M_0+S_0l+\left(\frac{8kH}{l^2}-w\right)\frac{l^2}{2}-\frac{w'r^2l^2}{2}.$$

Substituting in these equations the values of S_0, M_0, given above, we have

$$S_1=-\frac{wl}{2}-w'rl\left(1-r^2+\frac{r^3}{2}\right)+\frac{4kH}{l}, \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and

$$M_1=-\frac{wl^2}{12}-w'l^2r^2\left(\frac{1}{2}-\frac{2}{3}r+\frac{r^2}{4}\right)+\frac{2}{3}kH. \quad . \quad . \quad . \quad . \quad (41)$$

To find the greatest intensity of stress, etc.—The intensity of the stress due to direct compression $=\frac{T}{A}=\frac{H}{A_1}$.

The intensity of the stress in the outside layers of the rib due to *bending* is the same as that in the outside layers of a horizontal beam of uniform section A_1, acted upon by the same moments as act on the rib, for the deflections of the beam and rib are equal at every point (eq. (19)). Also, since the rib is fixed at both ends, the bending moment due to that portion of the load which produces flexure is a *maximum* at the loaded end, i.e., at Q. Hence the maximum intensity of stress (p_1) occurs at Q, and $p_1=\frac{H}{A_1}\pm M_1\frac{z_1}{I_1}$, z_1 being the distance of the layers from the neutral axis.

H and M_3 are both functions of r, and therefore p_1 is an absolute maximum when

$$\frac{dp_1}{dr}=0=\frac{1}{A}\frac{dH}{dr}\pm\frac{z_1}{I_1}\frac{dM_1}{dr}. \qquad (42)$$

But

$$\frac{dH}{dr}=\frac{15}{4}\frac{w'l^2}{k}\frac{r^2(1-r)^2}{1+\frac{45}{4}\frac{I_1}{Ak^2}}, \qquad (43)$$

and

$$\frac{dM_1}{dr}=-w'l^2r(1-r)^2+\frac{2}{3}k\frac{dH}{dr}. \qquad (44)$$

Hence p_1 is an absolute maximum when

$$0=w'l^2r(1-r)^2\left\{\frac{15}{4}\frac{r\left(\frac{1}{A_1}\pm\frac{2}{3}\frac{kz_1}{I_1}\right)}{k\left(1+\frac{45}{4}\frac{I_1}{A_1k^2}\right)}\mp\frac{z_1}{I_1}\right\}.$$

The roots of this equation are

$$\left.\begin{aligned}&r=1\\ \text{and}\qquad &r=\pm\frac{2}{5}\frac{1+\frac{45}{4}\frac{I_1}{A_1k^2}}{\frac{3}{2}\frac{I_1}{A_1z_1k}\pm1}.\end{aligned}\right\} \qquad (45)$$

$r=1$ makes $\frac{d^2p}{dr^2}$ *zero*, so that the maximum value of p corresponds to one of the remaining roots.

Thus,

$$\text{the } \textit{max. thrust}=\frac{1}{A_1}\left(H+\frac{A_1z_1}{I_1}M_1\right)=p_1'. \qquad (46)$$

and

$$\text{the } \textit{max. tension}=\frac{1}{A_1}\left(-H+\frac{A_1z_1}{I_1}M_1\right)=p_1'', \qquad (47)$$

the values of H and M_3 being found by substituting in eqs. (39) and (41)

$$\left.\begin{aligned}&r=\frac{2}{5}\frac{1+\frac{45}{4}\frac{I_1}{A_1k^2}}{1-\frac{3}{2}\frac{I_1}{A_1z_1k}}\\ \text{or}\qquad &r=\frac{2}{5}\frac{1+\frac{45}{4}\frac{I_1}{A_1k^2}}{1+\frac{3}{2}\frac{I_1}{A_1z_1k}},\end{aligned}\right\} \qquad (48)$$

according as the stress is a thrust or a tension.

If eq. (47) gives a *negative* result, there is no tension at any point of the rib.

Note.—The moment of inertia may be expressed in the form

$$I = qz_1^2A_1,$$

q being a coefficient depending upon the *form* of the section.

Hence

$$\text{the maximum intensity of stress} = \frac{1}{A_1}\left(\pm H + \frac{M_1}{qz_1}\right). \quad . \quad . \quad . \quad (49)$$

Cor. 1.—If the depth of the rib is small as compared with k, the fraction $\frac{z_1}{k}$ will be a small quantity, and the maximum intensity of stress will approximately correspond to $r=\frac{2}{5}$. The denominator in eq. (39) may be taken to be k, and it may be easily shown that the values of p_1', p_1'' are

$$p_1' = \frac{1}{A_1}\left\{\frac{wl^2}{8}\left(\frac{1}{k} + \frac{15}{2}\frac{z_1}{k^2}\right) \mp \frac{5}{4}\frac{\epsilon tEI_1}{qz_1k} + \frac{54}{3125}\frac{w'l^2}{qz_1}\right\}; \quad . \quad . \quad . \quad (50)$$

$$p_1'' = \frac{1}{A_1}\left\{\frac{wl^2}{8}\left(-\frac{1}{k} + \frac{15}{2}\frac{z_1}{k^2}\right) \mp \frac{5}{4}\frac{\epsilon tEI_1}{qz_1k} + \frac{54}{3125}\frac{w'l^4}{qz_1}\right\}. \quad . \quad . \quad . \quad (51)$$

Cor. 2.—If the numerator in eqs. (48) is greater than the denominator, then r must be *unity*. Hence, by eq. (39) and putting

$$b = 1 + \frac{45}{4}\frac{I_1}{A_1k^2} = 1 + \frac{45}{4}\frac{qz_1^2}{k^2},$$

$$H = \frac{l^2}{8}\frac{w+w'}{bk} \pm \frac{45}{4}\frac{\epsilon tEI_1}{bk^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

and by eqs. (38) and (41),

$$M_1 = M_0 = \frac{l^2}{12}(w+w')\frac{1-b}{b} \pm \frac{15}{2}\frac{\epsilon tEI_1}{bk} = -\frac{15}{16}l^2\frac{w+w'}{b}q\frac{z_1^2}{k^2} \pm \frac{5}{4}\frac{\epsilon t}{b}\frac{EI_1}{k}. \quad . \quad (53)$$

Thus, p_1', p_1'' can be found by substituting these values of H and M_1 in eqs. (46) and (47).

Ex. 14.—*Parabolic Rib of Uniform Stiffness, hinged at the Ends.*

Let the rib be similar to that of the preceding article.

Since the ends are hinged, $M_0 = 0 = M_1$, while i is an undetermined constant.

The following equations apply:

$$\text{(A)} \qquad S=S_0+\left(\frac{8kH}{l^2}-w\right)x; \quad \ldots \ldots \ldots \ldots \quad (54)$$

$$\text{(B)} \qquad S=S_0+\left(\frac{8kH}{l^2}-w\right)x-w'\{x-(1-r)l\}; \quad \ldots \ldots \quad (55)$$

$$\text{(A)} \qquad M=S_0x+\left(\frac{8kH}{l^2}-w\right)\frac{x^2}{2}; \quad \ldots \ldots \ldots \quad (56)$$

$$\text{(B)} \qquad M=S_0x+\left(\frac{8kH}{l^2}-w\right)\frac{x^2}{2}-\frac{w'}{2}\{x-(1-r)l\}^2. \quad \ldots \quad (57)$$

$$i=i_0-\frac{1}{EI}\left\{S_0\frac{x^2}{2}+\left(\frac{8kH}{l^2}-w\right)\frac{x^3}{6}\right\}; \quad \ldots \ldots \ldots \quad (58)$$

$$i=i_0-\frac{1}{EI}\left\{S_0\frac{x^2}{2}+\left(\frac{8kH}{l^2}-w\right)\frac{x^3}{6}-\frac{w'}{6}\{x-(1-r)l\}^3\right\}. \quad \ldots \quad (59)$$

Assume that the horizontal and vertical displacements of the loaded end are *nil.*

Substitute in eqs. (20) and (21) the value of i given by eq. (59). Integrate and reduce, neglecting the term involving the temperature. Then

$$0=i_0-\frac{1}{EI_1}\left\{S_0\frac{l^2}{12}+\left(\frac{8kH}{l^2}-w\right)\frac{l^3}{60}-w'l^3\frac{r^5}{60}\right\}-\frac{H}{4}\frac{1}{EA_1}\frac{l}{k}; \quad \ldots \quad (60)$$

$$0=i\ -\frac{1}{EI_1}\left\{S_0\frac{l^2}{6}+\left(\frac{8kH}{l^2}-w\right)\frac{l^3}{24}-w'l^3\frac{r^4}{24}\right\}. \quad \ldots \ldots \ldots \quad (61)$$

From (57), since $M_1=0$,

$$0=S_0+\left(\frac{8kH}{l^2}-w\right)\frac{l}{2}-w'l\frac{r^2}{2}. \quad \ldots \ldots \ldots \quad (62)$$

Equations (60), (61), and (62) are the equations of condition. Subtract (61) from (60). Then

$$0=\frac{1}{EI_1}\left\{S_0\frac{l^2}{12}+\left(\frac{8kH}{l^2}-w\right)\frac{l^3}{40}-w'l^3\left(\frac{r^4}{24}-\frac{r^5}{60}\right)\right\}-\frac{H}{4}\frac{1}{EA_1}\frac{l}{k},$$

which may be written

$$0=S_0+\left(\frac{8kH}{l^2}-w\right)\frac{3l}{10}-w'l\left(\frac{r^4}{2}-\frac{r^5}{5}\right)-3H\frac{I_1}{A_1}\frac{1}{kl}. \quad \ldots \quad (63)$$

Subtract (63) from (62). Then

$$0=\left(\frac{8kH}{l^2}-w\right)\frac{l}{5}-w'l\left(\frac{r^2}{2}-\frac{r^4}{2}+\frac{r^5}{5}\right)+3H\frac{I_1}{A_1}\frac{1}{kl}. \quad . \quad . \quad . \quad (64)$$

Hence,

$$H=\frac{l^2\left\{w+\frac{w'}{2}(5r^2-5r^4+2r^5)\right\}}{8k\left(1+\frac{15}{8}\frac{I_1}{A_1}\frac{1}{k^2}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Eliminating S_0 between (61) and (62),

$$i_0=-\frac{1}{EI}\left\{\left(\frac{8kH}{l^2}-w\right)\frac{l^3}{24}-w'l^3\left(\frac{r^2}{12}-\frac{r^4}{24}\right)\right\}. \quad . \quad . \quad . \quad . \quad (66)$$

Also, by (55),

$$S_1=S_0+\left(\frac{8kH}{l^2}-w\right)l-w'rl=-P, \text{ suppose}. \quad . \quad . \quad . \quad . \quad (67)$$

Eliminating S_0 between (62) and (67),

$$-P=S_1=\left(\frac{8kH}{l^2}-w\right)\frac{l}{2}-w'l\left(r-\frac{r^2}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Eqs. (62), (65), (66), and (68) give the values of H, S_0, S_1, and i_0.

Again, the maximum bending moment M' occurs at a point given by $\frac{dM}{dx}=0$ in (57), i.e.,

$$0=S_0+\left(\frac{8kH}{l^2}-w\right)x-w'\{x-(1-r)l\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Subtract (69) from (67). Then

$$-P_1=S_1=\left(\frac{8kH}{l^2}-w\right)(l-x)-w'(l-x).$$

Hence, the distance from the loaded end of the point at which the bending moment is greatest is

$$l-x=\frac{P}{w+w'-\frac{8kH}{l^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Substitute this value of x in (57), and, for convenience, put

$$w+w'-\frac{8kH}{l^2}=m.$$

Then

$$M' = S_0\left(l - \frac{P}{m}\right) + \frac{w'-m}{2}\left(l - \frac{P}{m}\right)^2 - \frac{w'}{2}\left(-\frac{P}{m} + rl\right)^2$$

$$= l\left(S_0 + \frac{w'-m}{2}l - \frac{w'}{2}r^2 l\right) - \frac{P}{m}\{S_0 + (w'-m)l - w'rl\} + \frac{P^2}{m^2}\left(\frac{w'-m}{2} - \frac{w'}{2}\right).$$

But by (62), $0 = S_0 + \frac{w'-m}{2}l - \frac{w'}{2}r^2 l$. Therefore,

$$M' = l(0) - \frac{P}{m}(-P) + \frac{P^2}{m^2}\left(-\frac{m}{2}\right) = \frac{P^2}{2m}.$$

Hence, M', the *maximum bending moment*,

$$= \frac{P^2}{2\left(w + w' - \frac{8kH}{l^2}\right)}. \qquad (71)$$

As before, the greatest stress (a *thrust*)

$$= \frac{1}{A_1}\left(H + \frac{A_1 z_1}{I_1}M'\right) = p_1', \qquad (72)$$

and the value of r which makes p_1' an *absolute* maximum is given by $\frac{dp_1'}{dr} = 0$. But by (71), M' involves r^{10} in the numerator and r^5 in the denominator, so that $\frac{dp_1'}{dr} = 0$ will be an equation involving r^{14}.

One of its roots is $r = 1$, which generally gives a *minimum* value of p_1'. Dividing by $r-1$, the equation reduces to one of the *thirteenth* order, but is still far too complex for use. It is found, however, that $r = \frac{1}{2}$ gives a *close approximation* to the *absolute* maximum thrust.

With this value of r, and, for convenience, putting.

$$1 + \frac{15}{8}\frac{I_1}{A_1}\frac{1}{k^2} = n,$$

By (65),

$$H = \frac{l^2}{8kn}\left(w + \frac{w'}{2}\right). \qquad (73)$$

By (62),

$$S_0 = \frac{l}{2}\left\{\left(w + \frac{w'}{2}\right)\frac{n-1}{n} - \frac{w'}{4}\right\}. \qquad (74)$$

By (68),

$$-S_1 = P = \frac{l}{2}\left\{\left(w + \frac{w'}{2}\right)\frac{n-0}{n} + \frac{w'}{4}\right\}. \qquad (75)$$

By (66 ,

$$\Delta\theta_0=\frac{l^3}{24EI}\left\{\left(w+\frac{w'}{2}\right)\frac{n-1}{n}-\frac{w'}{16}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

By (70),

$$l-x=\frac{\frac{l}{2}\left\{\left(w+\frac{w'}{2}\right)\frac{n-1}{n}+\frac{w'}{4}\right\}}{\left(w+\frac{w'}{2}\right)\frac{n-1}{n}+\frac{w'}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

By (71),

$$M'=\frac{\frac{l^2}{8}\left\{\left(w+\frac{w'}{2}\right)\frac{n-1}{n}+\frac{w'}{4}\right\}^2}{\left(w+\frac{w'}{2}\right)\frac{n-1}{n}+\frac{w'}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

Note.—If the rib is merely supported at the ends but not fixed, the *horizontal* displacement of the loaded end may be represented by μH (Art. 9). Thus the term $-\mu H$ must be added to the right-hand side of eq. (15).

Ex. 15.—*Parabolic Rib of Uniform Stiffness, hinged at the Crown and also at the Ends.*—In this case $M=0$ at the crown, which introduces a *fourth* equation of condition.

By (57),

$$0=S_0\frac{l}{2}+\left(\frac{8kH}{l^2}-w\right)\frac{l^2}{8}-\frac{w'l^2}{2}\left(-\frac{1}{2}+r\right)^2,$$

which may be written

$$0=S_0+\left(\frac{8kH}{l^2}-w\right)\frac{l}{4}-w'l\left(r^2-r+\frac{1}{4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

Eliminating S_0 between (79) and (62).

$$\frac{8kH}{l^2}-w=w'(-2r^2+4r-1).$$

Hence,

$$H=\frac{l^2}{8k}\{w-w'(2r^2-4r+1)\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

By (79),

$$S_0=\frac{w'l}{2}(3r^2-4r+1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

By (68),

$$P=S_1=\frac{w'l}{2}(r-1)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

By (66),

$$i_0=\frac{w'l}{24EI_1}(1-4r+4r^2-r^4). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

By (70) and (82),

$$l-x=\frac{\frac{w'l}{2}(r-1)^2}{2w'(r-1)^2}=\frac{l}{4}. \quad \ldots\ldots\ldots \quad (84)$$

By (71),

$$M'=\frac{w'l^2}{16}(r-1)^2. \quad \ldots\ldots\ldots\ldots \quad (85)$$

When $r=\frac{1}{2}$,

$$\left.\begin{array}{l} H=\frac{l^3}{8k}\left(w+\frac{w'}{2}\right), \quad S_0=-\frac{wl}{8}, \quad P=-S_1=\frac{w'l}{8}, \\ i_0=-\frac{1}{384}\frac{w'l^3}{EI_1}, \qquad \text{and} \qquad M'=\frac{w'l^2}{64}. \end{array}\right\} \quad \ldots\ldots \quad (86)$$

These results agree with those of (73) to (78), if $n=1$.

In general, when $n=1$,

$$w+\frac{w'}{2}(5r^2-5r^4+2r^5)=w-w'(2r^2-4r+1),$$

by (65) and (80). Hence,

$$2r^5-5r^4+9r^2+8r+2=0=(2r-1)(r-1)^2(r^2-2),$$

and the roots are $r=\frac{1}{2}$, $r=1$, $r=\pm\sqrt{2}$.

Hence, $n=1$ only renders the expressions in (86) identical with the corresponding expressions of the preceding article when $n=\frac{1}{2}$ or 1.

Again the intensity of thrust is greatest at the outer flange of the loaded and the inner flange of the unloaded half of the rib, and is

$$=\frac{l^2}{8A_1}\left\{\frac{z_1}{I_1}\frac{w'}{8}+\frac{1}{k}\left(w+\frac{w'}{2}\right)\right\}.$$

The intensity of tension is greatest at the inner flange of the loaded and the outer flange of the unloaded half of the rib, and is

$$=\frac{l^2}{8A_1}\left\{\frac{z_1}{I_1}\frac{w'}{8}-\frac{1}{k}\left(w+\frac{w'}{2}\right)\right\}.$$

The *greatest total horizontal thrust* occurs when $r=1$, and its value is

$$\frac{l^2}{8k}(w+w').$$

16. Maximum Deflection of an Arched Rib.—The deflection must necessarily be a maximum at a point given by $i=0$. Solve for x and substitute in (16) to find the deflection $y'-y$; the deflection is an *absolute* maximum when $\frac{d}{dr}(y'-y)=0$. The resulting equation involves r to a high power, and is too intricate to be of use. It has been found by trial, however, that in all ordinary cases the absolute maximum deflection occurs at the middle of the rib, when the live load covers its whole length, i.e., when $x=\frac{l}{2}$, and $r=1$.

Ex. 16.—*Rib of Ex.* 13. For convenience, put $1+\frac{45}{4}\frac{I_1}{Ak^2}=s$.

Then, by (39),

$$H=\frac{l^2}{8k}\frac{w+w'}{s}\pm\frac{15}{8}\frac{\varepsilon t}{s}\frac{EI_1}{k^2}. \qquad (87)$$

By (38) and (41),

$$-M_0=\frac{l^2}{12}(w+w')\frac{s-1}{s}\mp\frac{5}{4}\frac{\varepsilon t}{s}\frac{EI_1}{k}=-M_1. \qquad (88)$$

By (36) and (38),

$$S_0=-6\frac{M_0}{l}. \qquad (89)$$

By (30), (38), (89),

$$i=-\frac{1}{EI_1}\left(M_0x-3M_0\frac{x^2}{l}+2M_0\frac{x^3}{l^2}\right). \qquad (90)$$

Hence, the maximum deflection

$$=-\int_{\frac{l}{2}}^{0}idx=-\frac{M_0}{EI}\int_0^x\left(x-3\frac{x^2}{l}+2\frac{x^3}{l^2}\right)dx=-\frac{M_0}{EI}\frac{l^2}{32}$$

$$=\frac{l^4}{384}\frac{w+w'}{EI_1}\frac{s-1}{s}\mp\frac{5}{128}\frac{\varepsilon t}{s}\frac{l^2}{k}=d_3,\text{ suppose.} \qquad (91)$$

The central deflection d_2 of a uniform straight horizontal beam of the same span, of the same section as the rib at the crown, and with its ends fixed, is

$$d_2=\frac{l^4}{384}\frac{w+w'}{EI_1}. \qquad (92)$$

Hence, neglecting the term involving the temperature,

$$d_1=\frac{s-1}{s}d_2. \qquad (93)$$

Ex. 17.—*Rib of Ex.* 14.

By (65),

$$H=\frac{l}{8k}\frac{w+w'}{n}. \qquad (94)$$

By (66) and (62)

$$i_0=\frac{l^3}{24EI_1}(w+w')\frac{n-1}{n}=\frac{S_0l^2}{12EI}. \qquad (95)$$

By (30), (94), and (95),

$$i=\frac{S_0}{EI_1}\left(\frac{l^2}{12}-\frac{x^2}{2}+\frac{x^3}{3l}\right). \qquad (96)$$

Hence, the maximum deflection

$$=\frac{S_0}{EI_1}\int_0^{\frac{l}{2}}\left(\frac{l^2}{12}-\frac{x^2}{2}-\frac{x^3}{3l}\right)dx=\frac{5}{384}\frac{l^4}{EI}(w+w')\frac{n-1}{n}=d_1'. \qquad (97)$$

If the ends of the beam in Case I are free, its central deflection

$$=\frac{5}{384}\frac{l^4(w+w')}{EI}=d_2',$$

and

$$d_1'=\frac{n-1}{n}d_2'. \qquad (98)$$

Thus, the deflection of the arched rib in both cases is less than that of the beam.

Ex. 18.—*Arched Rib of Uniform Stiffness fixed at the Ends and connected at the Crown with a Horizontal Distributing Girder.*—The load is transmitted to the rib by vertical struts so that the vertical displacements of corresponding points of the rib and girder are the same. The horizontal thrust in the loaded is not necessarily equal to that in the unloaded division of the rib, but the excess of the thrust in the loaded division will be borne by the distributing girder, if the rib and girder are connected in such a manner that the horizontal displacement of each at the crown is the same.

The formulæ of Ex. 13 are applicable in the present case with the modification that I_1 is to include the moment of inertia of the girder.

The maximum thrust and tension in the rib are given by equations (64) and (65).

Let z' be the depth of the girder, A' its sectional area.

$$\text{The greatest thrust in the girder }=\frac{H}{A_1+A'}+\frac{M_1z'}{2EI_1}. \qquad (99)$$

$$\text{The greatest tension in the girder}=\frac{M_1z'}{2EI_1}. \qquad (100)$$

H and M_1 being given by equations (66) and (67), respectively.

The girder must have its ends so supported as to be capable of transmitting a thrust.

17. Stresses in Spandril Posts and Diagonals.—Fig. 936 represents an arch in which the spandril consists of a series of vertical posts and diagonal braces.

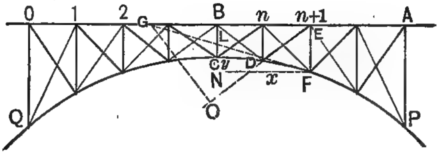


FIG. 936.

Let the axis of a curved rib be a parabola. The arch is then equilibrated under a uniformly distributed load, and the diagonals will be only called into play under a passing load.

Let x, y be the coordinates of any point F of the parabola with respect to the vertex C. Then

$$y=\frac{4k}{l^2}x^2.$$

Let the tangent at F meet CB in L, and the horizontal BE in G.

Let $BC=k'$. Then

$$BL=BC-CL=BC-CN=k'-y.$$

Let N be the total number of panels.

Consider any diagonal ED between the nth and $(n+1)$th posts.

Let w' be the greatest panel *live* load.

The greatest compression in ED occurs when the passing load is concentrated at the first $n-1$ panel-points.

Imagine a vertical section a little on the left of EF.

The portion of the frame on the right of this section is kept in equilibrium by the reaction R at P, and by the stresses in the three members met by the secant plane.

Taking moments about G,

$$D\cdot GE\cos\theta=R\cdot AG,$$

D being the stress in DE, and θ the angle DEF.

Now, $$R=\frac{w'}{2}\frac{n(n-1)}{N}.$$

Also, $$\frac{x+GB}{GB}=\frac{k'+y}{k'-y};\quad GB=\frac{k'x-xy}{2y};$$

and hence

$$GE=GB+x=\frac{k'x+xy}{2y},\quad \text{and}\quad GA=\frac{l}{2}+\frac{k'x-xy}{2y}.$$

Hence $$D=\frac{w'}{2}\frac{n(n-1)}{N}\frac{ly+k'x-xy}{k'x+xy}\sec\theta.$$

The stresses in the counterbraces (shown by dotted lines in the figure) may be obtained in the same manner.

The greatest thrust in $EF=w'+w$.

The greatest tension in $EF=D\cos\theta-w$, w being the *dead* load upon EF.

If the last expression is negative, EF is never in tension.

18. Clerk Maxwell's Method of determining the Resultant Thrusts at the Supports of a Framed Arch.—Let Δs be the change in the length s of any member of the frame under the action of a force P, and let a be the sectional area of the member. Then

$$\pm\frac{P}{Ea}s=\Delta s,$$

the sign depending upon the character of the stress.

Assume that all the members except the one under consideration are perfectly rigid, and let Δl be the alteration in the span l corresponding to Δs. The ratio $\frac{\Delta l}{\Delta s}$ is equal to a constant m, which depends only upon the geometrical form of the frame.

Therefore $$\Delta l=m\Delta s=\pm mP\frac{s}{Ea}.$$

Again, P may be supposed to consist of two parts, viz., f_1 due to a horizontal force H between the springings, and f_2 due to a vertical force V applied at one springing, while the other is firmly secured to keep the frame from turning.

By the principle of virtual velocities,

$$\frac{f_1}{H}=\frac{\Delta l}{\Delta s}=m.$$

Similarly, $\frac{f_2}{V}$ is equal to some constant n, which depends only upon the *form* of the frame. Also

$$P=f_1+f_2=mH+nV.$$

Therefore $$\Delta l= \pm(m^2H+mnV)\frac{s}{Ea}.$$

Hence the total change in l for all the members is

$$\Sigma\Delta l= \pm\Sigma\left(m^2H\frac{s}{Ea}\right)\pm\Sigma\left(mnV\frac{s}{Ea}\right).$$

If the abutments yield, let $\Sigma\Delta l=\mu H$, μ being some coefficient to be determined by experiment. Then

$$H=\frac{\pm\Sigma\left(mnV\frac{s}{Ea}\right)}{\mu\mp\Sigma\left(m^2\frac{s}{Ea}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

If the abutments are immovable, $\Sigma\Delta l$ is zero, and

$$H=-\frac{\Sigma\left(mnV\frac{s}{Ea}\right)}{\Sigma\left(m^2\frac{s}{Ea}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D)$$

V is the same as the corresponding reaction at the end of a girder of the same span and similarly loaded. The required thrust is the resultant of H and V, and the stress in each member may be computed graphically or by the method of moments. In any particular case proceed as follows:

(1) Prepare tables of the values of m and n for each member.

(2) Assume a cross-section for each member, based on a probable assumed value for the resultant of V and H.

(3) Prepare a table of the value of $m^2\frac{s}{Ea}$ for each member, and form the sum $\Sigma\left(m^2\frac{s}{Ea}\right)$.

(4) Determine, separately, the horizontal thrust between the springings due to the loads at the different joints. Thus let v_1, v_2 be the vertical reactions at the right and left supports due to any *one* of these loads. Form the sum $\Sigma\left(mnV\frac{s}{Ea}\right)$, using v_1 for all the members on the right of the load and v_2 for all those on its left. The corresponding thrust may then be found by eq. (C) or eq. (D), and the *total* thrust H is the sum of the thrusts due to all the weights taken separately.

(5) Repeat the process for each combination of live and dead load so as to find the maximum stresses to which any member may be subjected.

(6) If the assumed cross-sections are not suited to these maximum stresses, make fresh assumptions and repeat the whole calculation.

The same method may be applied to determine the resultant tensions at the supports of a framed suspension bridge.

Note.—The formulæ for a parabolic rib may be applied without material error to a rib in the form of a segment of a circle. More exact formulæ may be obtained for the latter in a manner precisely similar to that described in Exs. 13–17, *but the integrations will be much simplified by using polar coordinates, the centre of the circle being the pole.*

EXAMPLES.

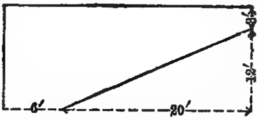


FIG. 937.

1. The arch represented in the figure is constructed of masonry weighing 150 lbs. cf. The span is 40 ft., the rise 8 ft., the depth of masonry above the centre 3 ft., the thickness of the abutments 6 ft. The centre of resistance at the middle point of upper key is 1 ft. below the crown surface. Deduce (*a*) the resultant pressure in the vertical joint at the key, (*b*) the resultant pressure in the horizontal joint at the springing, (*c*) the maximum stress in the vertical line coinciding with the side of the abutment. *Ans.* (*a*) 21,182 lbs.; (*b*) 38,700 lbs.; (*c*) 3852 lbs./sq. in. if $q=\frac{1}{6}$.

2. A masonry arch, of 48 ft. span and 17 ft. rise and having an intrados consisting of two plane faces, springs from abutments 6.2 ft. thick and with vertical faces. The outer thrusts of the abutments are produced to meet the extrados, which is horizontal; the depth of the key is 3 ft. and the specific weight of the masonry is 150 lbs. per cubic foot. The centre of resistance at the springing is at the middle point and one foot below the extrados at the crown. Find the resultant pressures at the crown and springing. Also find the maximum stresses of the vertical joint aligning with the inside of the abutment.

3. The intrados of an arch of 100 ft. span and 20 ft. rise is the segment of a circle. The arch ring has a uniform thickness of 3 ft. and weighs 140 lbs. per cubic foot; the superincumbent load may be taken at 480 lbs. per lineal foot of the ring. Determine the mutual pressures at the key and springing, their points of application being 2 ft. and $1\frac{1}{2}$ ft. respectively from the intrados. Also find the curve of the centres of pressure. *Ans.* 56,546 lbs.; 76,676 lbs.

4. Assuming that an arch may be divided into elementary portions by imaginary joint planes parallel to the direction of the load upon the arch, find the limiting span of an arch with a horizontal upper surface and a parabolic soffit (latus rectum $=40$ ft.), the depth over the crown being 6 ft. and the specific weight of the load 120 lbs. per cubic foot; the thrust of the crown is horizontal ($=P$) and 4 ft. above the soffit.

5. Fig. 938 represents one half of a masonry arch of 3 ft. rise and weighing 120 lbs. per cubic foot. The centres of resistance S and T are at the middle point of AB and at 1 ft. below D. Find the resultant thrusts at S and T, and determine the maximum intensity of stress in the vertical joints AC and EF.

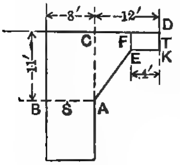


FIG 938.

6. A masonry arch of 96 ft. span and 24 ft. rise, with a parabolic intrados and a horizontal extrados, springs from abutments with vertical faces, the outside faces being carried up to meet the extrados. The depth of the masonry at the key is 6 ft. The centre of pressure is 2 ft. from the extrados at the key and at the middle of the joint at the springing. The masonry weighs $150\frac{1}{2}$ lbs. per cubic foot; width of abutment = 12 ft. Find the resultant pressures at the key and at the springing.

Ans. 83,592 lbs.; 177,100 lbs.

7. A masonry arch for a span of 40 ft. and a rise of 10 ft. springs from abutments with vertical faces and 10 ft. thick. The masonry has a depth of $3\frac{1}{3}$ ft. at the crown, is level from abutment to abutment, and weighs 150 lbs. per cubic foot. The intrados is a circular arc. The centres of resistance at the springing and at the crown are $4\frac{1}{2}$ ft. from the inside face of the abutment and 2 ft. above the crown respectively. Find the resultant pressures at the crown, at the springing, and in the vertical aligning with the inside face of an abutment.

8. A masonry arch for a span of 40 ft. and a rise of 10 ft. springs from abutments with vertical faces and 10 ft. thick. The masonry has a depth of $3\frac{1}{3}$ ft. at the crown, is level from abutment to abutment, and weighs 150 lbs. per cubic foot. The intrados is a circular arc. The centres of resistance at the springing and at the crown are $4\frac{1}{2}$ ft. from the inside face of the abutment and 2 ft. above the crown respectively. Find the resultant pressures at the crown, at the springing, and in the vertical aligning with the inside face of an abutment.

9. The soffit of an arch of 30 ft. span and 12 ft. rise is a transformed catenary. The masonry rises 12 ft. over the crown, and the specific weight of the load upon the arch may be taken at 120 lbs. per cubic foot. Determine the direction and amount of the thrust at the springing. *Ans.* 32,408 lbs.; 61° 16′.

10. A concrete arch has a clear spring of 15 ft. and a rise of 10 ft.; the height of masonry over crown = 15 ft.; the weight of the concrete = 144 lbs. per cubic foot. Determine the transformed catenary, the amount and direction of the thrust at the springing, and the curvatures at the crown and springing.

11. Determine the transformed catenary for an arch of 60 ft. span and 45 ft. rise, the masonry rising 18 ft. over the crown and weighing 120 lbs. per cubic foot. Also find the amount and direction of the thrust at the abutments. *Ans.* $m = 15.586$; 116,690 lbs.; 75° 32′.

12. A concrete arch for a span of 15 ft. and a rise of 6 ft. has a depth of 9 ft. of masonry over the crown. The concrete weighs 144 lbs. per cubic foot. Determine the transformed catenary and the thrusts at the crown and springing.

13. A concrete arch of 44 ft. span and 16 ft. rise weighs 144 lbs. per cubic foot; the masonry rises 24 ft. over the crown. Determine the transformed catenary and the thrust at the springings. (Take $\log_e 3 = 1.1$.)

14. The metal of an arch of 100 ft. span and 25 ft. rise has a modulus of elasticity of 28,000,000 lbs. per square inch and a coefficient of expansion of 0.0000055. If the temperature rise 50°, find the corresponding horizontal thrust. *Ans.* 92.4.

15. A concrete arch has a clear spring of 75 ft. and a rise of $37\frac{1}{2}$ ft.; the height of the masonry over the crown is 25 ft. and the weight of the concrete is 144 lbs. per cubic foot. Determine the transformed catenary, the amount and direction of the thrust at the springing and the curvatures at the crown and springing.

Ans. $m = 23.934$; 214,053 lbs.; 67° 20′; 22.914 ft.; 160 ft.

16. A 3-pin arch, 100 ft. span, 20 ft. rise, loaded with 2000 lbs. per horizontal foot run for 1st quarter span, 3000 lbs. for 2d quarter, 4000 lbs. for 3d quarter, and 1000 lbs. for 4th quarter.

Divide the load into eight parts and draw the line of resistance. Also determine the horizontal thrust and the maximum B.M. at any point of the arch.

17. A semicircular rib, pivoted at the crown and springings, is loaded

uniformly per horizontal unit of length. Determine the position and magnitude of the maximum bending moment, and show that the horizontal thrust on the rib is *one fourth* of the total load. *Ans.* $\frac{1}{8}wr^2$ at $.866r$ from support.

18. Draw the linear arch for a semicircular rib of uniform section under a load uniformly distributed per horizontal unit of length (*a*) when hinged at both ends; (*b*) when hinged at both ends and at the centre; (*c*) when fixed at both ends.

Ans. (*a*) A parabola z being $\frac{33}{28}r$; (*b*) a parabola through the three hinges; (*c*) $y_1 = \frac{29}{154}r = y_2$, $z = \frac{167}{154}r$.

19. A semicircular rib of 28 ft. span carries a weight of ¼ ton at 10 ft. and a weight of ½ ton at 21 ft. (measured horizontally) from the left support. Find the thrust and shear at the centre of the rib and at the point at which the weight is concentrated, (*a*) when both ends are hinged; (*b*) when both ends are fixed.

20. An arch (Fig. 939) of 52 ft. span, and 10⅓ ft. rise has a depth at the crown of 2½ ft. and at the springing line of 3⅔ ft. The loads per foot of breadth beginning with *a* are 23.04, 19.45, 16.35, 13.94, 12.01, 10.35, 8.98. 7.93, 6.32, 5.79, 5.52, 5.39 cwts. Find the horizontal thrust per foot of breadth,
Ans. 129 cwt.

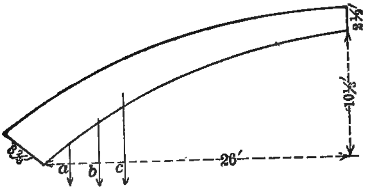


FIG. 939.

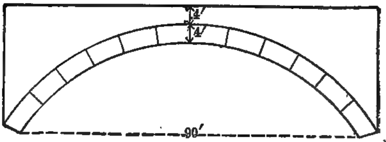


FIG. 940.

21. The arch (Fig. 940) has a clear span of 90 ft. and radius of 50 ft. Thickness of arch-ring is 4 ft. Draw the line of resistance, the load being 500 lbs. per foot run.

22. Determine the stability of a segmental arch of 40 ft. span and 25 ft. radius (Fig. 941); the loads in hundredweights being as shown. There is

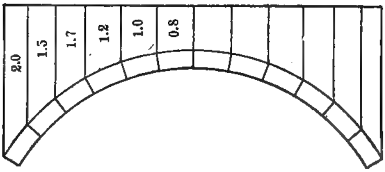


FIG. 941.

also a concentrated load of 2½ tons per foot of width at 10 ft. from the centre. Depth of arch = 2 ft.

23. A semi-elliptic rib (axes $2a$ and $2b$) is pivoted at the springings. Find the position and magnitude of the maximum bending moment, the load being uniformly distributed per horizontal unit of length.

How will the result be affected if the rib is also pivoted at the crown?

24. A pin-ended parabolic arch of 100 ft. span and 25 ft. rise carries a weight P at 25 ft. from the left support. Find the horizontal thrust.

Ans. $\frac{11}{20}P$.

25. A pin-ended parabolic arch of 100 ft. span and 25 ft. rise is acted upon by a horizontal force Q at 25 ft. from the left support. Find the horizontal thrust. *Ans.* $.574\,Q$.

26. In a parabolic arch of 50 ft. span and 10 ft. rise, hinged at both ends, a weight of 1 ton is concentrated at a point whose horizontal distance from the crown is 10 ft. Find the total thrust along the axis of the rib on each side of the given point, allowing for a change of 60° from the mean temperature ($\varepsilon = .0000694$). *Ans.* $.7941 \mp .1195I$.

27. Solve the preceding example, assuming both ends to be fixed.

28. An arched rib with a parabolic axis, of 100 ft. span and 12½ ft. rise, is loaded with 1 ton at the centre and 1 ton at 20 ft. from the centre, measured horizontally. Determine the thrusts and shears along the rib at the latter point, and show how they will be affected by a change of 100° F. from the mean; the coefficient of linear expansion being .00125 for 180° F. Take both ends hinged.

29. Solve the preceding example assuming both ends to be fixed.

30. A parabolic arched rib hinged at the ends, of 64 ft. span and 16 ft. rise, is loaded with 1 ton at each of the points of division of eight equal horizontal divisions. Find the horizontal thrust on the rib, allowing for a change of 60° F. from the mean temperature. Also find the maximum flange stresses, the rib being of double-tee section and 12 ins. deep throughout. (Coefficient of linear expansion per 1° F. $= 1 \div 144000$.)

31. The axis of an arched rib of 50 ft. span, 10 ft. rise, and hinged at both ends is a parabola. Draw the linear arch when the rib is loaded with two weights each equal to 2 tons concentrated at two points 10 ft. from the centre of the span. If the rib is of double-tee section and 24 ins. deep, find the maximum flange stresses. *Ans.* $\frac{3.176}{A} \pm \frac{254}{I}$.

If the arch is loaded so as to produce a stress of 10,000 lbs. per square inch in the metal, show that the rib will deflect .029 ft., E being 25,000,000 lbs.

32. Solve the preceding example, assuming both ends to be fixed.

33. A steel parabolic arched rib of 50 ft. span and 10 ft. rise is hinged at both ends and loaded at the centre with a weight of 12 tons. Find the horizontal thrust on the rib when the temperature varies 60° F. from the mean,

and also find the maximum flange stresses, the rib being of double-tee section and 12 ins. deep. *Ans.* $11\frac{23}{32} \mp \frac{7EI}{453120}$: $\frac{H}{A} \pm \frac{720}{I}(15-H)$.

34. Solve the preceding example, assuming both ends to be fixed.

35. A parabolic rib of 48 ft. span and 12 ft. rise carries a weight or 1 ton at the centre and at two points each 12 ft. (measured horizontally) from the centre. The rib is pin-ended.

Determine the horizontal thrust on the rib and draw the linear arch. Find the thrust and shears at the points at which the weights are concentrated.

Ans. 1.8945 ton; *thrusts* 1.72 and 2.2 ton; *shears* .39 and .49 tons.

36. An arch with fixed ends of 100 ft. span and 25 ft. rise carries a weight P at 25 ft. from the left support. Determine the reactions, the horizontal thrust and the B.M. ordinates, at the ends and at P.

Ans. $.844P$; $.156P$; $.5265\ P$; $-10'$, $7'.78$, $30'$.

37. In the preceding example determine how the results are modified if P is replaced by a horizontal force Q.

Ans. $+.1053Q$; $-.1053Q$; $.633Q$; $8'.33$, $7'.97$.

38. An arch with fixed ends of 100 ft. span and 25 ft. rise is constructed of metal having a modulus of elasticity of 28,000,000 lbs per square inch, and a coefficient of expansion of .0000055. If the temperature rises 50°, determine the ordinates of the linear arch at the springings and at the centre and also find the horizontal thrust. *Ans.* $16'\frac{2}{3}$: 554.4 lbs.

39. An arched parabolic rib of 64 ft. span and 8 ft. rise, carries a load of 2 tons at the centre and at 8 and 16 ft. from the centre measured horizontally. Determine the axial thrusts and shears at the points at which the weights are concentrated and also the *abs.* max. B.M. (*a*) When the arch is pin-ended; (*b*) when the arch is fixed at both ends.

Also determine (*c*) the deflection in each case.

40. If a flat parabolic arched rib, of 60 ft. span and 16½ ft. rise, is loaded at 10 ft. from the centre, measured horizontally, with 650 lbs., draw the linear arch and find the B.M. and shear at the point at which the weight is concentrated.

What weight at the centre of the rib will give the same horizontal thrust?

41. A parabolic arched rib, of 80 ft. span and 13⅓ ft. rise, with both ends fixed, carries three weights of 2 tons, 4 tons, and 6 tons at 10, 20, and 30 ft. from one end. Draw the equilibrium polygon and determine the thrust and shear on each side of the point at which the 6-ton load is concentrated.

42. A semicircular arched rib of 40 ft. span is loaded at the centre and at two points, each 12 ft. from the centre, measured horizontally, with a weight of 1 ton. Find the axial thrusts on the rib at the centre and at the points where the weights are concentrated.

43. Draw the equilibrium polygon for a flat parabolic arch, hinged at both

ends, of 100 ft. span and 20 ft. rise, loaded with a weight of 2 tons at 25 ft., measured horizontally from the centre.

44. A flat parabolic arched rib of 100 ft. span and 18 ft. rise is hinged at the springing and carries a load of 4000 lbs. concentrated 10 ft., measured horizontally, from the centre. Find the horizontal thrust on the rib and also find the axial thrust and shears at the point at which the load is concentrated. Determine the three points at which the B.M. is *nil.*

45. An arched parabolic rib of 64 ft. span and 8 ft. rise carries a load of 2 tons at the centre and at 8 ft. and 16 ft., measured horizontally from the centre. Determine the thrusts and shears at the load 8 ft. from the centre.

46. A flat parabolic-arch rib of 100 ft. span and 18 ft. rise is fixed at the springing and carries a load of 4000 lbs. concentrated 10 ft., measured horizontally, from the centre. Find the horizontal thrust on the rib and also find the axial thrust and shears at the point at which the load is concentrated. Determine the three points at which the B.M. is *nil.*

47. Draw the linear arch for a flat parabolic arch with both ends fixed of 22½ ft. rise and 90 ft. span, loaded at the centre and at one-quarter span with a weight of 1 ton. If both ends were hinged, what should be the weight at the centre and at the quarter span to produce the same horizontal thrust?

48. Find the skin stress due to change of curvature in a two-hinged arch rib, on account of its own dead weight, which produces a mean compressive stress of 6 tons per square inch. ($E=12{,}000$ tons per square inch.) Span, 550 ft.; rise, 114 ft.; depth of rib, 15 ft. Also find the deflection due to a stationary test load which produces a further mean compressive stress f_c of one ton per square inch.

49. A pin-ended arch of 100 ft. span and 25 ft. rise is in the form of a circular arch. Find the horizontal thrust due to (*a*) a weight of 100 lbs. at 25 ft. from the left support; (*b*) a horizontal force of 100 lbs. at the same point. *Ans.* (*a*) 57 lbs.; (*b*) 55.7 lbs.

50. How will the results in the preceding example be modified if both ends are fixed? *Ans.* (*a*) 54.6 lbs.

51. Draw the equilibrium polygon for a parabolic arch of 100 ft. span and 20 ft. rise when loaded with weights of 3, 2, 4, and 2 tons, respectively, at the end of the third, sixth, eighth, and ninth division from the left support, of ten equal horizontal divisions. (Neglect the weight of the rib.) If the rib consist of a web and of two flanges 2½ ft. from centre to centre, determine the maximum flange stress. Find the flange stresses at the ends of the rib, and also at the points at which the weights are concentrated. Both ends are absolutely fixed. *Ans.* $y_1=1.98$; $y_2=-3.434$; $H=6.8644$ tons; $\frac{8.7933}{A}\pm\frac{2450.25}{I}$, $\frac{7.2076}{A}\pm\frac{743.4}{I}$, $\frac{6.96}{A}\pm\frac{629.28}{I}$, $\frac{7.6147}{A}\pm\frac{3236.76}{I}$, $\frac{8.1494}{A}\pm\frac{310.356}{I}$.

52. The axis of an arched rib hinged at both ends, for a span of 50 ft. and a rise of 10 ft., is a parabola. Draw the equilibrium polygon when the arch is loaded with two equal weights of 2 tons concentrated at two points 10 ft. from the centre of the span. Also determine the maximum flange stress in the rib, which is a double-tee section 2 ft. deep.

53. Solve the preceding example when both ends are fixed.

Ans. $y_1 = y_2 = \frac{20}{3}$; $H = 3.3075$ tons; $\frac{3.3075}{A} \pm \frac{291.6}{I}$; max. B.M = 2.025 ft.-tons.

54. The load upon a parabolic rib of 50 ft. span and 15 ft. rise, hinged at both ends, consists of weights of 1, 2, and 3 tons at points 15, 25, and 40 ft., respectively, from one end. Find the axial thrusts and the shears at these points.

Ans. $H = 2.9915$ tons; *axial thrusts,* 3.2595, 3.6924, 2.8365, and 4.59 tons; *shears,* 0.1231, 0.8179, 1.1806, and 1.2536 tons.

55. Solve the preceding example when both ends are fixed.

56. A parabolic arched rib of 100 ft. span and 20 ft. rise is fixed at the springings. The uniformly distributed load upon one half of the arch is 100 tons, and upon the other 200 tons. Find the bending moment and shearing force at 25 ft. from each end.

Ans. B.M. $-\frac{5}{8}H + \frac{8750}{3}, -\frac{5}{8}H + \frac{4875}{6}$. S.F. $87\frac{1}{2} - .4H, -112\frac{1}{2} + .4H$.

57. A wrought-iron parabolic rib of 96 ft. span and 16 ft. rise is hinged at the two abutments; it is of a double-tee section uniform throughout, and 24 in. deep from centre to centre of the flanges. Determine the compression at the centre, and also the position and amount of the maximum bending moment (*a*) when a load of 48 tons is concentrated at the centre; (*b*) when a load of 96 tons is uniformly distributed per horizontal unit of length.

Determine (*c*) the deflection of the rib in each case.

Ans. (*a*) $\frac{56.25}{A} + \frac{36288}{I}$; max. B.M. is at crown and = 252 ft.-tons.

(*b*) $\frac{86.4}{A}$.

58. Design a parabolic arched rib of 100 ft. span and 20 ft. rise, hinged at both ends and at the middle joint; dead load = 40 tons uniformly distributed per horizontal unit of length, and live load = 1 ton per horizontal foot.

59. Show how the calculations in the preceding question are affected when both ends are absolutely fixed.

60. In the framed arch represented by the figure, the span is 120 ft., the rise 12 ft., the depth of the truss at the crown 5 ft., the fixed load at each top joint 10 tons, and the moving load 10 tons. Determine the maximum stress in each member with any distribution of load. Show that, approximately, the amount of metal required for the arch: the amount required for a bowstring lattice girder of the same span

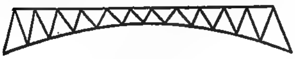


FIG. 942.

and 17 ft. deep at the centre: the amount required for a girder of the same span and 12 ft. deep : : 100:155:175.

61. Weights W_1, W_2, W_3 . . . are concentrated at a series of points in horizontal girder of span $2l$ resting upon supports at the ends, the corresponding B.M.s being M_1, M_2, M_3, . . . respectively. If a flat parabolic arch, fixed at both ends, of the same span and of rise k, carry the same weights at points vertically above the points in the girder, show that the horizontal thrust on the arch is

$$\frac{15}{8lk}\sum\frac{M^2}{W}.$$

62. Show that a weight at the crown of a flat parabolic arch fixed at both ends will produce the same horizontal thrust as a weight at any other point dividing the horizontal span into two segments, ml, nl, if the two weights are in the ratio of $(mn)^2$ to 1.

63. The steel parabolic ribs for one of the Harlem River bridges has a clear opening of 510 ft., a rise of 90 ft., a depth of 13 ft., and are spaced 14 ft. centre to centre. The dead weight per lineal foot is estimated at 33,000 lbs. and the live load at 8000 lbs.; a variation in temperature of 75° F. from the mean is also to be allowed for. Determine the maximum bending moment (assuming I constant) and the maximum deflection. (E=26,000,000 lbs.) Show how to deduce the play at the hinges.

64. A cast-iron arch (see figure) whose cross sections are rectangular and uniformly 3 ins. wide has a straight horizontal extrados and is hinged at the centre and at the abutments. Calculate the normal intensity of stress at the top and bottom edges D, E of the vertical section, distant 5 ft. from the centre of the span, due to a vertical load of 20 tons concentrated at a point distant 5 ft. 4 in. horizontally from B. Also find the maximum intensity of the shearing stress on the same section, and state the point at which it occurs. (AB=21 ft. 4 in.)

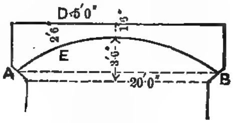


FIG. 943.

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